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DEFINITIONS FOR UNIFORM DOMAINS

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1. Introduction

The concept of an (α, β)-uniform domain D in R^n was introduced in [MS].

Although useful in several applications, this definition has no immediate conformally invariant meaning in $\bar{R}^n = R^n \cup \{\infty\}$. Here we present a very simple condition, called δ-uniformity and based on the concept of the cross ratio, which turns out to be equivalent to the (α, β)-uniformity in the case $D \subset R^n$.

A domain $D \subset \bar{R}^n$ is called δ-uniform, $0 < \delta \leqq 1$, if for all $x_1, x_2 \in D$, $x_1 \neq x_2$, there is a continuum K in D connecting x_1 to x_2 with

$$\frac{|x-y|}{|x-x_i|} \frac{|x_1-x_2|}{|y-x_j|} \geqq \delta, \quad i \neq j, \; i, j = 1, 2$$

for all $x \in K \setminus \{x_1, x_2\}$ and $y \in \bar{R}^n \setminus D$.

In order to prove that a domain $D \subset R^n$ is δ-uniform if and only if it is (α, β)-uniform we consider two other definitions for the (α, β)-uniformity in Chapters 3 and 4. The main equivalence is then proved in Chapter 5. Chapter 6 is devoted to some general properties of uniform domains, e.g. if D is a δ-uniform domain, the Hausdorff-dimension of ∂D satisfies $\dim_H \partial D \leqq c < n$, where c depends only on δ and n.

In $\bar{R}^2$ a simply connected domain D is δ-uniform if and only if $D = \bar{R}^2$ or $D = \bar{R}^2 \setminus \{z\}$ or D is a quasiconformal disc. In the case of Jordan domains this gives a new and simple characterization of quasiconformal discs in $\bar{R}^2$.

Notation will be as in [MS] and generally standard. A rectifiable path is always parametrized by means of arc length. If $\gamma\colon [a, b] \to \bar{R}^n$ is a path, then $|\gamma| = \gamma[a, b]$ is the locus of γ.

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2. John domains and (α, β)-uniform domains

First we recall the definitions presented in [MS].

2.1. Definition. A domain $G\subset R^n$ is called an (α, β)-John domain, $0<\alpha\leqq\beta<\infty$, if there is $x_0\in D$ such that every $x\in D$ can be joined to x_0 by a rectifiable path γ: $[0, d]\to D$ such that $d\leqq\beta$ and

$$\text{dist}\left(\gamma(t), \partial D\right) \geqq \frac{\alpha}{d}t \quad \text{for} \quad t\in[0, d]. \tag{2.2}$$

The point x_0 is called a center of D.

2.3. An alternative characterization for John domains was given in [MS, Lemma 2.7]. Here we shall need the following fact included in the proof of that lemma: Suppose that $G\subset R^n$ is a domain, $x, x_0\in G$ and x can be joined to x_0 using a path γ: $[0, 1]\to D$ such that for some $\delta\in(0, 1]$

$$\gamma[0, t] \subset B^n\left(\gamma(t), \frac{1}{\delta}\,\text{dist}\left(\gamma(t), \partial G\right)\right), \quad 0 \leqq t \leqq 1. \tag{2.4}$$

Then there exists a rectifiable path γ_1: $[0, d]\to G$ joining x to x_0 and satisfying

$$d \leqq |x-x_0|/\varphi^2 \tag{2.5}$$

$$\text{dist}\left(\gamma_1(t), \partial D\right) \geqq \varphi t, \quad t\in[0, d], \tag{2.6}$$

and φ depends only on n and δ.

2.7. Definition. A domain $D\subset R^n$ is called (α, β)-uniform, $0<\alpha\leqq\beta<\infty$, if for each pair of points $x_1, x_2\in D$, $x_1\neq x_2$, there is an $(\alpha|x_1-x_2|, \beta|x_1-x_2|)$-John domain G such that $x_1, x_2\in G\subset D$.

2.8. Remark. The definition for uniformity in the above sense can easily be extended for domains in $\overline{R}^n$; for instance, a domain $D\subset\overline{R}^n$ can be called (α, β)-uniform if $D\cap R^n$ is (α, β)-uniform in the sence of 2.7. However, we only consider (α, β)-uniform domains in R^n since the definition for δ-uniform domains applies to $\overline{R}^n$ and reduces to (α, β)-uniformity in R^n; see Chapter 5.

3. Domains of type (α, β)

3.1. Definition. A domain $D\subset R^n$ is of type (α, β), $0<\alpha\leqq\beta<\infty$, if for each pair of points $x_1, x_2\in D$, $x_1\neq x_2$, there is a rectifiable path γ: $[0, d]\to D$ joining x_1 to x_2 and

$$d \leqq \beta\,|x_1-x_2| \tag{3.2}$$

$$\text{dist}\left(\gamma(t), \partial D\right) \geqq \frac{\alpha}{\beta}\,p_d(t),\ t\in[0, d], \quad \text{where} \quad p_d(t) = d/2-|t-d/2|,\ t\in[0, d]. \tag{3.3}$$

3.4. Theorem. (a) *If D is (α, β)-uniform, then D is of type $(\alpha, 2\beta)$.* (b) *If D is of type (α, β), then D is $\big((\alpha/\beta)^2/32, \beta\big)$-uniform.*

Proof. For (a) suppose that D is (α, β)-uniform. Let $x_1, x_2 \in D$, $x_1 \neq x_2$. Then there exist rectifiable paths γ_i: $[0, d_i] \to D$ joining x_i to a point x_0 of D and

$$d_i \leqq \beta |x_1 - x_2|, \quad \operatorname{dist}\big(\gamma_i(t), \partial D\big) \geqq \alpha |x_1 - x_2| t/d_i.$$

Since $\alpha |x_1 - x_2| t/d_i \geqq \alpha t/\beta$, the composed path $\gamma = \gamma_2^{-1}\gamma$: $[0, d_1 + d_2] \to D$ clearly satisfies (3.2) and (3.3) for the required α and β.

For (b) assume that D is of type (α, β). Let $x_1, x_2 \in D$, $x_1 \neq x_2$, and let γ be a path as in 3.1. Write for $t > 0$

$$A_t = \bigcup_{s \in (0, d)} B^n\big(\gamma(s), t p_d(s)\big).$$

Then A_t is a domain in D provided $t \in (0, \alpha/\beta)$.

Let $\varepsilon = (1/4) \min\big(d, \operatorname{dist}(A_{\alpha/2\beta}, \partial D)\big) > 0$. Set $G = A_{\alpha/2\beta} + B^n(\varepsilon)$. We claim that G is an (α', β')-John domain with

$$\alpha' = (\alpha/\beta)^2 |x_1 - x_2|/32, \quad \beta' = \beta |x_1 - x_2|$$

and $x_1, x_2 \in G \subset D$. The last two assertions are trivial and it remains to prove the first.

Set $x_0 = \gamma(d/2)$ and let $y \in G$. Fix $s \in [0, d]$ such that $y \in B^n\big(\gamma(s), \alpha p_d(s)/2\beta\big) + B^n(\varepsilon)$. By symmetry we may assume $s \leqq d/2$. Let γ_1: $[0, d_1] \to G$ represent a straight line segment joining y to $\gamma(s)$. The composed path $\gamma_2 = (\gamma | [s, d/2])\gamma_1$ joins y to x_0 in G. Let γ_2: $[0, l] \to G$ with arc length as parameter. Now

$$l \leqq d_1 + d/2 \leqq \alpha s/2\beta + \varepsilon + d/2 \leqq \alpha d/4\beta + \varepsilon + d/2 \leqq d \leqq \beta |x_1 - x_2|. \tag{3.5}$$

Choose $t \in [0, l]$. If $t \in [0, d_1]$, then clearly

$$\operatorname{dist}\big(\gamma_2(t), \partial G\big) \geqq t/2 \geqq \alpha t/2\beta.$$

Suppose $t > d_1$. Pick $s' \in [s, d/2]$ with $d/2 - s' = l - t$. Since $l - d/2 \leqq \varepsilon$, the inequality $s' + \varepsilon \geqq t$ holds. If $t/2 \geqq \varepsilon$,

$$\operatorname{dist}\big(\gamma_2(t), \partial G\big) = \operatorname{dist}\big(\gamma(s'), \partial G\big) \geqq \frac{\alpha}{2\beta} s' \geqq \frac{\alpha}{2\beta}(t - \varepsilon) \geqq \frac{\alpha}{4\beta} t. \tag{3.6}$$

On the other hand, if $t/2 < \varepsilon$, then

$$\operatorname{dist}\big(\gamma_2(t), \partial G\big) \geqq \varepsilon + \frac{\alpha}{2\beta} s' \geqq \frac{\alpha}{2\beta}(\varepsilon + s') \geqq \frac{\alpha}{2\beta} t. \tag{3.7}$$

The conclusion now follows from

3.8. Lemma. *Suppose that $G \subset R^n$ is a domain and $x_0 \in G$. If there exist numbers $0 < \alpha \leqq \beta < \infty$ such that every point $x \in G$ can be joined to x_0 using a rectifiable path γ: $[0, d] \to G$ with $d \leqq \beta$ and*

$$\operatorname{dist}\big(\gamma(t), \partial G\big) \geqq \frac{\alpha}{\beta} t, \tag{3.9}$$

then G is an $\big((\alpha/\beta)^2 \operatorname{dia}(G)/2, \beta\big)$-John domain.

3.10. Remark. The only difference is in the lower bounds of (2.2) and (3.9).

Proof for Lemma 3.8. Set $p=\alpha\,\mathrm{dia}\,(G)/2\beta$. Clearly $\mathrm{dia}\,(G)\leqq 2\beta$ and $B^n(x_0, p)\subset G$. Note also that $(\alpha/\beta)^2\,\mathrm{dia}\,(G)/2\leqq\beta$.

Let $x\in G$ and let $\gamma\colon [0, d]\to G$ be a path as in (3.9). If $d<p$, then instead of γ we can use a straight line segment γ' connecting x to x_0, and it is not difficult to see that

$$\mathrm{dist}\,(\gamma'(t), \partial G) \geqq pt/d', \quad t\in[0, d'],$$

where d' is the length of γ'. On the other hand, if $d\geqq p$, then

$$\mathrm{dist}\,(\gamma(t), \partial G) \geqq \alpha t/\beta = \alpha dt/\beta d \geqq (\alpha/\beta)^2\,\mathrm{dia}\,(G)\,t/2d.$$

The lemma follows.

To finish the proof of (b) note that by (3.6) and (3.7)

$$\mathrm{dist}\,(\gamma_2(t), \partial G) \geqq \frac{\alpha}{4\beta}\,t = \frac{\alpha}{4\beta}\,\frac{|x_1-x_2|}{|x_1-x_2|}\,t. \tag{3.11}$$

Since $\mathrm{dia}\,(G)\geqq|x_1-x_2|$, Lemma 3.8 implies by (3.5) and (3.11) that G is a $\big(32^{-1}(\alpha/\beta)^2|x_1-x_2|, \beta|x_1-x_2|\big)$-John domain. This completes the proof for (b).

4. Condition A(δ)

For uniform domains this condition is a counterpart of [MS, Lemma 2.7] given for John-domains.

4.1. Definition. A domain $D\subset R^n$ satisfies the condition A(δ), $0<\delta\leqq 1$, if for all $x_1, x_2\in D$ there is a path $\gamma\colon [0, s]\to D$ (not necessarily rectifiable) joining x_1 to x_2 and

$$\mathrm{dia}\,|\gamma| \leqq |x_1-x_2|/\delta \tag{4.2}$$

$$\gamma[0, t] \subset \bar{B}^n\Big(\gamma(t), \frac{1}{\delta}\,\mathrm{dist}\,(\gamma(t), \partial D)\Big), \quad 0\leqq t\leqq s/2 \tag{4.3}$$

$$\gamma[t, s] \subset \bar{B}^n\Big(\gamma(t), \frac{1}{\delta}\,\mathrm{dist}\,(\gamma(t), \partial D)\Big), \quad s/2\leqq t\leqq s. \tag{4.4}$$

4.5. Theorem. (a) *If a domain $D\subset R^n$ satisfies the condition* A(δ), $0<\delta\leqq 1$, *then D is of type (α, β), where $0<\alpha\leqq\beta<\infty$ depend only on δ and, possibly, on n.* (b) *If D is of type (α, β), then D satisfies* A(δ) *with* $\delta=\min\,(1/\beta, \alpha/\beta)$.

Proof. The proof for (a) rests on 2.3. Suppose that D satisfies the condition A(δ). Let $x_1, x_2\in D$ and let $\gamma\colon [0, s]\to D$ be a path as in 4.1. We may assume

$s=2$. Write $x_0=\gamma(1)$. By 2.3 there are rectifiable paths γ_i: $[0, d_i]\to D$ connecting x_i to x_0, $i=1, 2$, with

$$\text{dist}\,(\gamma_i(t), \partial D) \geqq \varphi t \tag{4.6}$$

$$d_i \leqq |x_i-x_0|/\varphi^2, \tag{4.7}$$

where $\varphi\in(0, 1]$ depends only on δ and n. Letting $\gamma=\gamma_2^{-1}\gamma_1$ it is not difficult to see that γ satisfies (3.2) and (3.3) with $\beta=2/\varphi^2\delta$, $\alpha=1$.

The proof for (b) is easy. Suppose that D is of type (α, β). Pick $x_1, x_2\in D$, $x_1\neq x_2$, and let γ: $[0, d]\to D$ be a rectifiable path joining x_1 to x_2 as in (3.2) and (3.3). Now $\text{dia}\,(|\gamma|)\leqq d\leqq\beta|x_1-x_2|$ and for $t\in[0, d/2]$

$$\gamma[0, t] \subset \bar{B}^n(\gamma(t), t) \subset \bar{B}^n(\gamma(t), (\beta/\alpha)\,\text{dist}\,(\gamma(t), \partial D)),$$

and, by symmetry, the same holds for $\gamma[t, d]$ in the interval $[d/2, d]$. Thus γ satisfies (4.2)—(4.4) with $\delta=\min\,(1/\beta, \alpha/\beta)$. The proof is complete.

5. δ-uniform domains

5.1. Definition. A domain $D\subset\bar{R}^n$ is called δ-uniform, $0<\delta\leqq 1$, if for all $x_1, x_2\in D$, $x_1\neq x_2$, there is a continuum K joining x_1 to x_2 such that the cross ratio

$$(x, y, x_i, x_j) = \frac{|x-y|}{|x-x_i|}\frac{|x_i-x_j|}{|x_j-y|} \geqq \delta, \quad i, j = 1, 2, \ i \neq j, \tag{5.2}$$

for all $x\in K\setminus\{x_1, x_2\}$ and $y\in\complement D$.

5.3. Remark. The cross ratio (x, y, x_1, x_2) is defined whenever all four points are distinct in $\bar{R}^n$. Especially, if $y=\infty$, then

$$(x, \infty, x_1, x_2) = |x_1-x_2|/|x-x_1|.$$

Observe that the cross ratio is a conformal invariant in $\bar{R}^n$. If x, y, x_1, x_2 are four distinct points of $\bar{R}^n$, then there is a Möbius transformation T with $T(x_1)=0$, $T(y)=\infty$ and $T(x)=e_1$ and we have

$$(x, y, x_1, x_2) = |T(x_2)|.$$

5.4. Theorem. (a) *If a domain* $D\subset R^n$ *is δ-uniform, then D is (α, β)-uniform and* $0<\alpha\leqq\beta<\infty$ *depend only on δ and n.*
(b) *If* $D\subset R^n$ *is (α, β)-uniform, then D is δ-uniform and* $\delta\in(0, 1]$ *depends only on α and β.*

Proof. For (a) fix $x_1, x_2\in D$, $x_1\neq x_2$, and let K be a connecting continuum as in 5.1. Let K_1 be a subcontinuum of K such that K_1 joins $S^{n-1}(x_1, s)$ to $S^{n-1}(x_2, s)$, $s=|x_1-x_2|/4$, in $D\setminus\bigcup_{i=1,2}B^n(x_i, s)$. Choose $x_0\in K_1\cap S^{n-1}(x_1, s)$.

By symmetry and trivial geometric considerations it suffices to show that x_1 can be joined to x_0 by a path ε such that

$$\varepsilon[0, t] \subset \bar{B}^n\left(\varepsilon(t), \frac{1}{\varkappa} \operatorname{dist}\big(\varepsilon(t), \partial D\big)\right) \tag{5.5}$$

where $\varkappa$ depends on δ and n, since then it is easy to see that D satisfies the A (δ')-condition and hence Theorems 4.5 and 3.4 show that D is (α, β)-uniform.

To construct the path ε define a sequence of points $z_0, z_1, \ldots$ and paths γ_i connecting z_i to z_{i+1} as follows. Set $z_0 = x_0$. Choose $z_1 \in K \cap S^{n-1}(x_1, s/2)$. Since D is δ-uniform, there is a continuum connecting z_0 to z_1 in D and satisfying (5.2). Without loss of generality we may assume that the continuum is a path. Call it γ_0. In general we pick $z_i \in S^{n-1}(x_1, s/2^i) \cap K$ and define γ_i similarly.

Fix i. For $y \in \complement D$ with $|z_i - y| = \operatorname{dist}(z_i, \complement D)$ the estimate

$$\operatorname{dist}(z_i, \complement D) \geqq \delta s |x_2 - y| / |x_1 - x_2| 2^i \geqq \delta |x_2 - y| / 2^{i+2} \tag{5.6}$$

holds. If now $|x_2 - y| < |x_1 - x_2|/2$ for some i, then ε can be chosen to be a straight line segment and the estimate (5.5) is trivial. Otherwise $|x_2 - y| \geqq |x_1 - x_2|/2$ for all i. Consequently (5.6) yields

$$\operatorname{dist}(z_i, \complement D) \geqq \delta |x_1 - x_2| / 2^{i+3} = r_i, \quad i = 0, 1, \ldots . \tag{5.7}$$

Consider γ_i. Let $x \in |\gamma_i| \setminus \bigcup_{j=i, i+1} B^n(z_j, r_j/2)$. For $y \in \complement D$

$$|x - y| \geqq \delta |x - z_i| \, |y - z_{i+1}| / |z_i - z_{i+1}|. \tag{5.8}$$

On the other hand, $|x - z_i| \geqq r_i/2$, $|y - z_{i+1}| \geqq r_{i+1}$ and $|z_i - z_{i+1}| \leqq 2s/2^i$; thus (5.8) yields

$$|x - y| \geqq \delta^3 |x_1 - x_2| / 2^{i+7}. \tag{5.9}$$

Clearly the same estimate holds if $x \in |\gamma| \cup \bigcup_{j=i, i+1} \bar{B}^n(z_j, r_j/2)$, since in this case (5.7) gives

$$|x - y| \geqq |y - z_j| - |x - z_j| \geqq r_j - r_j/2 = \delta |x_1 - x_2| / 2^{j+4}$$

for $j = i, i+1$. Compose the paths γ_i into a single path ε joining x_0 to x_1. It remains to show that ε satisfies (5.5).

To this end let $x \in |\varepsilon|$. Then $x \in |\gamma_i|$ for some i. For $z \in |\gamma_j|$

$$\begin{aligned} |z - x_1| &\leqq |x_1 - z_j| + |z - z_j| \leqq |x_1 - z_j| + s/\delta 2^{j-1} \\ &\leqq s/2^j + s/\delta 2^{j-1} \leqq 2s/\delta 2^{j-1}, \end{aligned} \tag{5.10}$$

since for $y = \infty$ (5.2) implies

$$|z_j - z| \leqq |z_j - z_{j+1}| / \delta \leqq 2s/\delta 2^j.$$

Let now $z \in |\gamma_j|$, $j = i, i+1, \ldots$ be arbitrary. By (5.9) and (5.10)

$$|z - x| \leqq |x - x_1| + |z - x_1| \leqq 4s/\delta 2^{i-1} \leqq \operatorname{dist}(x, \complement D)/\varkappa,$$

where $\varkappa = \delta^4 2^{-8}$. This proves (5.5).

To prove (b) suppose that the domain $D\subset R^n$ is (α, β)-uniform. Let $x_1, x_2\in D$, $x_1\neq x_2$. By Theorem 3.4, D is of type $(\alpha, 2\beta)$; hence there is a path $\gamma\colon [0, d]\to D$ joining x_1 to x_2 and satisfying (3.2) and (3.3). We shall show that $K=|\gamma|$ satisfies the condition (5.2), where δ depends only on α and β.

Let $x=\gamma(t)$, $t\in(0, d)$, $x\neq x_1$, x_2 and $y\in\complement D$. Suppose $t\leqq d/2$. Assume first $|x_2-y|\leqq 4\beta|x_1-x_2|$. Then

$$\frac{|x-y|\,|x_1-x_2|}{|x-x_1|\,|x_2-y|}\geqq\frac{\alpha p_d(t)\,|x_1-x_2|}{\beta t\,|x_2-y|}\geqq\frac{\alpha}{4\beta^2}.$$

If $|x_2-y|>4\beta|x_1-x_2|$, then

$$\frac{|x-y|}{|x_2-y|}\geqq\frac{|y-x_2|-|x-x_2|}{|x_2-y|}\geqq\frac{|y-x_2|-\beta|x_1-x_2|}{|x_2-y|}\geqq 3/4,$$

since the map $s\mapsto(s-\beta|x_1-x_2|)/s$ has the minimum $3/4$ in $[4\beta|x_1-x_2|, \infty)$. On the other hand,

$$\frac{|x_1-x_2|}{|x-x_1|}>\frac{|x_1-x_2|}{\beta|x_1-x_2|}=1/\beta$$

and thus

$$\frac{|x-y|}{|x-x_1|}\frac{|x_1-x_2|}{|x_2-y|}\geqq 3/4\beta.$$

Thus we have a lower bound for the cross ratio (x, y, x_1, x_2) in terms of α and β whenever $y\in\complement D$ and $x=\gamma(t)$, $t\leqq d/2$. Observe that the case $y=\infty$ is trivial.

Next consider (x, y, x_2, x_1) for $y\in\complement D$ and $x=\gamma(t)$, $t\leqq d/2$. Assume first $|x-y|\geqq c|y-x_1|$, $c=\alpha/\beta(1+\alpha/\beta)$. Now

$$(5.11)\qquad (x, y, x_2, x_1)\geqq c\,|x_1-x_2|/|x-x_2|\geqq c\,|x_1-x_2|/\beta\,|x_1-x_2|=c/\beta.$$

If $|x-y|<c|y-x_1|$, then

$$|x_1-x|\geqq|y-x_1|-|y-x|\geqq|y-x_1|-c\,|y-x_1|=|y-x_1|/(1+\alpha/\beta)$$

and hence

$$|y-x|\geqq\alpha t/\beta\geqq\alpha|x-x_1|/\beta\geqq c\,|y-x_1|,$$

a contradiction. Thus (5.11) holds in each case.

To complete the proof we observe that the above estimates also hold, by symmetry, in the case $x=\gamma(t)$, $d/2\leqq t\leqq d$. Hence we have the required lower bounds for the cross ratios in (5.2).

5.12. Remark. It is also possible to give alternative characterizations for uniform domains in terms of cross ratios. For instance, a domain $D\subset\bar{R}^n$ can be called t-uniform, $0<t\leqq 1$, if for all $x_1, x_2\in D$, $x_1\neq x_2$, there is a path $\gamma\colon [0, 1]\to D$ joining x_1 to x_2 such that for all $z_1, z_2\in|\gamma|$, $z_1\neq z_2$ and $x\in|\gamma|$ between z_1 and z_2 the estimate $(x, y, z_1, z_2)\geqq t$ holds for all $y\in\complement D$. To prove that this gives essentially the same concept as 5.1 requires lengthy technical constructions, which we omit here. Observe that this definition is also a conformal invariant in $\bar{R}^n$.

6. Properties of uniform domains

6.1. *Quasiconformal invariance.* It was shown in [MS, Theorem 2.15] that if a domain $D\subset R^n$ is (α, β)-uniform and $f\colon R^n\to R^n$ is a K-quasiconformal mapping, then fD is (α', β')-uniform and $0<\alpha'\leqq\beta'<\infty$ depend only on α, β, K and n. Since the cross ratio is invariant under Möbius transformations of $\bar{R}^n$, standard modulus estimates and a compactness argument give the following result.

6.2. Theorem. *There is a function* $\varphi_n\colon (0, 1]\times[1, \infty)\to(0, 1]$ *depending only on n such that*

$$\varphi_n(\delta,1) = \delta = \lim_{K\searrow 1} \varphi_n(\delta, K),$$

and if $D\subset\bar{R}^n$ *is a* δ*-uniform domain and* $f\colon \bar{R}^n\to\bar{R}^n$ *is* K*-quasiconformal, then* fD *is* $\varphi_n(\delta, K)$*-uniform.*

6.3. Remark. For $n=2$, because of Teichmüller's famous theorem, the change of the cross ratio under a quasiconformal map is known. Consequently, for $n=2$ the function $\varphi_n(\delta, K)$ can be calculated.

6.4. *Hausdorff dimension of* ∂D. It was proved in [GV] that if $f\colon \bar{R}^n\to\bar{R}^n$ is a quasiconformal mapping, then the Hausdorff dimension, $\dim_H$, of fS^{n-1} satisfies $\dim_H fS^{n-1}\leqq c<n$, where c depends only on n and K. Observe that $\dim_H fS^{n-1}$ can take values arbitrarily close to n. Thus the following theorem is sharp.

6.5. Theorem. *Suppose that* $D\subset\bar{R}^n$ *is a* δ*-uniform domain. Then* $\dim_H \partial D\leqq c<n$, *where* c *depends only on* δ *and* n.

Proof. For the proof we apply [S, Theorem 3.2]. Let $A\subset R^n$ and $z\in A$. Write

$$G(z, A) = \varliminf_{r\to 0} \sup_{x\in B^n(z,r)} \operatorname{dist}(x, A)/r$$

and $G(A)=\inf\{G(z, A)\colon z\in A\}$. If $G(A)>0$, then $\dim_H A\leqq c<n$, where c depends only on $G(A)$ and n; see [S, Theorem 3.2]. Thus it suffices to find a lower bound for $G(\partial D)$ depending only on δ.

We may assume $\partial D\cap R^n\neq\emptyset$. Fix $z\in\partial D\cap R^n$ and choose $r_0>0$ such that for all $r\in(0, r_0]$, $S^{n-1}(z, r)\cap D\neq\emptyset$.

Let $x_1\in S^{n-1}(z, r)\cap D$, $r\in(0, r_0]$. Pick $x_2\in D\cap S^{n-1}(z, r/4)$. Since D is δ-uniform, there is a continuum K joining x_1 to x_2 and $(x, y, x_1, x_2)\geqq\delta$ for all $x\in K\setminus\{x_1, x_2\}$ and $y\in\complement D$. Fix $x\in S^{n-1}(z, 3r/4)\cap K$ and then $y\in\complement D$ such that $|y-x|=\operatorname{dist}(\complement D, x)$. Now either $|y-x|\geqq r/16$ or $|y-x|<r/16$, in which case

$$|y-x|\geqq\delta\frac{|x-x_1|\,|y-x_2|}{|x_1-x_2|}\geqq\delta\frac{(r/4)(3r/4-r/16-r/4)}{r+r/4}>\frac{r\delta}{16}.$$

Hence in both cases

$$\sup_{x\in B^n(z,r)}\frac{1}{r}\operatorname{dist}(x, \partial D)\geqq\delta/16,\quad r\in(0, r_0].$$

This is the required lower bound.

6.6. *A metric property.* If $D \subset \bar{R}^n$ is a δ-uniform domain, it is easy to show that D is locally connected at boundary points (for the case $n=2$, see [MS, Lemma 2.29]). F. Gehring, see e.g. [G], has introduced an important metric property called b-locally connectedness. We recall the definition. A set $E \subset \bar{R}^n$ is said to be b-locally connected $1 \leqq b < \infty$ if, for all $z \in R^n$ and $r>0$, points in $E \cap \bar{B}^n(z, r)$ can be joined in $E \cap \bar{B}^n(z, br)$ and points in $E \setminus B^n(z, r)$ can be joined in $E \setminus B^n(z, r/b)$.

6.7. Theorem. *Suppose that $D \subset \bar{R}$ is a δ-uniform domain. Then D is b-locally connected and b depends only on δ.*

Proof. Let $z \in D \cap R^n$ and $r>0$. Since δ-uniform domains are invariant under the inversion in a ball, it suffices to show that points in $D \cap \bar{B}^n(z, r)$ can be joined in $D \cap \bar{B}^n(z, br)$, where b depends only on δ. Let $x_1, x_2 \in D \cap \bar{B}^n(z, r)$. Since D is δ-uniform, there is a continuum K joining x_1 to x_2 in D and satisfying (5.2). Set $b=1+5/\delta$ and $S=S^{n-1}(z, br)$. If $K \subset \bar{B}^n(z, br)$ or $S \subset D$, we have proved the claim. Otherwise let $x \in K \cap S$ and $y \in \complement D \cap S$. Now (5.2) yields

$$\frac{5}{\delta} r \leqq |x-x_1| \leqq \frac{1}{\delta} \frac{|x-y|\,|x_1-x_2|}{|y-x_2|} \leqq \frac{1}{\delta} \frac{2br\, 2r}{(b-1)r} = \frac{4}{5}\left(1+\frac{5}{\delta}\right) r,$$

clearly a contradiction since $\delta \in (0, 1]$. This proves the theorem.

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