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A CONTINUITY PROPERTY OF HOLOMORPHIC DIFFERENTIALS UNDER QUASICONFORMAL DEFORMATIONS

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Introduction

In this paper we shall investigate a continuity property of certain holomorphic Abelian differentials with respect to the Dirichlet norm under the quasiconformal deformation of Riemann surfaces. In the case of compact Riemann surfaces related studies have been made by L. Ahlfors, L. Bers and others. However, when we generalize such results to the class of open Riemann surfaces, we encounter many difficulties. For example, we do not generally know the existence of Teichmüller mappings, nor the existence and uniqueness of (square integrable) holomorphic differentials with prescribed periods along A_j-cycles. Hence we have to restrict our consideration to either certain classes of open Riemann surfaces over which some theorems used in the case of compact Riemann surfaces can be generalized, or certain classes of differentials on general surfaces with appropriate boundary behaviour.

In Chapter 1 we shall provide first some basic estimates for the variation under quasiconformal deformations of holomorphic differentials with fixed A-periods and holomorphic reproducing differentials. For such differentials we shall prove in Chapter 2 the continuity theorems with respect to the Dirichlet norm in the Teichmüller space of a given surface. In the case of compact Riemann surfaces these results are essentially due to L. Ahlfors [2], but we can show further in Chapter 3 that the continuity of holomorphic reproducing differentials still holds under the squeezing deformation about a non-dividing simple loop on a compact Riemann surface.

1. Variation of holomorphic differentials

1.1. For compact Riemann surfaces, L. Ahlfors [2] showed the continuity (or variation) of normal holomorphic differentials with respect to the Dirichlet norm, which played a fundamental role in his remarkable theory on Teichmüller spaces. To extend this continuity theorem to general Riemann surfaces we need

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Riemann's bilinear relation as in [2]. However, in the case of open Riemann surfaces, it does not generally hold in its classical form even if the differentials are square integrable. Thus to use a generalized bilinear relation we consider first a restricted class O'' of Riemann surfaces introduced by Kusunoki [10]. For the sake of convenience we recall the definition. Let R be an open Riemann surface and $E=\{R_n\}_{n=1}^{\infty}$ a canonical exhaustion (cf. [4]) of R. Let $\mathscr{L}_n$ be the set of 1-cycles γ in $R-\bar{R}_1$ such that each γ consists of (piecewise smooth) dividing curves on R and is freely homotopic to ∂R_n. This means that every $\gamma\in\mathscr{L}_n$ consists of the same number of connected components γ_i $(\gamma_i\cap\gamma_j=\emptyset,\ i\neq j)$ as $\partial R_n=\sum\tilde{\gamma}_i$ and dividing curves γ_i and $\tilde{\gamma}_i$ are freely homotopic. We denote $\mathscr{L}_E=\bigcup_{n=1}^{\infty}\mathscr{L}_n$ and by O'' the class of Riemann surfaces whose element admits a canonical exhaustion E for which the extremal length $\lambda(\mathscr{L}_E)$ vanishes. It is known ([10]) that $O''\subsetneqq O_G$ in general, but $O''=O_G$ whenever the genus is finite, and that on every Riemann surface R of class O'' a generalized bilinear relation holds for harmonic differentials with finite Dirichlet norm.

For later use we shall extend it slightly as follows.

Proposition 1. *Let R be an open Riemann surface of class O'' and $E=\{R_n\}$ be a canonical exhaustion of R such that $\lambda(\mathscr{L}_E)=0$, and let $\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis with respect to E modulo dividing cycles*[1]*, where g $(\leqq+\infty)$ is the genus of R. Then for any two square integrable closed C^1-differentials ω and σ there is a sequence $\{n_k\}$ of integers for which the bilinear relation*

$$(\omega, {}^*\sigma) = -\lim_{k\to\infty}\sum_{A_j, B_j\subset R_{n_k}}\left[\int_{A_j}\omega\int_{B_j}\bar{\sigma}-\int_{B_j}\omega\int_{A_j}\bar{\sigma}\right] \tag{1}$$

holds, where ${}^*\sigma$ is the conjugate differential of σ. In particular, if ω and σ have vanishing periods along all A-cycles, then we have

$$(\omega, {}^*\sigma) = -\int_R \omega\wedge\bar{\sigma} = 0. \tag{2}$$

Proof. By means of the orthogonal decomposition $\Gamma_c=\Gamma_h+\Gamma_{e0}$ (cf. Ahlfors—Sario [4]) we can write

$$\omega = \omega_h+\omega_{e0}, \quad \text{and} \quad \sigma = \sigma_h+\sigma_{e0},$$

where $\omega_h, \sigma_h\in\Gamma_h$ and $\omega_{e0}, \sigma_{e0}\in\Gamma_{e0}$. Since Γ_h, Γ_{e0} and ${}^*\Gamma_{e0}$ are mutually orthogonal, we have

$$(\omega, {}^*\sigma) = (\omega_h, {}^*\sigma_h).$$

Now, we know that the bilinear relation for ω_h and σ_h holds on $R\in O''$ because (1)

[1] For each R_n, the subset $\{A_j, B_j\}_{j=1}^{g_n}$, $g_n(<\infty)$ being the genus of R_n, forms a homology basis on R_n (mod ∂R_n), for which $A_i\times B_j=\delta_{ij}$, $A_i\times A_j=B_i\times B_j=0$ for every $i, j=1, \ldots, g_n$. The intersection number $A\times B$ of two cycles A and B is taken here so that it has the positive signature when B crosses A from right to left. Note that it has the opposite signature to that in [4].

is valid for $\omega=\omega_h$ and $\sigma=\sigma_h$ with suitable $\{n_k\}$ (cf. [10]), where $\{n_k\}$ may depend on ω and σ. Since ω and σ are C^1-differentials, ω_{e0} and σ_{e0} belong to $\Gamma_{e0}\cap\Gamma^1\subset\Gamma_e\cap\Gamma^1=\Gamma_e^1$. Consequently, they are written as $\omega_{e0}=df$ and $\sigma_{e0}=dg$ with C^2-functions f and g on R. It follows that

$$\int_{A_j}\omega_h=\int_{A_j}\omega \quad \text{and} \quad \int_{A_j}\sigma_h=\int_{A_j}\sigma,$$

and analogously for B-periods, which proves the assertion.

1.2. Let R_0 be a marked Riemann surface of class O'' with a canonical homology basis $\{A_j, B_j\}_{j=1}^g$ modulo dividing curves as in Proposition 1. We consider a C^2-quasiconformal mapping f_R of R_0 onto another Riemann surface R, where C^2 means the property of being continuous up to the second derivatives. Then f_R induces on R canonical homology basis modulo dividing curves, which we denote also by $\{A_j, B_j\}_{j=1}^g$. It is easy to see that R also belongs to the class O''. Let θ_0 be a holomorphic Abelian differential on R_0 with finite norm, that is, $\theta_0\in\Gamma_a(R_0)$. We shall now show that there exists on R a unique differential $\theta_R\in\Gamma_a(R)$ having the same A-periods as θ_0. Actually, we have more generally the following

Proposition 2. *Let S be an arbitrary open Riemann surface and ω be a closed C^1-differential square integrable on S. Then there exists a square integrable Abelian differential θ having the same periods with ω along all A_j-cycles where $\{A_j, B_j\}_{j=1}^g$ is a canonical homology basis on S modulo dividing curves. In particular, if S belongs to O'', then θ is uniquely determined.*

Proof. Let $\omega=\omega_h+\omega_{e0}$ be the orthogonal decomposition where $\omega_h\in\Gamma_h$ and $\omega_{e0}\in\Gamma_{e0}$. Since $\omega\in\Gamma^1$, $\omega_{e0}\in\Gamma^1$ hence ω_h has the same periods as ω. Set

$$\int_{A_j}\omega=\int_{A_j}\omega_h=a_j+\sqrt{-1}\cdot b_j$$

for every j with real numbers a_j and b_j. We write $\omega_h=\omega_1+\sqrt{-1}\cdot\omega_2$, where ω_1 and ω_2 are real harmonic differentials in Γ_h. Then $\varphi_k=\omega_k+\sqrt{-1}\cdot{}^*\omega_k$ $(k=1, 2)$ are holomorphic and

$$\int_{A_j}\varphi_1=a_j+\sqrt{-1}\cdot c_j, \quad \text{and} \quad \int_{A_j}\varphi_2=b_j+\sqrt{-1}\cdot d_j$$

for every j with some real c_j and d_j. Then by Virtanen—Kusunoki's theorem (cf. [11], [12]) there exist holomorphic differentials θ_1 and θ_2 in Γ_a such that

$$\int_{A_j}\theta_1=a_j \quad \text{and} \quad \int_{A_j}\theta_2=b_j \quad \text{for every } j.$$

Hence $\theta=\theta_1+\sqrt{-1}\cdot\theta_2$ is a holomorphic differential in Γ_a with the same A_j-periods $a_j+\sqrt{-1}\cdot b_j$ as ω.

Note that on $S\in O''$ every holomorphic differential with finite norm is uniquely determined by its A-periods, which follows from Proposition 1.

Corollary 1. *Let R_0, R and θ_0 be as before. There exists on R a unique differential $\theta_R\in\Gamma_a(R)$ having the same A-periods with θ_0.*

Proof. First we show that the pull-back $\omega=\theta_0\circ f_R^{-1}$ belongs to $\Gamma_c^1(R)$. Writing $f_R^{-1}=f$, $J=|f_z|^2-|f_{\bar z}|^2$, and $\theta_0=a(w)\,dw$, we have

$$\|\theta_0\circ f\|_R^2 = 2\int_R |(a\circ f)|^2[|f_z|^2+|f_{\bar z}|^2]\,dx\,dy$$

$$= 2\int_R |(a\circ f)|^2\left[\frac{|f_z|^2+|f_{\bar z}|^2}{|f_z|^2-|f_{\bar z}|^2}\right] J\,dx\,dy \leqq \frac{1+k^2}{1-k^2}\|\theta_0\|_{R_0}^2,$$

where $z=x+\sqrt{-1}\cdot y$ stands for the generic local parameter on R and $k=\sup_R|f_{\bar z}|/|f_z|\ (<1)$ is the supremum of the modulus of the complex dilatation of f. Hence $\omega\in\Gamma^1(R)$, and it is easily checked that $\omega\in\Gamma_c^1(R)$.

Next, noting that $\int_{A_j}\theta_0=\int_{A_j}\omega$ for every j, we see from Proposition 2 that there exists a holomorphic square integrable differential θ_R on R having the same periods as θ_0 along A-cycles. q.e.d.

Remark. Let $\omega_R\in\Gamma^k(R)$ be given $(1\leqq k\leqq\infty)$, and f be a C^{k+1}-quasiconformal mapping from R_0 onto R. Then as in the proof of Corollary 1, we can show that $\omega_R\circ f\in\Gamma^k(R_0)$. Moreover, if $\omega_R\in\Gamma_c(R)$, then it is easily seen that $\omega_R\circ f\in\Gamma_c^k(R_0)$.

1.3. Now let R_0 again be a non-planar Riemann surface of class O''. Let f be a C^2-quasiconformal mapping from R_0 onto R, $E=\{R_n\}_{n=1}^\infty$ be a canonical exhaustion of R such that $\lambda(\mathscr{L}_E)=0$, and $\{A_j, B_j\}_{j=1}^g$ be a canonical homology basis with respect to E. Suppose that a differential $\theta_{R_0}\in\Gamma_a(R_0)$ is arbitrarily given. Then by Corollary 1 there exists a unique $\theta_R\in\Gamma_a(R)$ having the same periods as θ_{R_0} along all A-cycles, and we can show the following

Theorem 1. *Let $R_0\in O''$ and f be a C^2-quasiconformal mapping from R_0 onto R. Then, given $\theta_{R_0}\in\Gamma_a(R_0)$, there exists a unique $\theta_R\in\Gamma_a(R)$ with the same A-periods as θ_{R_0} and we have*

$$\|\theta_R\circ f-\theta_{R_0}\|_{R_0}\leqq\frac{2k}{1-k}\|\theta_{R_0}\|_{R_0},$$

where $k=\sup_{R_0}|f_{\bar z}|/|f_z|\ (<1)$, i.e. $K=(1+k)/(1-k)$ is the maximal dilatation of f.

Proof. Write $\theta_R=a(w)\,dw$ on R, and set $\omega=\theta_R\circ f-\theta_{R_0}$. Then, since ω has vanishing periods along all A_j-cycles, we have by Proposition 1

$$(\omega, {}^*\omega)=0,$$

which implies immediately

$$\|(a\circ f)\cdot f_z(z)\,dz-\theta_{R_0}\|_{R_0}^2-\|(a\circ f)\cdot f_{\bar z}(z)\,d\bar z\|_{R_0}^2=0.$$

As $|f_{\bar z}| \leqq k|f_z|$, we have

$$\|(a\circ f)\cdot f_z(z)\,dz - \theta_{R_0}\|_{R_0} \leqq k\|(a\circ f)\cdot f_z(z)\,dz\|_{R_0},$$

and hence

$$\|(a\circ f)\cdot f_z(z)\,dz\|_{R_0} \leqq \frac{1}{1-k}\|\theta_{R_0}\|_{R_0}.$$

Thus the assertion follows from the inequality

$$\|\theta_R\circ f - \theta_{R_0}\|_{R_0} \leqq \|(a\circ f)\cdot f_z(z)dz - \theta_{R_0}\|_{R_0} + \|(a\circ f)\cdot f_{\bar z}\,d\bar z\|_{R_0}.$$

1.4. Next let R be an arbitrary Riemann surface, and a simple closed curve c on R be given. There now exists a unique differential $\theta_{c,R}\in\Gamma_a(R)$ which satisfies the condition

$$\int_c \omega = (\omega, \operatorname{Re}\theta_{c,R}) \quad \text{for every} \quad \omega\in\Gamma_h(R).$$

We call $\theta_{c,R}$ the holomorphic reproducing differential for c on R (cf. [4], [12]). This differential has several extremal properties, and especially $\|\operatorname{Re}\theta_{c,R}\|^2$ $(=1/2\,\|\theta_{c,R}\|^2)$ is equal to the extremal length of the homology class of c on R.

For simplicity, we write hereafter a curve corresponding to c on another surface again as c. As for $\theta_{c,R}$ we can show the following

Theorem 2. *Let $R_0\in O_{HD}$ and a simple closed curve c be given (on R_0), and f be a C^2-quasiconformal mapping from R_0 onto a surface R. Then, if we let $k=\sup_{R_0}|f_{\bar z}|/|f_z|$, it holds that*

$$\|\theta_{c,R}\circ f - \theta_{c,R_0}\|_{R_0} \leqq \frac{2k}{1-k}\|\theta_{c,R_0}\|_{R_0}.$$

Proof. Write $\omega=\theta_{c,R}\circ f-\theta_{c,R_0}$. Now we find $\omega\in\Gamma_c^1(R_0)$ as before. So ω can be decomposed in the form

$$\omega = \omega_1 + dg_1 + \sqrt{-1}\cdot(\omega_2 + dg_2)$$

with real $\omega_j\in\Gamma_h(R_0)$ and real $dg_j\in\Gamma_{e0}^1(R_0)$ $(j=1,2)$.

Note that for every 1-cycle d on R_0

$$\operatorname{Im}\int_d \theta_{c,R_0} = c\times d = \operatorname{Im}\int_d \theta_{c,R}.$$

Hence $\omega_2\in\Gamma_{he}(R_0)$, and, further, $\omega_2=0$ because $R_0\in O_{HD}$ $(\Gamma_{he}(R_0)=\{0\})$, and we have

$$(\omega, {}^*\omega) = (\omega_1 + dg_1 + \sqrt{-1}\cdot dg_2,\ {}^*\omega_1 + {}^*dg_1 + \sqrt{-1}\,{}^*dg_2)$$
$$= (\omega_1, {}^*\omega_1) = -\int_{R_0}\omega_1\wedge\omega_1 = 0.$$

Thus by the same argument as in the proof of Theorem 1, we can prove the assertion.

1.5. Finally, to obtain a similar result as Theorem 2 for an arbitrary Riemann surface, we must restrict ourselves to reproducing differentials for a suitable subclass of Γ_h. Namely, we consider the holomorphic Γ_{h0}-reproducing differential $\theta_c(\Gamma_{h0}(R))$ for a given c on R, which is characterized by the condition that $\mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\in\Gamma_{h0}(R)$, and

$$\int_c \omega = (\omega, \mathrm{Re}\,\theta_c(\Gamma_{h0}(R))) \quad \text{for every} \quad \omega\in\Gamma_{h0}(R).$$

Note here that $\mathrm{Im}\,\theta_c(\Gamma_{h0}(R))={}^*\mathrm{Re}\,\theta_c(\Gamma_{h0}(R))$.

Now let R_0 be an arbitrary Riemann surface, and f be a C^2-quasiconformal mapping from R_0 onto a surface R. First we show the following

Lemma 1. $\mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\circ f\in\Gamma_{c0}(R_0)\cap\Gamma^1(R_0)$.

Proof. It is clear that $\theta_c(\Gamma_{h0}(R))\circ f\in\Gamma_c^1(R_0)$. Next note that the mapping F defined by

$$F(dg) = (dg)\circ f \quad \text{for every} \quad dg\in\Gamma_e^1(R)$$

maps $\Gamma_e^1(R)$ onto $\Gamma_e^1(R_0)$.

Also we have

$$(\mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\circ f, {}^*F(dg))_{R_0} = -\int_{R_0} \mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\circ f\wedge \overline{dg\circ f}$$

$$= -\int_R \mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\wedge\overline{dg} = (\mathrm{Re}\,\theta_c(\Gamma_{h0}(R)), {}^*dg)_R = 0$$

for every $dg\in\Gamma_e^1(R)$. And since $\Gamma_e^1(R_0)$ is dense in $\Gamma_e(R_0)$, we conclude that $\mathrm{Re}\,\theta_c(\Gamma_{h0}(R))\circ f\in{}^*\Gamma_e(R_0)^\perp=\Gamma_{c0}(R_0)$.

Now we can show the following

Theorem 3. *Let R_0, R and f be as above and $k=\sup_{R_0}|f_{\bar z}|/|f_z|$. Then it holds that*

$$\|\theta_c(\Gamma_{h0}(R))\circ f-\theta_c(\Gamma_{h0}(R_0))\|_{R_0} \leqq \frac{2k}{1-k}\|\theta_c(\Gamma_{h0}(R_0))\|_{R_0}.$$

Proof. Write $\omega=\theta_c(\Gamma_{h0}(R))\circ f-\theta_c(\Gamma_{h0}(R_0))$. Then by Lemma 1 $\mathrm{Re}\,\omega\in\Gamma_{c0}(R_0)\cap\Gamma^1(R_0)$, and by the orthogonal decomposition $\Gamma_{c0}(R_0)=\Gamma_{h0}(R_0)+\Gamma_{e0}(R_0)$ we can decompose ω in the form

$$\omega = \omega_1+dg_1+\sqrt{-1}\cdot(\omega_2+dg_2),$$

where ω_j and dg_j are real, $\omega_1\in\Gamma_{h0}(R_0)$, $\omega_2\in\Gamma_h(R_0)$ and $dg_j\in\Gamma_{e0}(R_0)\cap\Gamma^1(R_0)\subset\Gamma_e^1(R_0)$ $(j=1, 2)$. Moreover, we know that $\omega_2\in\Gamma_{he}(R_0)$, for it holds as before that

$$\int_d \omega_2 = \mathrm{Im}\int_d \theta_c(\Gamma_{h0}(R))-\mathrm{Im}\int_d \theta_c(\Gamma_{h0}(R_0)) = 0$$

for every 1-cycle d. So noting that $(\omega_1, {}^*\omega_2)=0$ because $\omega_1 \in \Gamma_{h0}(R_0)$ and ${}^*\omega_2 \in {}^*\Gamma_{he}(R_0)=\Gamma_{h0}(R_0)^{\perp} \cap \Gamma_h(R_0)$, we have

$$
\begin{aligned}
(\omega, {}^*\omega) &= \left(\omega_1+dg_1+\sqrt{-1}\cdot(\omega_2+dg_2), {}^*\omega_1+{}^*dg_1+\sqrt{-1}\cdot({}^*\omega_2+{}^*dg_2)\right)\\
&= -2\sqrt{-1}\cdot(\omega_1, {}^*\omega_2) = 0.
\end{aligned}
$$

Thus by the same argument as in the proof of Theorem 1, we have the assertion.

Corollary 2. *Let R, R_0 and f be as in Theorem 3, and $\theta_c(\Gamma_{hse}(R_0))$ and $\theta_c(\Gamma_{hse}(R))$ be the holomorphic Γ_{hse}-reproducing differentials for c on R_0 and R respectively. Further suppose that $R_0 \in O_{KD}$. Then it holds that*

$$\|\theta_c(\Gamma_{hse}(R))\circ f-\theta_c(\Gamma_{hse}(R_0))\|_{R_0} \leqq \frac{2k}{1-k}\|\theta_c(\Gamma_{hse}(R_0))\|_{R_0}.$$

This follows from Theorem 3, since $R_0 \in O_{KD}$ if and only if $\Gamma_{hse}(R_0)=\Gamma_{h0}(R_0)$.

2. Continuity theorem on the Teichmüller spaces

2.1. Let R^* be a fixed Riemann surface with the hyperbolic universal covering surface, and consider all pairs (R, f), where R is a Riemann surface and f is a quasiconformal mapping from R^* onto R. We say that (R_1, f_1) and (R_2, f_2) are equivalent if $f_2\circ f_1^{-1}$ is homotopic to a conformal mapping from R_1 onto R_2. The equivalence classes are, by definition, the points of the Teichmüller space $T(R^*)$ with the base point $\bar{R}^*=(R^*, \mathrm{id})$, which are called marked Riemann surfaces and denoted simply by $\bar{R}$ etc. The space $T(R^*)$ has the natural Teichmüller metric. (See for example [3].) Now let G^* be a Fuchsian group representing R^*, then $T(R^*)$ can be canonically identified with the reduced Teichmüller space $T^{\#}(G^*)$, which coincides with the Teichmüller space $T(G^*)$ if and only if G^* is of the first kind. Also note that $\dim T(R^*)$ is finite if and only if G^* is finitely generated (cf. [8]) and that if $R^* \in O_{HD}$ then G^* is of the first kind ([13]).

In the case of general open Riemann surfaces, the existence of the so-called Teichmüller mappings is not known. Actually, for given $\bar{R}_1$ and $\bar{R}_2$ in $T(R^*)$ there exists an extremal quasiconformal mapping f from $\bar{R}_1$ to $\bar{R}_2$ (which preserves the markings), but such an f is not unique in general, and even if it were, it would not be known whether f has such regularity as Teichmüller mappings. Hence we want to begin by showing the existence of a smooth quasiconformal mapping between points on $T(R^*)$ sufficiently near each other.

2.2. Let $\bar{R}_0 \in T(R^*)$ be fixed in the sequel, and $\lambda_{\bar{R}_0}$ be the Poincaré metric on R_0. We say that a Beltrami differential μ on R_0 is *canonical* if the quadratic differential $\bar{\mu}\cdot\lambda^2_{\bar{R}_0}$ is holomorphic on R_0, and that a quasiconformal mapping f from R_0 onto another surface is *canonical* if the complex dilatation $\mu(f)$ of f is a canonical Beltrami differential. In particular, a canonical Beltrami differential has a coefficient of C^ω-class and hence a canonical quasiconformal mapping is of C^ω-class (cf. for example [6]). Now let $d(\ ,\ )$ be the Teichmüller distance on $T(R^*)$, and set

$$k(\bar{R})=\frac{\exp[d(\bar{R}_0,\bar{R})]-1}{\exp[d(\bar{R}_0,\bar{R})]+1}$$

for every $\bar{R}\in T(R^*)$. Then by definition $k(\bar{R})$ is equal to the L^∞-norm of the complex dilatation of any extremal quasiconformal mapping from $\bar{R}_0$ to $\bar{R}$.

We do not know whether a canonical quasiconformal mapping from $\bar{R}_0$ to $\bar{R}$ exists for every $\bar{R}\in T(R^*)$, but we can show the following

Lemma 2. *Suppose that* $k(\bar{R})<1/3$ $\left(\textit{i.e. } d(\bar{R}_0,\bar{R})<\log 2\right)$. *Then there exists a canonical quasiconformal mapping from* $\bar{R}_0$ *to* $\bar{R}$ *whose maximal dilatation is not greater than* $(1+3k(\bar{R}))/(1-3k(\bar{R}))$.

Proof. Let f be an extremal quasiconformal mapping from $\bar{R}_0$ to $\bar{R}$, and μ be the complex dilatation of f. Recall that $\operatorname{ess\,sup}_{R_0}|\mu|=k(\bar{R})$. Let G_0 be a Fuchsian group acting on the unit disk U such that $R_0=U/G_0$. Then μ can be lifted to a Beltrami coefficient for G_0 on U, which is also denoted by μ. Now there exists a unique quasi-conformal automorphism F of the extended z-plane $\bar{C}$ fixing 1, $\sqrt{-1}$ and $-\sqrt{-1}$ whose complex dilatation is μ on U and zero on $\bar{C}-U$. Next let φ_F be the Schwarzian derivative of F considered as a schlicht function on $\bar{C}-\bar{U}$. Then Kühnau—Lehto's theorem ([9], [15]) states that

$$(1-|z|^2)^2|\varphi_F(z)|\leqq 6\cdot k(\bar{R})\ (<2)\quad\text{on }\ \bar{C}-\bar{U}.$$

This in turn implies (cf. [3], [5]) that there exists a unique quasiconformal automorphism g on $\bar{C}$ such that

(i) the complex dilatation of g is equal to

$$\begin{cases}-\dfrac{1}{2}(1-|z|^2)^2\varphi_F\left(\dfrac{1}{\bar{z}}\right)\left(\dfrac{1}{\bar{z}}\right)^4 & \text{on}\quad U,\ \text{and}\\ 0 \text{ on }\ \bar{C}-U,\ \text{and}\end{cases}$$

(ii) $g(z)\equiv F(z)$ on $\bar{C}-U$.

Now by (i) it is seen that the complex dilatation of g is a Beltrami coefficient for G_0, and (ii) implies that $F\circ G_0\circ F^{-1}=g\circ G_0\circ g^{-1}$ on $\bar{C}$. Thus g can be projected to a quasiconformal mapping from R_0 onto R $\left(=F(U)/F\circ G_0\circ F^{-1}\right)$, which is canonical by (i). And noting that

$$\begin{aligned}&\sup_U\left|-\frac{1}{2}(1-|z|^2)^2\varphi_F\left(\frac{1}{\bar{z}}\right)\left(\frac{1}{\bar{z}}\right)^4\right|\\ &=\sup_{\bar{C}-U}\left|-\frac{1}{2}(1-|z|^2)^2\varphi_F(z)\right|\leqq 3\cdot k(\bar{R}),\end{aligned}$$

we have the assertion.

2.3. By Lemma 2 above one can restate Theorem 1 as follows.

Theorem 1′. *Let $R^* \in O''$, and a point $\bar{R}_0 \in T(R^*)$ and $\theta_{\bar{R}_0} \in \Gamma_a(R_0)$ be given. Suppose that $d(\bar{R}_0, \bar{R}) < \log 2$, and let $\theta_{\bar{R}}$ be the differential in $\Gamma_a(R)$ having the same A-periods with $\theta_{\bar{R}_0}$. Then there exists a canonical quasiconformal mapping $f_{\bar{R}}$ from $\bar{R}_0$ to $\bar{R}$ such that*

$$\|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0} \leqq \frac{6 \cdot k(\bar{R})}{1 - 3 \cdot k(\bar{R})} \|\theta_{\bar{R}_0}\|_{R_0}.$$

Corollary 3. *Using the same notation as in Theorem 1′, we have*

1) $$\lim_{\bar{R} \to \bar{R}_0} \|\theta_{\bar{R}}\|_R = \|\theta_{\bar{R}_0}\|_{R_0}, \quad \text{and}$$

2) $$\lim_{\bar{R} \to \bar{R}_0} \int_d \theta_{\bar{R}} = \int_d \theta_{\bar{R}_0} \quad \textit{for every 1-cycle } d.$$

Proof. For $\bar{R}$ sufficiently near $\bar{R}_0$ there is a canonical quasiconformal mapping $f_{\bar{R}}$ of R_0 onto R. Let $\theta_{\bar{R}} = a_{\bar{R}}(w)dw$ with $w = f_{\bar{R}}(z)$. Then

$$\|\theta_{\bar{R}}\|_R = \left[2 \int_{R_0} |a_{\bar{R}} \circ f_{\bar{R}}|^2 (|(f_{\bar{R}})|_z^2 - |(f_{\bar{R}})_{\bar{z}}^2|)\, dx\, dy\right]^{1/2}$$

$$\leqq \|\theta_{\bar{R}} \circ f_{\bar{R}}\|_{R_0} \leqq \|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0} + \|\theta_{\bar{R}_0}\|_{R_0}.$$

Hence by Theorem 1′ we see that

$$\limsup_{\bar{R} \to \bar{R}_0} \|\theta_{\bar{R}}\|_R \leqq \|\theta_{\bar{R}_0}\|_{R_0}.$$

On the other hand, writing $k = k(\bar{R})$, we have

$$\|\theta_{\bar{R}_0}\|_{R_0} \leqq \|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0} + \|\theta_{\bar{R}} \circ f_{\bar{R}}\|_{R_0}$$

$$\leqq \|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0} + \left[\frac{1+k^2}{1-k^2}\right]^{1/2} \|\theta_{\bar{R}}\|_R,$$

and consequently the inequality

$$\liminf_{\bar{R} \to \bar{R}_0} \|\theta_{\bar{R}}\|_R \geqq \|\theta_{\bar{R}_0}\|_{R_0}.$$

It is well-known that for a fixed 1-cycle d, the linear functional $L(\omega) = \int_d \omega$ is bounded on $\Gamma_c^1(R_0)$, that is, there is a constant C_d such that

$$\left|\int_d \omega\right| \leqq C_d \|\omega\|_{R_0} \quad \text{for every} \quad \omega \in \Gamma_c^1(R_0).$$

Noting that $\theta_{\bar{R}} \circ f_{\bar{R}} \in \Gamma_c^1(R_0)$ we have therefore

$$\left|\int_d \theta_{\bar{R}} - \int_d \theta_{\bar{R}_0}\right| \leqq C_d \|\theta_{\bar{R}} \circ f_{\bar{R}} - \theta_{\bar{R}_0}\|_{R_0}.$$

Thus the assertion 2) follows from Theorem 1′, Remark. The equation 2) in Corollary 3 implies that when $R \in O''$ and a canonical homology basis $\{A_j, B_j\}$ on R is suitably chosen, then the period matrix with respect to this basis varies continuously on $T(R)$ (equipped with the Teichmüller topology).

2.4. Similarly we have the following results from Theorems 2 and 3 respectively.

Theorem 2′. *Let $R^* \in O_{HD}$ and a simple closed curve c be given. Suppose that $d(\bar{R}_0, \bar{R}) < \log 2$. Then there exists a canonical quasiconformal mapping $f_{\bar{R}}$ from $\bar{R}_0$ to $\bar{R}$ for which*

$$\|\theta_{c,\bar{R}} \circ f_{\bar{R}} - \theta_{c,\bar{R}_0}\|_{R_0} \leqq \frac{6 \cdot k(\bar{R})}{1 - 3 \cdot k(\bar{R})} \|\theta_{c,\bar{R}_0}\|_{R_0}.$$

Corollary 4. *Using the same notation as in Theorem 2′, we have*

1) $$\lim_{\bar{R} \to \bar{R}_0} \|\theta_{c,\bar{R}}\|_R = \|\theta_{c,\bar{R}_0}\|_{R_0}, \quad \text{and}$$

2) $$\lim_{\bar{R} \to \bar{R}_0} \int_d \theta_{c,\bar{R}} = \int_d \theta_{c,\bar{R}_0} \quad \text{for every 1-cycle } d.$$

Theorem 3′. *Let R^* and a simple closed curve c be arbitrarily given. Suppose that $d(\bar{R}_0, \bar{R}) < \log 2$. Then there exists a canonical quasiconformal mapping $f_{\bar{R}}$ from $\bar{R}_0$ to $\bar{R}$ for which*

$$\left\|\theta_c(\Gamma_{h0}(R)) \circ f_{\bar{R}} - \theta_c(\Gamma_{h0}(R_0))\right\|_{R_0} \leqq \frac{6 \cdot k(\bar{R})}{1 - 3 \cdot k(\bar{R})} \left\|\theta_c(\Gamma_{h0}(R))\right\|_{R_0}.$$

Corollary 5. *Using the same notation as in Theorem 3′, we have*

1) $$\lim_{\bar{R} \to \bar{R}_0} \left\|\theta_c(\Gamma_{h0}(R))\right\|_R = \left\|\theta_c(\Gamma_{h0}(R_0))\right\|_{R_0}, \quad \text{and}$$

2) $$\lim_{\bar{R} \to \bar{R}_0} \int_d \theta_c(\Gamma_{h0}(R)) = \int_d \theta_c(\Gamma_{h0}(R_0)) \quad \text{for every 1-cycle } d.$$

2.5. Here we shall consider quasiconformal mappings not necessarily of class C^2. Let $\bar{R}_n \in T(R^*)$ converge to $\bar{R}_0 \in T(R^*)$, where R^* is arbitrarily given, and let f_n be a quasiconformal mapping of $\bar{R}_0$ to $\bar{R}_n$. We call a sequence $\{f_n\}_{n=1}^{\infty}$ *admissible* if the maximal dilatation of f_n converges to 1. Then we can show the following

Proposition 3. *If we let $\theta_{\bar{R}_n}$ be a holomorphic Abelian differential on R_n with finite Dirichlet norm for every n, then the following two conditions are equivalent;*

1) *For some admissible sequence $\{f_n\}_{n=1}^{\infty}$ we have*

(3) $$\lim_{n \to \infty} \|\theta_{\bar{R}_n} \circ f_n - \theta_{\bar{R}_0}\|_{R_0} = 0.$$

2) *For any admissible sequence $\{f_n\}_{n=1}^{\infty}$, the above (3) holds.*

Proof. It is clear that 2) implies 1). Now suppose that for an admissible sequence $\{f_n\}_{n=1}^{\infty}$ the condition (3) holds, and let any admissible $\{g_n\}_{n=1}^{\infty}$ be given. First note that

$$(4)\qquad \|\theta_{\bar{R}_n}\circ g_n-\theta_{\bar{R}_0}\|_{R_0}^2$$
$$\leqq 2\frac{1+k_n^2}{1-k_n^2}\|\theta_{\bar{R}n}\circ g_n\circ(g_n^{-1}\circ f_n)-\theta_{\bar{R}_0}\circ(g_n^{-1}\circ f_n)\|_{R_0}^2,$$

where $k_n=\mathrm{ess}_{R_0}\sup|(g_n^{-1}\circ f_n)_{\bar{z}}|/|(g_n^{-1}\circ f_n)_z|$. In fact, for any square integrable differential $\theta=a(w)\,dw+b(w)\,d\bar{w}$, and any quasi-conformal mapping $w=F(z)$ of R_0 onto itself, it holds that

$$\|\theta\circ f\|_{R_0}^2=\int_{R_0}2(|a\circ F(z)\cdot F_z(z)+b\circ F(z)\cdot\bar{F}_z(z)|^2$$
$$+|a\circ F(z)\cdot F_{\bar{z}}(z)+b\circ F(z)\cdot\bar{F}_{\bar{z}}(z)|^2)\,dx\,dy$$
$$\leqq 4\int_{R_0}(|a\circ F|^2+|b\circ F|^2)(|F_z|^2+|F_{\bar{z}}|^2)\,dx\,dy$$
$$\leqq 4\frac{1+k^2}{1-k^2}\int_{R_0}(|a\circ F|^2+|b\circ F|^2)\cdot J\,dx\,dy$$
$$=4\frac{1+k^2}{1-k^2}\int_{R_0}(|a(w)|^2+|b(w)|^2)\,du\,dv=2\frac{1+k^2}{1-k^2}\|\theta\|_{R_0}^2,$$

where $k=\mathrm{ess}_{R_0}\sup|F_{\bar{z}}|/|F_z|$, $J=|F_z|^2-|F_{\bar{z}}|^2$ with generalized L^2-derivatives F_z and $F_{\bar{z}}$. So, putting $\theta=\theta_{\bar{R}_n}\circ g_n\circ(g_n^{-1}\circ f_n)-\theta_{\bar{R}_0}\circ(g_n^{-1}\circ f_n)$, and $F=(g_n^{-1}\circ f_n)^{-1}$, we have (4).

Next

$$\|\theta_{\bar{R}_n}\circ g_n\circ(g_n^{-1}\circ f_n)-\theta_{\bar{R}_0}\circ(g_n^{-1}\circ f_n)\|_{R_0}$$
$$\leqq\|\theta_{\bar{R}_n}\circ f_n-\theta_{\bar{R}_0}\|_{R_0}+\|\theta_{\bar{R}_0}-\theta_{\bar{R}_0}\circ(g_n^{-1}\circ f_n)\|_{R_0},$$

and the first term of the right hand side converges to 0 by assumption. To complete the proof it therefore suffices to show that

$$(5)\qquad \lim_{n\to\infty}\|\theta_{\bar{R}_0}-\theta_{\bar{R}_0}\circ(g_n^{-1}\circ f_n)\|_{R_0}=0.$$

For this purpose, let G_0 be a Fuchsian group acting on $U=\{|z|<1\}$ such that $R_0=U/G_0$, let $a(z)\,dz$ with holomorphic $a(z)$ be the lift of $\theta_{\bar{R}_0}$ and $F_n(z)$ be the lift of $(g_n^{-1}\circ f_n)$ such that $F_n\circ g=g\circ F_n$ for every $g\in G_0$. Note that g_n and f_n are homotopic. Fix a normal fundamental region D for G_0 and any positive ε. Then there exists an $r_0<1$ such that $\|a(z)\|_{2,D-D_{r_0}}<\varepsilon$, where $D_r=D\cap\{|z|<r\}$ and $\|f\|_{2,E}^2=2\int_E|f|^2\,dx\,dy$ for any subset E in U. Since $\bar{D}_r$ is compact in U for any $r<1$, $|a(z)|$ is bounded on D_r. Since $\{f_n\}$ and $\{g_n\}$ are admissible and homotopic, $F_n(z)$ converges uniformly to the identity mapping on D_r, and

$$\lim_{n\to\infty}\|(F_n)_z-1\|_{2,D_r}=\lim_{n\to\infty}\|(F_n)_{\bar{z}}\|_{2,Dr}=0.$$

(Cf. [14] IV Theorem 5.2 and V Theorem 5.3.) Hence we conclude that

$$\limsup_{n\to\infty} \|\theta_{\bar{R}_0} - \theta_{\bar{R}_0} \circ (g_n^{-1} \circ f_n)\|_{R_0}$$

$$\leqq \limsup_{n\to\infty} [\|a \circ F_n \cdot [(F_n)_z - 1]\|_{2,D_r} + \|a \circ F_n \cdot (F_n)_{\bar{z}}\|_{2,D_r}$$

$$+ \|a - a \circ F_n\|_{2,D_r} + \|a\|_{2,D-D_r}$$

$$+ \Big(\int_{D-D_r} 2|a \circ F_n|^2 (|(F_n)_z|^2 + |(F_n)_{\bar{z}}|^2)\, dx\, dy \Big)^{1/2}]$$

$$\leqq \|a(z)\|_{2,D-D_r} + \limsup_{n\to\infty} \left(\frac{1+k_n^2}{1-k_n^2}\right)^{1/2} \cdot \|a(z)\|_{2,F_n(D-D_r)}$$

$$\leqq 2\varepsilon,$$

where $r_0 < r < 1$. Since ε is arbitrary, we have (5).

Thus using Proposition 3 we obtain from Theorem 1′, 2′ and 3′ the following theorems.

Theorem 4. *Let $R^* \in O''$, $\bar{R}_n \in T(R^*)$ converge to $\bar{R}_0 \in T(R^*)$, and $\theta_{\bar{R}_n}$ be as in Theorem 1′. Then for every admissible sequence $\{f_n\}_{n=1}^{\infty}$*

$$\lim_{n\to\infty} \|\theta_{\bar{R}_n} \circ f_n - \theta_{\bar{R}_0}\|_{R_0} = 0.$$

Theorem 5. *Let $R^* \in O_{HD}$, and $\bar{R}_n \in T(R^*)$ converge to $\bar{R}_0 \in T(R^*)$. Then for every admissible sequence $\{f_n\}_{n=1}^{\infty}$*

$$\lim_{n\to\infty} \|\theta_{c,\bar{R}_n} \circ f_n - \theta_{c,\bar{R}_0}\|_{R_0} = 0.$$

Theorem 6. *Let R^* be arbitrary and $\bar{R}_n \in T(R^*)$ converge to $\bar{R}_0 \in T(R^*)$. Then for every admissible sequence $\{f_n\}_{n=1}^{\infty}$*

$$\lim_{n\to\infty} \big\|\theta_c(\Gamma_{h0}(R_n)) \circ f_n - \theta_c(\Gamma_{h0}(R_0))\big\|_{R_0} = 0.$$

2.6. Finally we note that Theorem 5 can be extended to arbitrary Riemann surfaces. That is, we can prove the following

Theorem 5′. *Let R^* be an arbitrary Riemann surface, a simple closed curve c be fixed, and $\bar{R}_n \in T(R^*)$ converge to $\bar{R}_0 \in T(R^*)$. Then for every admissible sequence $\{f_n\}_{n=1}^{\infty}$*

$$\lim_{n\to\infty} \|\theta_{c,\bar{R}_n} \circ f_n - \theta_{c,\bar{R}_0}\|_{R_0} = 0. \tag{6}$$

Proof. Suppose first that every f_n is canonical, and let $\sigma_{c,\bar{R}_n} = \mathrm{Re}\,\theta_{c,\bar{R}_n}$ for every n. Then one can see that $\|\sigma_{c,\bar{R}_n}\|_{R_n}$ converges to $\|\sigma_{c,\bar{R}_0}\|_{R_0}$, for $\|\sigma_{c,\bar{R}}\|_R^2$ is equal to the extremal length of the homology class of c on R. Hence it holds that

$$K_n^{-1} \|\sigma_{c,\bar{R}_0}\|_{R_0}^2 \leqq \|\sigma_{c,\bar{R}_n}\|_{R_0}^2 \leqq K_n \|\sigma_{c,\bar{R}_0}\|_{R_0}^2,$$

where K_n is the maximal dilatation of f_n. Note also that

$$\|\sigma_{c,R_n}\circ f_n-\sigma_{c,R_0}\|_{R_0}^2$$
$$=\|\sigma_{c,R_n}\circ f_n\|_{R_0}^2-2\int_c \sigma_{c,R_n}+\|\sigma_{c,R_0}\|_{R_0}^2$$
$$=\|\sigma_{c,R_n}\circ f_n\|_{R_0}^2-2\|\sigma_{c,R_n}\|_{R_n}^2+\|\sigma_{c,R_0}\|_{R_0}^2.$$

Now set $\theta_{c,R_n}=a_n(z_n)\,dz_n$, (hence $2\sigma_{c,R_n}=a_n(z_n)\,dz_n+\overline{a_n}(z_n)\,d\bar{z}_n$), on R_n with $z_n=f_n(z)$. Then $\sigma_{c,R_n}\circ f_n=(1/2)[(a_n\circ f_n\cdot(f_n)_z+\overline{a_n\circ f_n\cdot(f_n)_{\bar z}})\,dz+(\overline{a_n\circ f_n\cdot(f_n)_z}+a_n\circ f_n\cdot(f_n)_{\bar z})\,d\bar z]$, and thus we have

$$\|\sigma_{c,R_n}\circ f_n\|_{R_0}^2$$
$$=\frac{1}{2}\left\|[(a_n\circ f_n\cdot(f_n)_z+a_n\circ f_n\cdot(f_n)_{\bar z}]\,dz\right\|_{R_0}^2$$
$$\leqq\frac{1}{2}\int_{R_n}|a_n\circ f_n|^2(|(f_n)_z|+|(f_n)_{\bar z}|)^2 2\,dx\,dy$$
$$\leqq\frac{1}{2}K_n\|\theta_{c,R_n}\|_{R_n}^2=K_n\|\sigma_{c,R_n}\|_R^2\ .$$

Hence, we conclude from above that

$$\lim_{n\to\infty}\|\sigma_{c,R_n}\circ f_n-\sigma_{c,R_0}\|_{R_0}=0.$$

Since

$$\sqrt{2}\,\|\sigma_{c,R_n}\circ f_n-\sigma_{c,R_0}\|_{R_0}$$
$$=\|[a_n\circ f_n\cdot(f_n)_z+\overline{a_n}\circ f_n\cdot\overline{(f_n)_{\bar z}}-a_0]\,dz\|_{R_0},$$

we have

$$\|\theta_{c,R_n}\circ f_n-\theta_{c,R_0}\|_{R_0}$$
$$\leqq\|[a_n\circ f_n\cdot(f_n)_z-a_0]\,dz\|_{R_0}+\|a_n\circ f_n\cdot(f_n)_{\bar z}\,d\bar z\|_{R_0}$$
$$\leqq\sqrt{2}\|\sigma_{c,R_n}\circ f_n-\sigma_{c,R_0}\|_{R_0}+2\|a_n\circ f_n\cdot(f_n)_{\bar z}\,d\bar z\|_{R_0}.$$

Here $\|a_n\circ f_n\cdot(f_n)_{\bar z}d\bar z\|_{R_0}$ also converges to 0, because

$$\|a_n\circ f_n\cdot(f_n)_{\bar z}\,d\bar z\|_{R_0}\leqq k_n\|a_n\circ f_n\cdot(f_n)_z\,dz\|_{R_0}$$
$$\leqq k_n\|\theta_{c,R_n}\circ f_n\|_{R_0}\leqq\sqrt{2}k_n\left(\frac{1+k_n^2}{1-k_n^2}\right)^{1/2}\|\sigma_{c,R_n}\|_{R_n},$$

where $k_n=\sup_{R_0}|(f_n)_{\bar z}|/|(f_n)_z|$.

Thus if every f_n is canonical, we have (6), and the assertion follows if we use Proposition 3.

3. A remark for the case of compact Riemann surfaces

3.1. For a compact Riemann surface R^* of genus g ($\geqq 2$), $T(R^*)$ is usually denoted by T_g, and we have the following corollary to Theorem 5, which is essentially due to Ahlfors.

Corollary 6. *Let $\bar{R}_n \in T_g$ converge to $\bar{R}_0 \in T_g$, and a non-dividing simple closed curve c be given. Then for every admissible sequence $\{f_n\}_{n=1}^{\infty}$ we have*

$$\lim_{n\to\infty} \|\theta_{c,\bar{R}_n}\circ f_n - \theta_{c,\bar{R}_0}\|_{R_0} = 0.$$

If we let G^* be a Fuchsian group corresponding to R^*, T_g can be identified with the Teichmüller space of G^*, and is considered to be embedded in the (finite dimensional) space of G^*-invariant bounded holomorphic quadratic forms on the lower half plane (cf. [3], [7]). In the sequel we fix a non-dividing simple closed curve c on R^* (hence for every $\bar{R} \in T_g$), and by using the same notation and terminology as in [17], we shall consider the space $\partial_c T_g$ for c and the fine topology on ${}_c\hat{T}_g = T_g \cup \partial_c T_g$. Here we recall some definitions. First, $\partial_c T_g$ is the set of marked Riemann surfaces with one single node corresponding to c. Next, by letting S_c be the set of points $\bar{R}$ of T_g on which $\theta_{c,\bar{R}}^2$ has a closed trajectory freely homotopic to c, we can construct a mapping F_1 from S_c onto $T_{g-1,2}$, and set

$$F_2(\bar{R}) = \operatorname{Re}\int_{B_1} \frac{2\cdot\theta_{c,\bar{R}}}{\|\theta_{c,\bar{R}}\|^2} + \sqrt{-1}\cdot m_{\bar{R}} \tag{7}$$

for every $\bar{R} \in S_c$, where $m_{\bar{R}}$ is the modulus of the characteristic ring domain of $\theta_{c,\bar{R}}^2$ for c on R and $\{A_j, B_j\}_{j=1}^{g}$ is a canonical homology basis on R such that A_1 is freely homotopic to c. Then $F=(F_1, F_2)$ can be extended to a bijection from $S_c \cup \partial_c T_g$ onto $T_{g-1}\times\hat{U}$, where $\hat{U}=\{z\colon \operatorname{Im} z>0\}\cup\{\infty\}$ is equipped with the usual fine (cusp) topology. The fine topology is, by definition, the induced topology by F from $T_{g-1,2}\times\hat{U}$.

In this case we say that a sequence $\{\langle \bar{R}_n, \bar{R}_0, f_n\rangle\}_{n=1}^{\infty}$ of deformations (cf. [1]) is *admissible* if for every neighbourhood K of the (single) node of R_0 and every positive ε there exists an N such that $f_n^{-1}|_{(R_0-K)}$ is $(1+\varepsilon)$-quasiconformal for every $n \geqq N$ (cf. Chapter 2), and set

$$\theta_{\bar{R}} = \frac{2\cdot\theta_{c,\bar{R}}}{\|\theta_{c,\bar{R}}\|_R^2} \quad \text{for every} \quad \bar{R}\in T_g, \text{ and} \tag{8}$$

$$\theta_{\bar{R}} = \frac{\sqrt{-1}}{2\pi}\cdot\varphi_{\bar{R}'} \quad \text{for every} \quad \bar{R}\in\partial_c T_g,$$

where $\bar{R}'=F_1(\bar{R})$ and $\varphi_{\bar{R}'}$ is the elementary differential of the third kind on $\bar{R}'$ with poles at two punctures.

It is known that if $\bar{R}_n \in T_g$ converges to $\bar{R}_0 \in \partial_c T_g$ in the sense of the conformal topology (cf. [1]), then it holds that

$$\lim_{n\to\infty} \|\theta_{\bar{R}_n}\|_{R_n} = +\infty, \quad \text{that is,} \quad \lim_{n\to\infty} \|\theta_{c,\bar{R}_n}\|_{R_n} = 0. \tag{9}$$

Now using the fine topology on ${}_c\hat{T}_g$, we can extend this slightly by stating the main theorem of this section.

Theorem 7. *Suppose that $\bar{R}_n \in {}_c\hat{T}_g$ converges to $\bar{R}_0 \in {}_c\hat{T}_g$ in the sense of the fine topology. Then for every admissible sequence $\{\langle \bar{R}_n, \bar{R}_0, f_n\rangle\}_{n=1}$ of deformations we have*

$$\lim_{n\to\infty} \|\theta_{\bar{R}_n} \circ f_n^{-1} - \theta_{\bar{R}_0}\|_{(R_0 - K)} = 0 \tag{10}$$

for every neighbourhood K of the node of R_0, where we assume that $K=\emptyset$ if $\bar{R}_0 \in T_g$.

Note that (9) is derived from (10), because

$$\|\theta_{\bar{R}_0}\|_{(R_0-K)} \leqq \|\theta_{\bar{R}_n} \circ f_n^{-1} - \theta_{\bar{R}_0}\|_{(R_0-K)} + \|\theta_{\bar{R}_n} \circ f_n^{-1}\|_{(R_0-K)},$$

$$\|\theta_{\bar{R}_n} \circ f_n^{-1}\|^2_{(R_0-K)} \leqq \frac{1+k_n^2}{1-k_n^2} \|\theta_{\bar{R}_n}\|^2_{(R_n - f_n^{-1}(K))},$$

and $\|\theta_{\bar{R}_0}\|_{(R_0-K)}$ tends to $+\infty$ as K shrinks to the node of R_0.

3.2. The proof of Theorem 7 will be given in Section 3.3 after preparing the lemmas below. First we construct an admissible sequence satisfying the condition (10). Our construction of such a sequence is based on Corollary 6 and for later use we modify it as follows.

Lemma 3. *Suppose that $\bar{R}_n \in T_g$ converges to $\bar{R}_0 \in T_g$. Let $f_{\bar{R}_n}$ be the Teichmüller mapping of $\bar{R}_0$ to $\bar{R}_n$, and $f_n = f_{\bar{R}_n}^{-1}$. Then the admissible sequence $\{\langle \bar{R}_n, \bar{R}_0, f_n\rangle\}_{n=1}$ satisfies the condition* (10).

Proof. It is clear that the given sequence is admissible, and noting that $\lim_{n\to\infty} \|\theta_{c,\bar{R}_n}\|_{R_n} = \|\theta_{c,\bar{R}_0}\|_{R_0}$ (cf. Corollary 4), the assertion follows from Corollary 6 and the following inequality:

$$\begin{aligned} &\|\theta_{\bar{R}_n} \circ f_n^{-1} - \theta_{\bar{R}_0}\|_{R_0} \\ &\leqq \left| \frac{2}{\|\theta_{c,\bar{R}_n}\|^2_{R_n}} - \frac{2}{\|\theta_{c,\bar{R}_0}\|^2_{R_0}} \right| \cdot \|\theta_{c,\bar{R}_0}\|_{R_0} \\ &+ \frac{2}{\|\theta_{c,\bar{R}_n}\|^2_{R_n}} \cdot \|\theta_{c,\bar{R}_n} \circ f_n^{-1} - \theta_{c,\bar{R}_0}\|_{R_0}. \end{aligned}$$

Now let a positive y_0 be fixed, and write

$$X = \{\bar{R} \in S_c \colon F_2(\bar{R}) = \sqrt{-1} \cdot y_0\}.$$

Let $\{\bar{S}_n\} \subset X$ be a sequence converging to $\bar{S}_0 \in X$, and $\bar{R}_n' = F_1(\bar{S}_n)$ for every n.

Then $\bar{R}'_n$ is canonically embedded in a compact Riemann surface, say $\bar{\bar{R}}_n$, of genus $g-1$, and $\bar{\bar{R}}_n - R'_n$ consists of two points, say $\{p_n^i\}_{i=1}^2$, for every n. Next let $\bar{R}_n = F^{-1}(\bar{R}'_n, \infty)$. Then $\theta_{\bar{R}_n}$ is, by definition, $(\sqrt{-1}/2\pi)\cdot\varphi_{\bar{R}'_n}$. Denote the characteristic disk of $\theta_{\bar{R}_n}$ at p_n^i on $\bar{\bar{R}}_n$ by D_n^i $(i=1, 2)$. Map each D_n^i conformally onto $D=\{|z|<1\}$ so that p_n^i corresponds to $z=0$. Then under the natural embedding π_n of $S_n - C_{\bar{S}_n}$ into R'_n (where $C_{\bar{S}_n}$ is the center trajectory of the characteristic ring domain $W_{\bar{S}_n}$ of $\theta_{c,\bar{S}_n}$ for c on $\bar{S}_n$), the images of the boundaries of $S_n - C_{\bar{S}_n}$ are loops in D_n^1 and D_n^2 corresponding to the circle $\{|z|=r_0\}$ in D for every n with $r_0 = \exp[-\pi y_0]$. Finally let $W_{r,n}$ be the ring domain in $W_{\bar{S}_n} \subset S_n$ containing $C_{\bar{S}_n}$ such that the boundaries of $W_{r,n}$ are mapped by π_n to the loops in D_n^1 and D_n^2 corresponding to $\{|z|=r\}$ in D, where $r_0<r<1$. (See [16], [17] for more detailed preliminaries.)

Lemma 4. *Let $\bar{S}_n$, $\bar{R}_0$ and $W_{r,n}$ be as above, and let a finite number of constants $\{\delta_j\}_{j=1}^s$ such that $0<\delta_1<\delta_2<\ldots<\delta_s<(1-r_0)$ be given. Then there exists a sequence $\{\langle\bar{S}_n, \bar{R}_0, f_n\rangle\}_{n=1}^\infty$ of deformations satisfying the following condition: for every positive ε $(<1-r_0-\delta_s)$ we can find an N such that for every $n \geqq N$*

(a) $f_n^{-1}|_{(R_0-K_1)}$ *is $(1+\varepsilon)$-quasiconformal,*

(b) $\|\theta_{\bar{S}_n}\circ f_n^{-1} - \theta_{\bar{R}_0}\|_{(R_0-K_1)} < \varepsilon$, *and*

(c) $f_n^{-1}(K_j) \supset W_{r_j,n}$ $(j=1, \ldots, s)$ *and* $f_n^{-1}(N(R_0)) = C_{\bar{S}_n}$,

where K_j is the neighbourhood of the node $N(R_0)$ of R_0 corresponding to $\{|z|<r_0+\delta_j+\varepsilon\}$ in D, and $r_j = r_0+\delta_j$.

Proof. Let $f_{\bar{S}_n}$ be the Teichmüller mapping from $\bar{S}_0$ to $\bar{S}_n$. Then it is known ([16] Corollary 2) that we can find and N_1 such that

(11) $$f_{\bar{S}_n}(W_{r'_j,0}) \supset W_{r_j,n} \quad (j=1, \ldots, s)$$

for every $n \geqq N_1$, where $r'_j = r_j + \varepsilon$. Hence in particular we can consider each $\pi_0 \circ f_{\bar{S}_n}^{-1}|_{(S_n - C_{\bar{S}_n})}$ $(n \geqq N_1)$ as a quasiconformal embedding from $S_n - C_{\bar{S}_n}$ into $R'_0 \subset R_0$, which we denote by g_n. Here we can assume (cf. [16]) that π_0 is conformally extended to a suitable neighbourhood of the boundaries of $S_0 - C_{\bar{S}_0}$ in $W_{\bar{S}_0}$.

By deforming g_n in $W_{r_1,n} - C_{\bar{S}_n}$, we can now make a deformation $\langle\bar{S}_n, \bar{R}_0, f_n\rangle$ such that $f_n^{-1}(N(R_0)) = C_{\bar{S}_n}$ and f_n coincides with g_n on $S_n - W_{r_1,n}$ for every $n \geqq N_1$. By taking any suitable deformations for $n<N_1$, we arrive at a sequence of deformations. We show that this sequence satisfies the conditions in Lemma 4.

First by Lemma 3 for a given ε we can find an N $(\geqq N_1)$ such that

(12) $$f_{\bar{S}_n} \text{ is } (1+\varepsilon)\text{-quasiconformal, and}$$

(13) $$\|\theta_{\bar{S}_n}\circ f_{\bar{S}_n} - \theta_{\bar{S}_0}\|_{S_0} < \varepsilon \quad \text{for every} \quad n \geqq N.$$

Note that $\pi_0(S_0 - W_{r'_j,n}) = R_0 - K_j$ $(j=1, \ldots, s)$ and $\theta_{\bar{S}_0}\circ\pi_0^{-1} = \theta_{\bar{R}_0}$ on $R_0 - K_1$. Hence from (11) we have

$$f_n(S_n - W_{r_j,n}) = g_n(S_n - W_{r_j,n}) \supset \pi_0(S_0 - W_{r'_j,0}) = R_0 - K_j,$$

that is, $f_n^{-1}(R_0-K_j)\subset S_n-W_{r_j,n}$ $(j=1,\dots,s)$ for every $n\geqq N_1$. Thus we conclude that $f_n^{-1}(K_j)\supset W_{r_j,n}$ $(j=1,\dots,s)$ for every $n\geqq N_1$, which implies that $f_n^{-1}=f_{\bar{S}n}^{-1}\circ\pi_0^{-1}$ on R_0-K_1, and hence $f_n^{-1}|_{(R_0-K_1)}$ is $(1+\varepsilon)$-quasiconformal for every $n\geqq N$ by (12).

Next, from (13) we have

$$\|\theta_{\bar{S}_n}\circ f_n^{-1}-\theta_{\bar{R}_0}\|_{(R_0-K_1)}=\|\theta_{\bar{S}_n}\circ f_{\bar{S}_n}\circ\pi_0^{-1}-\theta_{\bar{S}_0}\circ\pi_0^{-1}\|_{(R_0-K_1)}$$
$$=\|\theta_{\bar{S}_n}\circ f_{\bar{S}_n}-\theta_{\bar{S}_0}\|_{(S_0-W_{r'_1,0})}<\varepsilon$$

for every $n\geqq N$, which completes the proof.

Now we can show the following

Lemma 5. *Suppose that $\bar{R}_n\in\partial_c T_g$ and converges to $\bar{R}_0\in\partial_c T_g$. Then there exists an admissible sequence $\{\langle\bar{R}_n,\bar{R}_0,f_0\rangle\}_{n=1}^{\infty}$ of deformations satisfying the condition* (10).

Proof. Let K_n^k be the neighbourhood of the node of R_n corresponding to $\{|z|<1/k\}$ in D for every n. Fix a positive integer k arbitrarily, and let $\bar{R}_{k,n}=F^{-1}(\bar{R}'_n,(\sqrt{-1}/\pi)\log(k+1))$, where $\bar{R}'_n=F_1(\bar{R}'_n)$. Then using Lemma 4 with $\bar{S}_n=\bar{R}_{k,n}$, $y_0=(1/\pi)\log(k+1)$ (, hence $r_0=1/(k+1)$), $s=1$, and $\delta_1=\varepsilon=\varepsilon_k=(1/2)(1/k-1/(k+1))$, we have a sequence $\{\langle\bar{R}_{k,n},\bar{R}_0,f_{k,n}\rangle\}_{n=1}^{\infty}$ of deformations satisfying the condition; there exists an N_k such that for every $n\geqq N_k$

(i) $f_{k,n}^{-1}|_{(R_0-K_0^k)}$ is $(1+\varepsilon_k)$-quasiconformal,

(ii) $\|\theta_{\bar{R}_{k,n}}\circ f_{k,n}^{-1}-\theta_{\bar{R}_0}\|_{(R_0-K_0^k)}<\varepsilon_k$, and

(iii) $f_{k,n}^{-1}(K_0^k)\supset W_{r_1,n}$ $\left(\text{with } r_1=\dfrac{1}{2}\left(\dfrac{1}{k}+\dfrac{1}{k+1}\right)\right)$ and

$f_{k,n}^{-1}(N(R_0))=C_{\bar{R}_{k,n}}$.

Now let $\pi_{k,n}$ be the natural embedding of $R_{k,n}-C_{\bar{R}_{k,n}}$. Then we have an embedding $f_{k,n}\circ\pi_{k,n}^{-1}$ from $R_n-K_n^{k+1}$ into R_0 for every $n\geqq N_k$, and deforming $\pi_{k,n}$ in $W_{r_1,n}-C_{\bar{R}_{k,n}}$ we can make a sequence $\{\langle\bar{R}_n,\bar{R}_0,g_{k,n}\rangle\}_{n=N_k}^{\infty}$ of deformations such that $g_{k,n}=f_{k,n}\circ\pi_{k,n}^{-1}$ on $\pi_{k,n}(R_{k,n}-W_{r_1,n})$, which contains $g_{k,n}^{-1}(R_0-K_0^k)$ by (iii). As in the proof of Lemma 4, it is easily seen from (i) and (ii) that

(iv) $g_{k,n}^{-1}|_{(R_0-K_0)}$ is $(1+\varepsilon_k)$-quasiconformal, and

(v) $\|\theta_{\bar{R}_n}\circ g_{k,n}^{-1}-\theta_{\bar{R}_0}\|_{(R_0-K_0^k)}<\varepsilon_k$

for every $n\geqq N_k$. Here we assume that $N_{k+1}\geqq N_k$ for every k.

Finally define a sequence $\{\langle\bar{R}_n,\bar{R}_0,f_n\rangle\}_{n=1}^{\infty}$ of deformations by taking as f_n the above $g_{k,n}$ if $N_k\leqq n\leqq N_{k+1}-1$. By (iv) and (v) we find that it is a desired sequence.

Lemma 6. *Suppose that $\bar{S}_n\in T_g$ and converges to $\bar{R}_0\in\partial_c T_g$. Then there exists an admissible sequence $\{\langle\bar{S}_n,\bar{R}_0,f_n\rangle\}_{n=1}^{\infty}$ of deformations satisfying the condition* (10).

Proof. From the definition of the fine topology ([17]), we may assume that $\{\bar{S}_n\}_{n=1}^{\infty}$ is contained in S_c. Let $F(\bar{S}_n)=(\bar{R}'_n, z_n)$ and $\bar{R}_n=F^{-1}(\bar{R}'_n, \infty)$ for every n, and K_n^k and ε_k be as in the proof of Lemma 5. Then since $\bar{R}_n$ converges to $\bar{R}_0$, using Lemma 4 as in the proof of Lemma 5, we can construct a sequence $\{\langle\bar{R}_n, \bar{R}_0, g_n\rangle\}_{n=1}^{\infty}$ of deformations satisfying the condition: for every positive integer k, there exists an N_k such that for every $n\geqq N_k$

(i) $g_n^{-1}|_{(R_0-K_0^k)}$ is $(1+\varepsilon_k)$-quasiconformal,

(ii) $\|\theta_{\bar{R}n}\circ g_n^{-1}-\theta_{\bar{R}_0}\|_{(R_0-K_0^k)}<\varepsilon_k$, and

(iii) $g_n^{-1}(K_0^j)\supset K_n^{j+1}$ $(j=1, \dots, k)$.

Next, let k_n be the largest integer such that $\pi_n(S_n-C_{\bar{S}_n})\cap K_n^k\neq\emptyset$ (that is, $\pi_n(S_n-C_{\bar{S}_n})\supset R_n-K_n^k$) for every n. Then k_n tends to $+\infty$, for $1/k_n>\exp(-\pi\cdot\operatorname{Im} z_n)$ for every n and $\lim_{n\to\infty}\operatorname{Im} z_n=+\infty$. Deform each π_n in $\pi_n^{-1}(K_n^{k_n})$ as before, and we have a sequence $\{\langle\bar{S}_n, \bar{R}_n, h_n\rangle\}_{n=1}^{\infty}$ of deformations such that h_n $(=\pi_n)$ is conformal on $S_n-\pi_n^{-1}(K_n^{k_n})$.

Finally let $f_n=g_n\circ h_n$ for every n. We can now show that $\{\langle\bar{S}_n, \bar{R}_0, f_n\rangle\}_{n=1}^{\infty}$ is a desired sequence. In fact, for every neighbourhood K of the node of R_0 and every positive ε, there exists a k_0 such that $K\supset K_0^{k_0}$ and $\varepsilon_{k_0}<\varepsilon$. Thus we can find an N $(\geqq N_{k_0})$ such that $k_n\geqq k_0+1$ (, hence $f_n^{-1}(K)\supset h_n^{-1}(K_n^{k_n})$ by (iii)), and (i) and (ii) hold with $k=k_0$ for every $n\geqq N$. Then noting that $f_n^{-1}=\pi_n^{-1}\circ g_n^{-1}$ and $\theta_{\bar{S}_n}\circ\pi_n^{-1}=\theta_{\bar{R}_n}$ on $S_n-f_n^{-1}(K)$ for every $n\geqq N$, we have the assertion.

3.3. Proof of Theorem 7. We consider only the case in Lemma 6, for other cases can be treated similarly as in Proposition 3. Let $\bar{S}_n$, $\bar{R}_0$ and f_n be as in Lemma 6, $G_n\in T(G^*)$ correspond to $\bar{S}_n$, and let U_n be the component of G_n such that $\bar{S}_n=U_n/G_n$. Also let $G_0\in\partial T(G^*)$ correspond to $\bar{R}_0$, U_0 be a non-invariant component of G_0, and F_n be the lift of f_n^{-1} from U_0 into U_n. Then it is seen (cf. [1]) that F_n converges locally uniformly on U_0 to the identity, and U_0 is a component of the Carathéodory kernel of $\{U_n\}_{n=1}^{\infty}$.

Let $a_n(z)\,dz$ and $a_0(z)\,dz$ be the lifts of $\theta_{\bar{S}_n}$ on U_n and $\theta_{\bar{R}_0}$ on U_0, respectively, and fix $z_0\in U_0$ and a small ϱ such that $D_{2\varrho}=\{|z-z_0|\leqq 2\varrho\}$ is contained in U_0 (hence, in U_n for every sufficiently large n) arbitrarily. Then similarly as in the proof of Lemma 2 in [2], we can show that there exists an M such that $\sup_{z\in D_{2\varrho}}|a_n(z)|\leqq M$ for every sufficiently large n. It also holds that

$$|a_n(z_0)-a_0(z_0)|\leqq\frac{1}{\sqrt{2\pi\varrho^2}}\|a_n(z)-a_0(z)\|_{2,D_\varrho}$$

$$\leqq\frac{1}{\sqrt{2\pi\varrho^2}}\Big[\big\|a_n(z)-a_n(F_n(z))\big\|_{2,D_\varrho}+\big\|a_n(F_n(z))[(F_n)_z-1]\big\|_{2,D_\varrho}$$

$$+\big\|a_n(F_n(z))(F_n)_z-a_0(z)\big\|_{2,D_\varrho}\Big].$$

Hence using Lemma 6 and Theorem V 5—3 in [14], we can show that $a_n(z)$ converges to $a_0(z)$ pointwise, hence locally uniformly on U_0.

Finally take any admissible sequence $\{\langle \bar{S}_n, \bar{R}_0, f_n' \rangle\}_{n=1}^{\infty}$ and neighbourhood K of the node of R_0, and fix them. Let F_n' be the lift of $f_n'^{-1}$ from U_0 into U_n, and E be a relatively compact region in U_0 which covers $R_0 - K$. It is then easily seen from above that $a_n(F_n'(z))$ is uniformly bounded on E and $a_n(F_n'(z))$ converges to $a_0(z)$ uniformly on E. Hence from the inequality

$$\|\theta_{\bar{S}_n} \circ f_n'^{-1} - \theta_{\bar{R}_0}\|_{(R_0-K)} \leqq \big\|a_n(F_n'(z))[(F_n')_z - 1]\big\|_{2,E}$$

$$+ \big\|a_n(F_n'(z)) - a_0(z)\big\|_{2,E} + \big\|a_n(F_n'(z))(F_n')_{\bar{z}}\big\|_{2,E}$$

and Theorem V 5—3 in [14], we can conclude that

$$\lim_{n\to\infty} \|\theta_{\bar{S}_n} \circ f_n'^{-1} - \theta_{\bar{R}_0}\|_{(R_0-K)} = 0.$$

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