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# EXTENSIONS OF ISOMETRIC AND SYMMETRIC LINEAR RELATIONS IN A KREIN SPACE

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## Introduction

This paper continues the study of linear relations in an *indefinite* inner product space begun in [6]. Here we consider isometric linear relations and their extensions as well as symmetric linear relations and their extensions.

The first chapter summarizes briefly the terminology used in this paper: the first section recalls the basic definitions from the theory of indefinite inner product spaces; for more complete treatment, see [2]. The second section gives the notation used for linear relations; see [6].

In Chapter 2 we analyse isometric linear relations. After presenting the basic properties in Section 1 we investigate various ways of reducing an isometric linear relation to an operator in Section 2. Section 3 introduces formally unitary linear relations, which form a natural generalization of unitary operators. In Section 4 we study another kind of generalization, namely rectangular isometric linear relations. These relations have some useful topological properties which are needed e.g. to introduce the defect numbers in Section 5.

In Chapter 3 we investigate extensions of a rectangular isometric linear relation in a Krein space. Section 1 examines the existence of various kinds of extensions and Section 2 characterizes these extensions. These characterizations may be new also in the case of rectangular isometric operators in a Krein space.

In Chapter 4 we use the Cayley transformation to analyse rectangular symmetric linear relations mostly in a Krein space. This method allows us to take advantage of the results in Chapters 2 and 3. After studying rectangular symmetric linear relations in Section 1 we prove the existence of various kinds of extensions and describe them in Section 2. These characterizations seem to be new also in the case of operators in a Krein space.

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## 1. Preliminaries

**1.1. Indefinite inner product spaces.** Throughout this paper $\mathfrak{H}$ denotes an *(indefinite) inner product space*, that is, $\mathfrak{H}$ is a complex vector space equipped with a non-degenerate hermitean sesquilinear form $[\cdot|\cdot]$.

A vector $f$ in $\mathfrak{H}$ is said to be *positive/non-negative/neutral/non-positive/negative* if $[f|f]>0/\geqq 0/=0/\leqq 0/<0$. A similar terminology holds also for subspaces of $\mathfrak{H}$. Specifically, a subspace which contains positive as well as negative vectors is *indefinite*; otherwise it is *semi-definite*.

For a subspace $\mathfrak{L}$ of $\mathfrak{H}$ we define

$$\mathfrak{L}^{\perp} := \{f\in\mathfrak{H}\,|\,[f|g]=0 \text{ for all } g\in\mathfrak{L}\}$$

and call it the *orthogonal companion* of $\mathfrak{L}$ in $\mathfrak{H}$. $\mathfrak{L}$ is said to be *ortho-complemented* if $\langle\mathfrak{L}, \mathfrak{L}^{\perp}\rangle=\mathfrak{H}$, i.e., $\mathfrak{L}$ and $\mathfrak{L}^{\perp}$ together span the whole space. The subspace $\mathfrak{L}^0:=\mathfrak{L}\cap\mathfrak{L}^{\perp}$ is called the *isotropic part* of $\mathfrak{L}$. If $\mathfrak{L}^0\neq\{0\}$, the subspace $\mathfrak{L}$ is *degenerate*; otherwise it is *non-degenerate*. We call $\mathfrak{L}$ *closed* in case $\mathfrak{L}=\mathfrak{L}^{\perp\perp}$.

The space $\mathfrak{H}$ is said to be *decomposable* if it has a *fundamental decomposition* $\mathfrak{H}=\mathfrak{H}_+[\dotplus]\mathfrak{H}_-$, where $\mathfrak{H}_+/\mathfrak{H}_-$ is a positive/negative subspace; here the symbol $[\dotplus]$ denotes a direct and orthogonal sum. If this decomposition has the property that $\mathfrak{H}_+/\mathfrak{H}_-$ is a Hilbert space with respect to the inner product $[\cdot|\cdot]/-[\cdot|\cdot]$, $\mathfrak{H}$ is called a *Krein space*. In this case $\mathfrak{H}$ is also a Hilbert space with respect to the inner product $(\cdot|\cdot)$:

$$(f_+ + f_-|g_+ + g_-) := [f_+|g_+]-[f_-|g_-] \quad (f_\pm, g_\pm\in\mathfrak{H}_\pm).$$

For a topology of a Krein space we always take the topology induced by this inner product; this agreement is consistent with the earlier terminology.

A Krein space $\mathfrak{H}$ is called a *Pontrjagin space (with $\varkappa$ negative squares)* or a *$\pi_\varkappa$-space* if $\dim\mathfrak{H}_-=\varkappa<\infty$.

Let $\mathfrak{H}$ and $\mathfrak{K}$ be inner product spaces. Then the product space $\mathfrak{H}\oplus\mathfrak{K}$ is also an inner product space with the form

$$[(f, g)|(h, k)] := [f|h]+[g|k] \quad ((f, g), (h, k)\in\mathfrak{H}\oplus\mathfrak{K}).$$

Furthermore, it is a Krein/Pontrjagin space if $\mathfrak{H}$ and $\mathfrak{K}$ are Krein/Pontrjagin spaces.

**1.2. Linear relations.** A *linear relation* in the inner product space $\mathfrak{H}$ is a subspace of the product space $\mathfrak{H}^2:=\mathfrak{H}\oplus\mathfrak{H}$. It is said to be *closed* if it is a closed subspace. Let $T$ and $S$ be linear relations in $\mathfrak{H}$. We use the following notation:

$$\mathfrak{D}(T) := \{f \in \mathfrak{H} \mid (f, g) \in T \text{ for some } g \in \mathfrak{H}\},$$
$$\mathfrak{R}(T) := \{g \in \mathfrak{H} \mid (f, g) \in T \text{ for some } f \in \mathfrak{H}\},$$
$$\mathfrak{N}(T) := \{f \in \mathfrak{H} \mid (f, 0) \in T\},$$
$$T(0) := \{g \in \mathfrak{H} \mid (0, g) \in T\},$$
$$T(\mathfrak{L}) := \{g \in \mathfrak{H} \mid (f, g) \in T \text{ for some } f \in \mathfrak{L}\} \quad (\mathfrak{L} \subset \mathfrak{H}),$$
$$T^{-1} := \{(g, f) \in \mathfrak{H}^2 \mid (f, g) \in T\},$$
$$T^{+} := \{(h, k) \in \mathfrak{H}^2 \mid [g \mid h] = [f \mid k] \text{ for all } (f, g) \in T\},$$
$$zT := \{(f, zg) \in \mathfrak{H}^2 \mid (f, g) \in T\} \quad (z \in \boldsymbol{C} := \text{complex numbers}),$$
$$T + S := \{(f, g+k) \in \mathfrak{H}^2 \mid (f, g) \in T, (f, k) \in S\}.$$

We identify (linear) operators with their graphs. Consequently, a linear relation $T$ is an operator if and only if $T(0)=\{0\}$.

## 2. Isometric linear relations

**2.1. Basic properties.** A linear relation $V$ in the inner product space $\mathfrak{H}$ is *isometric* if

$$[g \mid g] = [f \mid f] \quad \text{for all} \quad (f, g) \in V.$$

Note that isometric linear relations in $\mathfrak{H}$ are not necessarily operators as in the Hilbert space case; see [6], 2.1. It is easy to prove the following characterizations of isometric linear relations; see also [1], Proposition 4.4.

Proposition 2.1.1. *Let $V$ be a linear relation in $\mathfrak{H}$. Then the following assertions are equivalent:*

(i) *$V$ is isometric;*

(ii) *$V^{-1}$ is isometric;*

(iii) *$[g \mid k]=[f \mid h]$ for all $(f, g), (h, k) \in V$;*

(iv) *$V^{-1} \subset V^{+}$.*

The following proposition summarizes some useful properties of isometric linear relations; compare with [2], where isometric operators in an inner product space are considered.

Proposition 2.1.2. *Let $V$ be an isometric linear relation in $\mathfrak{H}$. Then*

(i) *$V(\mathfrak{N}(V))=V(0)$;*

(ii) *$V(\mathfrak{D}(V)^0)=\mathfrak{R}(V)^0$;*

(iii) *$\mathfrak{N}(V) \oplus V(0) \subset V^0 = V \cap (\mathfrak{D}(V)^0 \oplus \mathfrak{R}(V)^0)$;*

(iv) *$V$ is ortho-complemented if and only if $\mathfrak{D}(V)$ and $\mathfrak{R}(V)$ are ortho-complemented;*

(v) *for a closed $V$ the subspaces $\mathfrak{N}(V)$, $V(0)$ and $V^0$ are closed.*

*Proof.* (i) is obvious. In order to verify (ii) and (iii) use Proposition 2.1.1.

To prove (iv) suppose first that $V$ is ortho-complemented. Then for every $f$ in $\mathfrak{H}$ we have

$$(f, 0) = (f_1, g_1) + (f - f_1, -g_1),$$

where $(f_1, g_1) \in V$ and $(f - f_1, -g_1) \in V^{\perp}$. From Proposition 2.1.1 we get that the vector $f - 2f_1$ is in $\mathfrak{D}(V)^{\perp}$, and so $f$ has the decomposition $f = 2f_1 + (f - 2f_1)$e, where $2f_1 \in \mathfrak{D}(V)$ and $f - 2f_1 \in \mathfrak{D}(V)^{\perp}$; hence $\mathfrak{D}(V)$ is ortho-complemented. The proof for $\mathfrak{R}(V)$ is similar.

To prove the converse let $(f, g) \in \mathfrak{H}^2$ be arbitrary. Then $f = f_1 + f_2$ with $f_1 \in \mathfrak{D}(V)$, $f_2 \in \mathfrak{D}(V)^{\perp}$ and $g = g_1 + g_2$ with $g_1 \in \mathfrak{R}(V)$, $g_2 \in \mathfrak{R}(V)^{\perp}$. Furthermore, we can choose vectors $h \in \mathfrak{D}(V)$ and $k \in \mathfrak{R}(V)$ so that $(h, \frac{1}{2} g_1)$ and $(\frac{1}{2} f_1, k)$ are in $V$. This implies the decomposition

$$(f, g) = (\tfrac{1}{2} f_1 + h, k + \tfrac{1}{2} g_1) + (f - \tfrac{1}{2} f_1 - h, g - \tfrac{1}{2} g_1 - k),$$

where the first component is in $V$ and, by Proposition 2.1.1, the second in $V^{\perp}$. Thus $V$ is ortho-complemented.

(v) follows easily from the fact that $V$ is closed if and only if $V = V^{++}$; see [6]. $\square$

Note that the inclusion in (iii) can be proper: Let $V$ be the diagonal of $\mathfrak{L} \oplus \mathfrak{L}$, where $\mathfrak{L}$ is a non-zero neutral subspace in $\mathfrak{H}$. Then $V$ is an isometric linear relation with $\mathfrak{R}(V) \oplus V(0) = \{0\} \neq V^0$ $(= V)$.

**2.2. Reductions to an operator.** A decomposition of a linear relation to the operator part and multi-valued part was given in [6]. Unfortunately, this reduction is not useful in this context because for an isometric linear relation $V$ the basic assumption under which this reduction holds implies that $V$ is an operator; see [6], Theorem 2.9. So we must find another reduction for isometric linear relations.

Recall that a subspace $\mathfrak{L}$ in $\mathfrak{H}$ can be represented in the form $\mathfrak{L} = \mathfrak{L}^0 \dotplus \mathfrak{L}_c$, where $\mathfrak{L}_c$ is a complementary subspace for $\mathfrak{L}^0$ in $\mathfrak{L}$; see [2], Lemma I.5.1. In the sequel we shall use the symbol $\dotplus$ to denote the algebraic sum in $\mathfrak{H}^2$. The following result is easily proved by using Proposition 2.1.2.

Proposition 2.2.1. *Let $V$ be an isometric linear relation in $\mathfrak{H}$, and let $V_c$ be a complementary subspace for $V^0$ in $V$. Then $V_c$ is an injective isometric operator and*

$$V = V_c [\dotplus] V^0. \tag{2.2.1}$$

*Furthermore, $\mathfrak{D}(V_c)$ and $\mathfrak{R}(V_c)$ are non-degenerate and*

$$\mathfrak{D}(V) = \mathfrak{D}(V_c)[+]\mathfrak{D}(V)^0,$$

$$\mathfrak{R}(V) = \mathfrak{R}(V_c)[+]\mathfrak{R}(V)^0.$$

This decomposition has the advantage that the domain and the range of $V_c$ are non-degenerate. On the other hand, $V_c$ may be very small, even zero, although $V$ itself is an operator as in the example after the proof of Proposition 2.1.2. To overcome this disadvantage we look for another way of decomposing an isometric linear relation.

Let $V$ be an isometric linear relation in $\mathfrak{H}$. We call a linear relation $V_s$ in $\mathfrak{H}$ an *operator part* of $V$ if $V=V_s\dot{+}\mathfrak{N}(V)\oplus V(0)$. As to the existence and basic properties of $V_s$ we have

Theorem 2.2.2. *Every isometric linear relation $V$ in $\mathfrak{H}$ has an operator part. Let $V_s$ be an operator part of $V$. Then $V_s$ is an injective isometric operator with*

$$V=V_s[\dot{+}]\mathfrak{N}(V)\oplus V(0) \tag{2.2.2}$$

*and*

$$\mathfrak{D}(V)=\mathfrak{D}(V_s)[\dot{+}]\mathfrak{N}(V),$$

$$\mathfrak{R}(V)=\mathfrak{R}(V_s)[\dot{+}]V(0).$$

*Furthermore, there exist non-degenerate subspaces $\mathfrak{D}$ and $\mathfrak{R}$ such that*

$$\mathfrak{D}(V_s)=\mathfrak{D}[\dot{+}]\mathfrak{D}(V_s)^0,$$

$$\mathfrak{R}(V_s)=\mathfrak{R}[\dot{+}]\mathfrak{R}(V_s)^0.$$

*Proof.* The existence of an operator part is clear because one can choose as $V_s$ a complementary subspace for $\mathfrak{N}(V)\oplus V(0)$ in $V$. The existence of $\mathfrak{D}$ follows similarly; see [2], Lemma I.5.1. The subspace $V_s(\mathfrak{D})$ is then suitable for $\mathfrak{R}$. The other claims are easily verified by using Proposition 2.1.2. □

Note that we can always find a closed operator part for a closed isometric linear relation in a Krein space. If in addition the space is a Pontrjagin space, then all subspaces appearing in Theorem 2.2.2 are (or can be chosen to be) closed.

Let us find out when the decompositions (2.2.1) and (2.2.2) coincide.

Theorem 2.2.3. *Let $V$ be an isometric linear relation in $\mathfrak{H}$, let $V_s$ be its operator part, and let $V_c$ be a complementary subspace for $V^0$ in $V$. Then the following conditions are equivalent:*

(i) *$V_s$ is a complementary subspace for $V^0$ in $V$;*

(ii) *$V_c$ is an operator part of $V$;*

(iii) *$V^0=\mathfrak{N}(V)\oplus V(0)$;*

(iv) *$\mathfrak{N}(V)=\mathfrak{D}(V)^0$;*

(v) *$V(0)=\mathfrak{R}(V)^0$;*

(vi) *$V_s$ is non-degenerate;*

(vii) *$V$ does not have a proper isometric extension $V'$ such that $V'^0=\mathfrak{N}(V')\oplus V'(0)$, $\mathfrak{D}(V')=\mathfrak{D}(V)$ and $\mathfrak{R}(V')=\mathfrak{R}(V)$.*

*Proof.* The conditions (i) and (ii) are obviously equivalent with (iii). The equivalences (iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) follow from Proposition 2.1.2. As $V^0$ is equal to $(V_s)^0[\dot{+}]\mathfrak{N}(V)\oplus V(0)$, (iii) and (vi) are equivalent.

Let us show that (v) implies (vii). Assume that there exists an isometric extension $V'$ of $V$ with the properties mentioned in (vii). Then $V'\supset V$, $\mathfrak{D}(V')=\mathfrak{D}(V)$ and, by applying the already proved equivalence (iii)$\Leftrightarrow$(v) to $V'$, we get $V'(0)=\mathfrak{R}(V')^0=\mathfrak{R}(V)^0=V(0)$. These imply $V'=V$, and so the extension $V'$ is not proper.

To complete the proof, suppose that (v) does not hold. Let $\mathfrak{N}\neq\{0\}$ be a complementary subspace for $V(0)$ in $\mathfrak{R}(V)^0$. Then it is easy to check that with the relation $V':=V+\{0\}\oplus\mathfrak{N}$ the converse of (vii) holds. □

**2.3. Formally unitary linear relations.** Recall first that an isometric linear relation $U$ in the inner product space $\mathfrak{H}$ is called *unitary* if $\mathfrak{D}(U)=\mathfrak{R}(U)=\mathfrak{H}$; see [6]. If $U$ is unitary, then $U^{-1}=U^+$, but the converse does not hold generally; see the second example after Proposition 2.3.3. We call a linear relation $U$ in $\mathfrak{H}$ *formally unitary* in case $U^{-1}=U^+$. These relations are of interest because they are exactly the Cayley transforms of the self-adjoint linear relations; see [1] or [6]. Formally unitary linear relations can be characterized in the following way.

Proposition 2.3.1. *Let $U$ be an isometric linear relation in $\mathfrak{H}$. Then the following assertions are equivalent:*

(i) *$U$ is formally unitary;*

(ii) *$\mathfrak{R}(U)=\mathfrak{D}(U^+)$ and $\mathfrak{N}(U)=\mathfrak{D}(U)^\perp$;*

(iii) *$\mathfrak{D}(U)=\mathfrak{R}(U^+)$ and $U(0)=\mathfrak{R}(U)^\perp$.*

*If in addition the range $\mathfrak{R}(U)$ is closed, then the assertions above are equivalent to*

(iv) *$U$ is closed, $\mathfrak{N}(U)=\mathfrak{D}(U)^\perp$ and $U(0)=\mathfrak{R}(U)^\perp$.*

*Proof.* The implications (i)$\Rightarrow$(ii)—(iv) are easy to prove by using the general equalities $U^+(0)=\mathfrak{D}(U)^\perp$ and $\mathfrak{N}(U^+)=\mathfrak{R}(U)^\perp$; see [6]. Recall also that two linear relations $S$ and $T$ are equal if and only if $S\subset T$, $S(0)=T(0)$ and $\mathfrak{D}(S)=\mathfrak{D}(T)$. Similarly, (ii) or (iii) implies (i). To prove the implication (iv)$\Rightarrow$(i), note that $\mathfrak{D}(U^+)\subset U(0)^\perp$. □

Corollary 2.3.2. *If $U$ is a formally unitary linear relation in $\mathfrak{H}$, then*

(i) *$U^0=\mathfrak{N}(U)\oplus U(0)$;*

(ii) *$U$ is maximal isometric.*

*Proof.* (i) follows from Proposition 2.3.1 and Theorem 2.2.3. To prove (ii), let $V$ be an isometric extension of $U$ in $\mathfrak{H}$. Then, by Proposition 2.1.1, $V^{-1}$ is also isometric and $V=(V^{-1})^{-1}\subset(V^{-1})^+$; hence $U=U^{++}=(U^{-1})^+\supset(V^{-1})^+\supset V$. □

We can also characterize unitary linear relations, which in fact are all operators, with the help of formally unitary linear relations.

Proposition 2.3.3. *Let $U$ be a formally unitary linear relation in $\mathfrak{H}$. Then the following conditions are equivalent:*

(i) *$U$ is unitary;*

(ii) *$\mathfrak{D}(U)=\mathfrak{R}(U)=\mathfrak{H}$;*

(iii) *$U$ is ortho-complemented.*

*Proof.* The implication (ii)⇒(i) is trivial. By [1], Proposition 4.4, (iii) implies (ii). The implication (i)⇒(iii) follows from Proposition 2.1.2. □

Let us give two examples, which show that the classes of the linear relations under current discussion are all different. Let $\mathfrak{H}$ be the $2\varkappa$-dimensional complex space $C^{2\varkappa}$, $\varkappa<\infty$, with the inner product

$$[f\,|\,g]:=\sum_{k=1}^{\varkappa} f_k\cdot\overline{g_k}-\sum_{k=\varkappa+1}^{2\varkappa} f_k\cdot\overline{g_k}.$$

Then $\mathfrak{H}$ is a Pontrjagin space with $\varkappa$ negative squares, and thus we can choose the components $\mathfrak{H}_\pm$ of a fundamental decomposition of $\mathfrak{H}$ so that $\dim\mathfrak{H}_\pm=\varkappa$. Let $\{e_k^\pm\}_{k=1,\ldots,\varkappa}$ be such a basis of $\mathfrak{H}_\pm$ that $[e_k^\pm\,|\,e_j^\pm]=\pm\delta_{kj}$, $k, j=1,\ldots,\varkappa$.

Define $\mathfrak{L}:=\langle e_1^+ + e_1^-\rangle$, i.e., the subspace spanned by the vector $e_1^+ + e_1^-$. Then $\mathfrak{L}$ is neutral, and so the linear relation $V:=\mathfrak{L}\oplus\mathfrak{L}$ in $\mathfrak{H}$ is isometric; but as $\mathfrak{L}\neq\mathfrak{L}^\perp$, $V$ is not formally unitary.

For the second example set $\mathfrak{M}:=\langle e_1^+ + e_1^-,\ldots,e_\varkappa^+ + e_\varkappa^-\rangle$. Then $\mathfrak{M}=\mathfrak{M}^\perp$, and thus the definition $U:=\mathfrak{M}\oplus\mathfrak{M}$ gives $U^{-1}=\mathfrak{M}\oplus\mathfrak{M}=\mathfrak{M}^\perp\oplus\mathfrak{M}^\perp=U^+$, that is, $U$ is a formally unitary linear relation in $\mathfrak{H}$. On the other hand, $U$ is not unitary because the domain is not the whole space.

**2.4. Rectangular isometric linear relations.** Recall that an isometric operator in a Krein space is said to be rectangular if both the domain and the range are ortho-complemented. This definition is of course meaningful also in a general inner product space. We extend this notion to linear relations by calling a closed isometric linear relation $V$ in the inner product space $\mathfrak{H}$ *rectangular* if it has a rectangular operator part.

Remarks. 1° The definition of the rectangularity is given only for closed linear relations. One could of course drop the requirement of closedness, but as we are mostly interested in closed linear relations we preferred this definition.

2° The apparently more natural definition "$V$ is rectangular if $\mathfrak{D}(V)$ and $\mathfrak{R}(V)$ are ortho-complemented" is not suitable for relations because it implies, by Proposition 2.1.2, that $V$ is an operator.

3° The definition of rectangularity is slightly inconvenient: we have agreed to call a linear relation $V$ an operator if $V(0)=\{0\}$. In this context one might think that a rectangular isometric linear relation $V$ with $V(0)=\{0\}$ is a rectangular isometric operator (in the sense of [2], p 128). But this is not necessarily true. For example, let $V_s$ be a rectangular isometric operator in a Krein space such that $\mathfrak{D}(V_s)^{\perp}$ is indefinite, let $\mathfrak{N}$ be a closed neutral subspace in $\mathfrak{D}(V_s)^{\perp}$, and define $V:=V_s[\dotplus]\mathfrak{N}\oplus\{0\}$. Then this $V$ is a rectangular isometric linear relation, which is an operator but which is not a rectangular isometric operator.

Using the results of [2] one can verify the following assertion: a rectangular isometric linear relation $V$ in a Krein space is a rectangular isometric operator if and only if $\mathfrak{N}(V)=V(0)=\{0\}$.

4° A rectangular isometric linear relation $V$ has the convenient property $V^0=\mathfrak{N}(V)\oplus V(0)$; see Theorem 2.2.3.

In studying rectangular linear relations the following general result is useful (cf [5], Lemma 5.1):

Lemma 2.4.1. *Let $\mathfrak{L}_1$ be an ortho-complemented subspace in $\mathfrak{H}$, let $\mathfrak{L}_2$ be a subspace orthogonal to $\mathfrak{L}_1$, and define $\mathfrak{L}:=\mathfrak{L}_1[\dotplus]\mathfrak{L}_2$. Then*

(i) $\overline{\mathfrak{L}}=\mathfrak{L}_1[\dotplus]\overline{\mathfrak{L}_2}$;

(ii) $\mathfrak{L}$ *is closed/ortho-complemented if and only if* $\mathfrak{L}_2$ *is closed/ortho-complemented.*

*Proof* (i) The assumptions imply that $\mathfrak{L}^{\perp\perp}=\mathfrak{L}_1[\dotplus]\mathfrak{L}^{\perp\perp}\cap\mathfrak{L}_1^{\perp}$ and

$$\mathfrak{L}^{\perp\perp}\cap\mathfrak{L}_1^{\perp} = (\mathfrak{L}_1^{\perp}\cap\mathfrak{L}_2^{\perp}+\mathfrak{L}_1)^{\perp} = \{(\mathfrak{L}_1^{\perp}+\mathfrak{L}_1)\cap\mathfrak{L}_2^{\perp}\}^{\perp} = \mathfrak{L}_2^{\perp\perp}.$$

Thus $\mathfrak{L}^{\perp\perp}=\mathfrak{L}_1[\dotplus]\mathfrak{L}_2^{\perp\perp}$, which is (i).

(ii) If $\mathfrak{L}_2$ is closed, then, by (i), $\mathfrak{L}$ is closed. If $\mathfrak{L}$ is closed, then, as shown above, $\overline{\mathfrak{L}_2}=\mathfrak{L}_2^{\perp\perp}=\overline{\mathfrak{L}}\cap\mathfrak{L}_1^{\perp}=\mathfrak{L}\cap\mathfrak{L}_1^{\perp}$, which is $\mathfrak{L}_2$. The rest of (ii) is proved in [2], Lemma I.9.2. □

The next result gives necessary and sufficient conditions for the rectangularity in two important special cases.

Proposition 2.4.2. *Let $V$ be an isometric linear relation in $\mathfrak{H}$.*

1° *In case $\mathfrak{H}$ is a Krein space, $V$ is rectangular if and only if it has an ortho-complemented operator part and $V^0$ is closed.*

2° *In case $\mathfrak{H}$ is a Pontrjagin space, $V$ is rectangular if and only if it has a closed operator part and $V^0=\mathfrak{N}(V)\oplus V(0)$.*

*Proof.* 1° follows if we combine Proposition 2.1.2, Theorem 2.2.3 and Lemma 2.4.1. For 2° recall that in a Pontrjagin space every neutral subspace is closed and that a closed, non-degenerate subspace is always ortho-complemented; see [2]. □

Formally unitary and unitary rectangular linear relations can be characterized with the help of the "deficiency spaces" of a rectangular operator part. Recall first that a maximal neutral subspace $\mathfrak{L}$ in $\mathfrak{H}$ is said to be *hypermaximal neutral* if it has neither non-positive nor non-negative proper extensions. This is equivalent to the condition $\mathfrak{L}=\mathfrak{L}^{\perp}$; see [2], Theorem I.7.4. As the hypermaximality plays a decisive role in the description mentioned above, we include here a result about the existence of hypermaximal neutral subspaces.

Proposition 2.4.3. *Let* $\mathfrak{H}$ *be a Krein space with the fundamental decomposition* $\mathfrak{H}=\mathfrak{H}_+[\dotplus]\mathfrak{H}_-$. *Then there exist hypermaximal neutral subspaces in* $\mathfrak{H}$ *if and only if* $\dim \mathfrak{H}_+=\dim \mathfrak{H}_-$.

*Proof.* If $\dim \mathfrak{H}_+=\dim \mathfrak{H}_-$, there exists an isometry $K\colon \mathfrak{H}_+\to\mathfrak{H}_-$. The definition $\mathfrak{L}:=\{f+Kf \mid f\in\mathfrak{H}_+\}$ gives then a hypermaximal neutral subspace. Conversely, a hypermaximal neutral subspace $\mathfrak{L}$ can be represented in the form $\mathfrak{L}=\{f+Kf \mid f\in\mathfrak{H}_+\}$, where the isometry $K$ is the angular operator of $\mathfrak{L}$ with respect to $\mathfrak{H}_+$; see [2]. Consequently, $\dim \mathfrak{H}_+=\dim \mathfrak{H}_-$. □

Theorem 2.4.4. *Let* $V$ *be a rectangular isometric linear relation in* $\mathfrak{H}$ *with a rectangular operator part* $V_s$. *Then*

(i) $V$ *is formally unitary if and only if* $\mathfrak{R}(V)$ *and* $V(0)$ *are hypermaximal neutral in* $\mathfrak{D}(V_s)^{\perp}$ *and* $\mathfrak{R}(V_s)^{\perp}$, *respectively;*

(ii) $V$ *is unitary if and only if* $\mathfrak{D}(V_s)=\mathfrak{R}(V_s)=\mathfrak{H}$.

*Proof.* (i) As $V$ is rectangular we have, by Theorems 2.2.2 and 2.2.3, the equality $\mathfrak{R}(V)=\mathfrak{R}(V_s)[\dotplus]V(0)$, which implies, by Proposition 2.1.2 and Lemma 2.4.1, that $\mathfrak{R}(V)$ is closed. Thus we can use the characterization given in Proposition 2.3.1 (iv). Owing to Theorem 2.2.2, $\mathfrak{D}(V)^{\perp}=\mathfrak{D}(V_s)^{\perp}\cap\mathfrak{N}(V)^{\perp}$ and $\mathfrak{R}(V)^{\perp}=\mathfrak{R}(V_s)^{\perp}\cap V(0)^{\perp}$. On the other hand, as noted above, $\mathfrak{N}(V)$ and $V(0)$ are hypermaximal neutral in $\mathfrak{D}(V_s)^{\perp}$ and $\mathfrak{R}(V_s)^{\perp}$, resp, if and only if $\mathfrak{N}(V)=\mathfrak{N}(V)^{\perp}\cap\mathfrak{D}(V_s)^{\perp}$ and $V(0)^{\perp}=V(0)^{\perp}\cap\mathfrak{R}(V_s)^{\perp}$, resp. Therefore Proposition 2.3.1 implies the result. (ii) is obvious. □

**2.5. Defect numbers.** Let $V$ be an isometric linear relation in the inner product space $\mathfrak{H}$. As in the operator case we call the subspaces $\mathfrak{D}(V)^{\perp}$ and $\mathfrak{R}(V)^{\perp}$ the *deficiency spaces* of $V$. The geometry of these spaces is reflected in the extensions of $V$ in the following way.

Theorem 2.5.1. *Let* $V$ *be an isometric linear relation in* $\mathfrak{H}$. *Then the following conditions are equivalent:*

(i) $\mathfrak{R}(V)^{\perp}$ *contains a non-zero neutral vector;*

(ii) $V$ *has an isometric extension which is not an operator;*

(iii) $V$ *has an isometric extension* $V'$ *in a larger inner product space* $\mathfrak{H}'$ *with* $V'(0)\cap\mathfrak{H}'\neq\{0\}$.

Theorem 2.5.2. *Let* $V$ *be an isometric linear relation in* $\mathfrak{H}$. *Then the following conditions are equivalent:*

(i) $\mathfrak{R}(V)^{\perp}$ *contains a positive/negative vector;*

(ii) *$V$ has an isometric extension $V'$ in a larger inner product space $\mathfrak{H}'$, which includes $\mathfrak{H}$ as an ortho-complemented subspace with $\mathfrak{H}^\perp$ negative/positive in $\mathfrak{H}'$, such that* $V'(0)\cap(\mathfrak{H}'\setminus\mathfrak{H})\neq\emptyset$.

These results can be proved by using the Cayley transformation and [6], Theorem 4.2. Note that one can get analogous theorems by considering $\mathfrak{D}(V)^\perp$ and $\mathfrak{R}(V)$ instead of $\mathfrak{R}(V)^\perp$ and $V(0)$, resp.

For the rest of this chapter we shall use the following *conventions*: $\mathfrak{H}$ is a Krein space, $V$ is a rectangular isometric linear relation in $\mathfrak{H}$, and $V_s$ is a rectangular operator part of $V$. Furthermore, we set $\mathfrak{D}:=\mathfrak{D}(V)$, $\mathfrak{R}:=\mathfrak{R}(V)$, $\mathfrak{D}_s:=\mathfrak{D}(V_s)$ and $\mathfrak{R}_s:=\mathfrak{R}(V_s)$.

As $\mathfrak{D}_s^\perp$ is ortho-complemented and hence a Krein space itself, it has a fundamental decomposition

(2.5.1) $$\mathfrak{D}_s^\perp = \mathfrak{D}_{s+}[\dotplus]\mathfrak{D}_{s-}$$

such that $\mathfrak{D}_{s+}$ and $\mathfrak{D}_{s-}$ are Hilbert spaces with respect to $[\cdot|\cdot]$ and $-[\cdot|\cdot]$, resp. The neutral subspace $\mathfrak{R}(V)$ in $\mathfrak{D}_s^\perp$ can be represented in the form

(2.5.2) $$\mathfrak{R}(V) = \{f+Kf\,|\,f\in\mathfrak{D}(K)\},$$

where $K$ is the angular operator of $\mathfrak{R}(V)$ with respect to $\mathfrak{D}_{s+}$; see [2]. Recall that $K$ is a J-*isometric* operator, i e,

$$-[Kf|Kf] = [f|f] \qquad (f\in\mathfrak{D}(K)),$$

and that $\mathfrak{D}(K)$ as well as $\mathfrak{R}(K)$ are ortho-complemented in $\mathfrak{D}_{s+}$ and $\mathfrak{D}_{s-}$, resp.

Set $\mathfrak{D}_+:=\mathfrak{D}(K)^\perp$ (in $\mathfrak{D}_{s+}$) and $\mathfrak{D}_-:=\mathfrak{R}(K)^\perp$ (in $\mathfrak{D}_{s-}$). Then these subspaces are ortho-complemented also in $\mathfrak{H}$, and we have the decompositions

(2.5.3) $$\begin{aligned}\mathfrak{D}_s^\perp &= \mathfrak{D}_+[\dotplus]\mathfrak{D}_-[\dotplus]\mathfrak{D}(K)[\dotplus]\mathfrak{R}(K),\\ \mathfrak{D}^\perp &= \mathfrak{D}_+[\dotplus]\mathfrak{D}_-[\dotplus]\mathfrak{R}(V).\end{aligned}$$

Indeed, the first formula is quite obvious, and the second follows from the first and from the representation

$$\mathfrak{R}(V) = \mathfrak{R}(V)^\perp\cap(\mathfrak{D}(K)[\dotplus]\mathfrak{R}(K)),$$

which in turn is a consequence of (2.5.2).

Likewise, a fundamental decomposition

(2.5.4) $$\mathfrak{R}_s^\perp = \mathfrak{R}_{s+}[\dotplus]\mathfrak{R}_{s-}$$

of $\mathfrak{R}_s^\perp$ and the representation

$$V(0) = \{f+Lf\,|\,f\in\mathfrak{D}(L)\}$$

of $V(0)$ with the angular operator $L$ with respect to $\mathfrak{R}_{s+}$ induce the decompositions

(2.5.5) $$\begin{aligned}\mathfrak{R}_s^\perp &= \mathfrak{R}_+[\dotplus]\mathfrak{R}_-[\dotplus]\mathfrak{D}(L)[\dotplus]\mathfrak{R}(L),\\ \mathfrak{R}^\perp &= \mathfrak{R}_+[\dotplus]\mathfrak{R}_-[\dotplus]V(0);\end{aligned}$$

here $\mathfrak{R}_+$ and $\mathfrak{R}_-$ are the orthogonal companions of $\mathfrak{D}(L)$ and $\mathfrak{R}(L)$, resp, in $\mathfrak{R}_{s+}$ and $\mathfrak{R}_{s-}$, resp.

Define $\alpha_\pm(V):=\dim \mathfrak{D}_\pm$, $\alpha_0(V):=\dim \mathfrak{N}(V)$, $\beta_\pm(V):=\dim \mathfrak{R}_\pm$ and $\beta_0(V):=\dim V(0)$. These numbers are independent of the decompositions; see [2]. The numbers $\alpha_\pm(V)$ and $\beta_\pm(V)$ are called the *defect numbers* of the rectangular isometric linear relation $V$. Several results below will justify this terminology. First, a useful lemma.

Lemma 2.5.3. *Let $V'$ be a rectangular isometric linear relation in the Krein space $\mathfrak{H}$. Then $V'$ is an extension of the rectangular isometric linear relation $V$ if and only if $V'^0$ extends $V^0$ and there exists a rectangular operator part $V_s'$ of $V'$ which extends $V_s$. In this case*

$$V_s' = V_s[\dotplus]W_s,$$

*where $W_s$ is a rectangular isometric operator in $\mathfrak{H}$.*

*Proof.* If $V'$ extends $V$, then $V'^0$ is a closed subspace of the Krein space $V_s^\perp$, and hence it has a closed complement $W'$ in $V_s^\perp$. Define $W_s:=V'\cap W'$ and $V_s':=V_s[\dotplus]W_s$. To prove the ortho-complementedness of $V_s'$, notice first that it is closed by Lemma 2.4.1, and it is obviously an operator part of $V'$. But as $\mathfrak{H}$ is a Krein space, all closed operator parts of $V'$ are ortho-complemented because they are isometrically isomorphic to an ortho-complemented complementary subspace of $V'^0$ in $V$; see [2]. The other assertions are easily established. □

The next result characterizes the various types of extensions of the rectangular isometric linear relation $V$ with the help of the defect numbers of $V$.

Theorem 2.5.4. (i) *$V$ has a rectangular isometric extension $V'$ with $V_s'\neq V_s$, $\mathfrak{N}(V')=\mathfrak{N}(V)$ and $V'(0)=V(0)$ if and only if* $\min\{\alpha_+(V), \beta_+(V)\}>0$ *or* $\min\{\alpha_-(V), \beta_-(V)\}>0$*;*

(ii) *$V$ has a rectangular isometric extension $V'$ with $\mathfrak{N}(V')\neq\mathfrak{N}(V)$, $V'(0)=V(0)$ and $V_s'=V_s$ if and only if* $\min\{\alpha_+(V), \alpha_-(V)\}>0$*;*

(iii) *$V$ has a rectangular isometric extension $V'$ with $V'(0)\neq V(0)$, $V_s'=V_s$ and $\mathfrak{N}(V')=\mathfrak{N}(V)$ if and only if* $\min\{\beta_+(V), \beta_-(V)\}>0$.

*Proof.* (i) If $V$ has an extension $V'$ with the mentioned properties, then we can suppose, by Lemma 2.5.3, that $V_s'=V_s[\dotplus]W_s$, where $W_s\neq\{0\}$ is a rectangular isometric operator. Hence $\mathfrak{D}(W_s)\neq\{0\}$ is ortho-complemented, and so there exists in $\mathfrak{D}(W_s)$ a positive or negative vector. As $W_s$ is isometric, $\mathfrak{R}(W_s)$ contains also a vector of the same kind. On the other hand, $\mathfrak{D}(W_s)\subset\mathfrak{D}(V_s)^\perp$ and $\mathfrak{R}(W_s)\subset\mathfrak{R}(V_s)^\perp$; consequently, we must have $\min\{\alpha_+(V), \beta_+(V)\}>0$ or $\min\{\alpha_-(V), \beta_-(V)\}>0$.

Conversely, let e.g. $\min\{\alpha_+(V), \beta_+(V)\}$ be greater than zero. Then we can find vectors $f\in\mathfrak{D}(V)^\perp$ and $g\in\mathfrak{R}(V)^\perp$ such that $[f|f]=[g|g]=1$. By setting $V':=V[\dotplus]\langle(f, g)\rangle$ we get a proper closed isometric extension with $\mathfrak{N}(V')=\mathfrak{N}(V)$

and $V'(0)=V(0)$. Furthermore, $V_s':=V_s[\dot{+}]\langle(f,g)\rangle$ is an operator part of $V'$. Lemma 2.4.1 guarantees that $V_s'$ is ortho-complemented, and thus $V'$ is rectangular by Proposition 2.4.2. The case $\min\{\alpha_-(V),\beta_-(V)\}>0$ can be handled similarly.

(ii) Suppose first that $V'$ has the given properties. With the agreed notation we have $\mathfrak{N}(V')=\mathfrak{N}(V)[+]\mathfrak{L}$, where $\mathfrak{L}:=\mathfrak{N}(V')\cap(\mathfrak{D}_+[+]\mathfrak{D}_-)\neq\{0\}$. Choose a non-zero vector $f$ from $\mathfrak{L}$; then $f=f_++f_-$ with $f_\pm\in\mathfrak{D}_\pm$. Furthermore, both $f_+$ and $f_-$ are different from zero; hence $\min\{\alpha_+(V),\alpha_-(V)\}>0$.

If $\min\{\alpha_+(V),\alpha_-(V)\}>0$, then there exist vectors $f_\pm$ in $\mathfrak{D}_\pm$ such that $[f_\pm|f_\pm]=\pm 1$. The formula

$$V':=V_s[\dot{+}](\mathfrak{N}(V)+\langle f_++f_-\rangle)\oplus V(0)$$

defines then a linear relation satisfying the required conditions.

(iii) can be proved in the same way as (ii). $\square$

We call a rectangular isometric linear relation in $\mathfrak{H}$ *maximal rectangular* if it has no rectangular isometric proper extensions in $\mathfrak{H}$. As a corollary of the previous theorem we get the following extension of [2], Theorem VI.4.3.

Theorem 2.5.5. *The rectangular isometric linear relation $V$ in the Krein space $\mathfrak{H}$ is maximal rectangular if and only if*

$$\min\{\alpha_+(V),\beta_+(V)\}=\min\{\alpha_-(V),\beta_-(V)\}$$
$$=\min\{\alpha_+(V),\alpha_-(V)\}=\min\{\beta_+(V),\beta_-(V)\}=0.$$

This result has a clear geometrical interpretation: $V$ is maximal rectangular if and only if the deficiency spaces $\mathfrak{D}(V)^\perp$ and $\mathfrak{R}(V)^\perp$ are semi-definite of opposite signs. The sign of a neutral subspace is here assumed to be different from the sign of every semi-definite subspace.

As in the operator case, we can characterize formally unitary and unitary linear relations by the defect numbers:

Theorem 2.5.6. *The rectangular isometric linear relation $V$ in the Krein space $\mathfrak{H}$ is formally unitary if and only if $\alpha_\pm(V)=\beta_\pm(V)=0$. In this case $V$ is unitary if and only if $\alpha_0(V)=\beta_0(V)=0$.*

*Proof.* If $V$ is formally unitary, then, by Theorem 2.4.4, the subspace $\mathfrak{N}(V)$ is hypermaximal neutral in the Krein space $\mathfrak{D}_s^\perp$. Consequently, for the angular operator $K$ of $\mathfrak{N}(V)$ we have $\mathfrak{D}(K)=\mathfrak{D}_{s+}$ and $\mathfrak{R}(K)=\mathfrak{D}_{s-}$; but this means that $\alpha_+(V)=\alpha_-(V)=0$. Similarly, $\beta_+(V)=\beta_-(V)=0$.

Conversely, from the assumptions we get $\mathfrak{D}^\perp=\mathfrak{N}(V)$ and $\mathfrak{R}^\perp=V(0)$; see (2.5.3) and (2.5.5). Proposition 2.3.1 (iv) implies now the desired result.

The second assertion is easily established on the first assertion and Proposition 2.3.3. $\square$

## 3. Extensions of rectangular isometric linear relations

**3.1. Existence.** We continue the study of rectangular isometric linear relations in a Krein space by showing the existence of various kinds of extensions. For the operator version of the following result, see [2], Theorem VI.4.4.

Theorem 3.1.1. *Every rectangular isometric linear relation in a Krein space admits maximal rectangular extensions.*

*Proof.* Let $V$ be a non-maximal rectangular isometric linear relation in a Krein space $\mathfrak{H}$, and let $V_s$ be its rectangular operator part.

1° Suppose first that $\alpha_+(V) \leqq \alpha_-(V)$ and $\beta_+(V) \geqq \beta_-(V)$. The subspaces $\mathfrak{N}(V)$ and $V(0)$ are neutral in the Krein spaces $\mathfrak{D}(V_s)^\perp$ and $\mathfrak{R}(V_s)^\perp$, resp., and thus they admit maximal neutral extensions $\mathfrak{N}_0$ and $\mathfrak{N}_\infty$, resp. By the assumption we then have

$$\dim \mathfrak{N}_0 = \dim \mathfrak{D}_{s+} = \alpha_0(V) + \alpha_+(V), \quad \dim \mathfrak{N}_\infty = \dim \mathfrak{R}_{s-} = \beta_0(V) + \beta_-(V);$$

see (2.5.1) and (2.5.4). The definition $V' := V_s[\dotplus]\mathfrak{N}_0 \oplus \mathfrak{N}_\infty$ gives a rectangular isometric extension of $V$. Furthermore, since $\alpha_+(V') = \beta_-(V') = 0$, Theorem 2.5.5 guarantees the maximality of $V'$.

2° The case $\alpha_+(V) \geqq \alpha_-(V)$ and $\beta_+(V) \leqq \beta_-(V)$ is analogous to 1°.

3° If $\alpha_+(V) \geqq \alpha_-(V)$ and $\beta_+(V) \geqq \beta_-(V)$, we can, as in 1°, construct a rectangular isometric extension $V' := V_s[\dotplus]\mathfrak{N}_0 \oplus \mathfrak{N}_\infty$ for which $\alpha_-(V') = \beta_-(V') = 0$. Hence, analogous to (2.5.3) and (2.5.5), we have

$$\mathfrak{D}(V')^\perp = \mathfrak{D}'_+ [\dotplus] \mathfrak{N}(V'), \quad \mathfrak{R}(V')^\perp = \mathfrak{R}'_+ [\dotplus] V'(0),$$

where $\mathfrak{D}'_+$ and $\mathfrak{R}'_+$ are Hilbert spaces. Consequently, there exists a Hilbert space isometry $V_+$ from $\mathfrak{D}'_+$ into $\mathfrak{R}'_+$ such that $\mathfrak{D}(V_+) = \mathfrak{D}'_+$ or $\mathfrak{R}(V_+) = \mathfrak{R}'_+$. This $V_+$ is obviously rectangular, and, as it is easy to see, the linear relation

$$V'' := (V_s[\dotplus]V_+)[\dotplus]\mathfrak{N}(V') \oplus V'(0)$$

is a rectangular isometric extension of $V$. The construction of $V''$ implies further that $\alpha_-(V'') = \beta_-(V'') = 0$ and $\alpha_+(V'') = 0$ or $\beta_+(V'') = 0$. Thus $V''$ is maximal rectangular by Theorem 2.5.5.

4° If $\alpha_+(V) \leqq \alpha_-(V)$ and $\beta_+(V) \leqq \beta_-(V)$, we can use the same method as in 3°. □

To examine the existence of formally unitary extensions we need the following lemma, the proof of which is straightforward.

Lemma 3.1.2. *Let there be given four cardinal numbers* $\alpha_\pm, \beta_\pm$. *Then* $\alpha_+ + \beta_- = \beta_+ + \alpha_-$ *if and only if there exist four cardinal numbers* $\gamma_\pm$, $\alpha$, $\beta$ *such that*

$$\alpha_+ = \gamma_+ + \alpha, \quad \beta_+ = \gamma_+ + \beta,$$
$$\alpha_- = \gamma_- + \alpha, \quad \beta_- = \gamma_- + \beta.$$

Theorem 3.1.3. *Let* $V$ *be a rectangular isometric linear relation in a Krein space* $\mathfrak{H}$. *Then* $V$ *has a rectangular formally unitary extension in* $\mathfrak{H}$ *if and only if*

$$\alpha_+(V) + \beta_-(V) = \beta_+(V) + \alpha_-(V). \tag{3.1.1}$$

*Proof.* 1° Suppose that $U$ is a rectangular formally unitary extension of $V$, and let $U_s$ be its rectangular operator part. By Lemma 2.5.3 we can assume that $U_s \supset V_s$, where $V_s$ is a rectangular operator part of $V$. Furthermore, we can choose the components $\mathfrak{D}_{s\pm}$ of a fundamental decomposition of $\mathfrak{D}(V_s)^\perp$ in such a way that the following scheme is true:

$$\begin{array}{ccc} \mathfrak{D}_{s+} & [\dot{+}] \, \mathfrak{D}_{s-} & = \mathfrak{D}(V_s)^\perp \\ \cup & \cup & \\ \mathfrak{D}(K') & [\dot{+}] \, \mathfrak{R}(K') & = \mathfrak{D}(U_s)^\perp; \\ \cup & \cup & \\ \mathfrak{D}(K) & \mathfrak{R}(K) & \end{array} \tag{3.1.2}$$

here $K$ and $K'$ $(\supset K)$ are the angular operators of $\mathfrak{R}(V)$ and $\mathfrak{R}(U)$, resp, with respect to $\mathfrak{D}_{s+}$. In addition, all the subspaces in this scheme are Hilbert spaces with respect to $[\cdot|\cdot]$ or $-[\cdot|\cdot]$.

From (3.1.2) we get

$$\mathfrak{D}_+ := \mathfrak{D}(K)^\perp = \mathfrak{D}(K')^\perp [\dot{+}] \, \mathfrak{D}(K)^\perp \cap \mathfrak{D}(K'),$$

where the orthogonal companions are formed in the space $\mathfrak{D}_{s+}$. Setting $\gamma_+ := \dim \mathfrak{D}(K')^\perp$ and $\alpha := \dim \mathfrak{D}(K)^\perp \cap \mathfrak{D}(K')$ we then have $\alpha_+(V) = \gamma_+ + \alpha$.

Likewise, from the analogous decomposition of $\mathfrak{D}_- := \mathfrak{R}(K)^\perp$ we get $\alpha_-(V) = \gamma_- + \alpha'$, where $\gamma_- := \dim \mathfrak{R}(K')^\perp$ and $\alpha' := \dim \mathfrak{R}(K)^\perp \cap \mathfrak{R}(K') = \alpha$. The last equality is implied by the fact that the spaces $\mathfrak{D}(K)^\perp \cap \mathfrak{D}(K')$ and $\mathfrak{R}(K)^\perp \cap \mathfrak{R}(K')$ are isometrically isomorphic.

Let $L$ and $L'$ be the angular operators of $V(0)$ and $U(0)$, resp., with respect to $\mathfrak{R}_{s+}$; here $\mathfrak{R}_{s+}$ is a component of a fundamental decomposition of $\mathfrak{R}(V_s)^\perp$ so chosen that the scheme analogous to (3.1.2) holds true. Then, as above, one can prove the equalities $\beta_\pm(V) = \delta_\pm + \beta$, where $\delta_+ := \dim \mathfrak{D}(L')^\perp$, $\delta_- := \dim \mathfrak{R}(L')^\perp$ and $\beta := \dim \mathfrak{D}(L)^\perp \cap \mathfrak{D}(L') = \dim \mathfrak{R}(L^\perp) \cap \mathfrak{R}(L')$.

Hence, in the light of Lemma 3.1.2, it is enough to prove the equalities $\delta_\pm = \gamma_\pm$. For this, decompose the space $\mathfrak{D}(V_s)^\perp$ in two ways:

$$\mathfrak{D}(V_s)^\perp = \mathfrak{D}(U_s)^\perp [\dot{+}] \, \mathfrak{D}(U_s) \cap \mathfrak{D}(V_s)^\perp$$

and

$$\begin{aligned} \mathfrak{D}(V_s)^\perp &= \{\mathfrak{D}(K') [\dot{+}] \, \mathfrak{D}(K')^\perp\} [\dot{+}] \{\mathfrak{R}(K') [\dot{+}] \, \mathfrak{R}(K')^\perp\} \\ &= \mathfrak{D}(U_s)^\perp [\dot{+}] \{\mathfrak{D}(K')^\perp [\dot{+}] \, \mathfrak{R}(K')^\perp\}, \end{aligned}$$

where the orthogonal companions are formed in suitable subspaces; see (3.1.2). Hence we get

$$\mathfrak{D}(U_s)\cap\mathfrak{D}(V_s)^{\perp} = \mathfrak{D}(K')^{\perp}[\dotplus]\mathfrak{R}(K')^{\perp}$$

and analogously

$$\mathfrak{R}(U_s)\cap\mathfrak{R}(V_s)^{\perp} = \mathfrak{D}(L')^{\perp}[\dotplus]\mathfrak{R}(L')^{\perp}.$$

Furthermore, one can easily check that these decompositions are fundamental decompositions. But the Krein spaces $\mathfrak{D}(U_s)\cap\mathfrak{D}(V_s)^{\perp}$ and $\mathfrak{R}(U_s)\cap\mathfrak{R}(V_s)^{\perp}$ are isometrically isomorphic, and thus $\gamma_+=\dim\mathfrak{D}(K')^{\perp}=\dim\mathfrak{D}(L')^{\perp}=\delta_+$; see [2], Theorem V.1.4. The equality $\gamma_-=\delta_-$ can be proved similarly. Hence (3.1.1) is established.

2° Conversely, suppose that (3.1.1) is valid. Then, by using Lemma 3.1.2, we can find the decompositions

$$\mathfrak{D}_{\pm} = \mathfrak{D}_{\gamma_{\pm}}[\dotplus]\mathfrak{D}^{\pm}_{\alpha}, \quad \mathfrak{R}_{\pm} = \mathfrak{R}_{\gamma_{\pm}}[\dotplus]\mathfrak{R}^{\pm}_{\beta};$$

here the Greek index denotes the dimension of the corresponding space. Thus there exist isometric (or J-isometric) operators $V_{\pm}$, $K'$ and $L'$ mapping $\mathfrak{D}_{\gamma_{\pm}}$ onto $\mathfrak{R}_{\gamma_{\pm}}$, $\mathfrak{D}^{+}_{\alpha}$ onto $\mathfrak{D}^{-}_{\alpha}$ and $\mathfrak{R}^{+}_{\beta}$ onto $\mathfrak{R}^{-}_{\beta}$, resp. Define

$$\mathfrak{N}_0 := \{f+K'f\mid f\in\mathfrak{D}^{+}_{\alpha}\}, \quad \mathfrak{N}_{\infty} := \{g+L'g\mid g\in\mathfrak{R}^{+}_{\beta}\}$$

and

$$U := V_s[\dotplus]V_+[\dotplus]V_-[\dotplus](\mathfrak{N}(V)[\dotplus]\mathfrak{N}_0)\oplus(V(0)[\dotplus]\mathfrak{N}_{\infty}),$$

where $V_s$ is a rectangular operator part of $V$. Then a straightforward calculation shows that this $U$ is a rectangular formally unitary extension of $V$. □

Corollary 3.1.4. *Let $V$ be a rectangular isometric linear relation in a Pontrjagin space $\mathfrak{H}$. Then $V$ has a rectangular formally unitary extension in $\mathfrak{H}$ if and only if*

(3.1.3) $$\alpha_+(V)+\alpha_0(V) = \beta_+(V)+\beta_0(V).$$

*Proof.* Let $\mathfrak{H}$ have $\varkappa$ negative squares, and let $V_s$ be a rectangular operator part of $V$. Then both $\mathfrak{D}(V_s)$ and $\mathfrak{R}(V_s)$ are Pontrjagin spaces with $\varkappa'$ $(\leqq\varkappa)$ negative squares. Consequently, $\mathfrak{D}(V_s)^{\perp}$ and $\mathfrak{R}(V_s)^{\perp}$ are both Pontrjagin spaces with $\varkappa-\varkappa'$ negative squares. This implies the equalities $\alpha_-(V)+\alpha_0(V)=\beta_-(V)+\beta_0(V)=\varkappa-\varkappa'$. In this case the formulae (3.1.1) and (3.1.3) are equivalent, which proves the result. □

As another corollary we have the following operator version; cf [2], Theorem VI.4.4.

Corollary 3.1.5. *Let $V$ be a rectangular isometric operator in a Krein space $\mathfrak{H}$. Then $V$ has a unitary extension in $\mathfrak{H}$ if and only if*

$$\alpha_+(V) = \beta_+(V) \quad \& \quad \alpha_-(V) = \beta_-(V).$$

*If in addition* $\mathfrak{H}$ *is a Pontrjagin space, then the equality* $\alpha_+(V)=\beta_+(V)$ *is a necessary and sufficient condition for* $V$ *to have unitary extensions in* $\mathfrak{H}$.

Although we do not characterize generalized extensions, i.e., extensions beyond the original space, we add the following result about the existence of such extensions.

Theorem 3.1.6. *Every rectangular isometric linear relation in a Krein space has rectangular formally unitary extensions in a possibly larger Krein space.*

*Proof.* We can suppose that a given rectangular isometric linear relation $V$ in a Krein space $\mathfrak{H}$ is not formally unitary and, by Theorem 3.1.1, maximal rectangular. Then, by Theorem 2.5.6, we can further assume that e.g. $\alpha_+(V)\neq 0$, and hence, by Theorem 2.5.5, $\alpha_-(V)=\beta_+(V)=0$.

Choose two Hilbert spaces $\mathfrak{K}_\pm$ with inner products $(\cdot|\cdot)_\pm$ and with infinite dimensions $\delta_+\geqq\alpha_+$ and $\delta_-\geqq\beta_-$, resp. Then $\mathfrak{K}:=\mathfrak{K}_+\oplus\mathfrak{K}_-$ is a Krein space with the inner product $[\cdot|\cdot]'$:

$$[f_+ + f_- | g_+ + g_-]' := (f_+|g_+)_+ - (f_-|g_-)_- \quad (f_\pm, g_\pm \in \mathfrak{K}_\pm).$$

Now we can regard $V$ as a rectangular isometric linear relation in the Krein space $\mathfrak{H}':=\mathfrak{H}\oplus\mathfrak{K}$. Denote by $V_s$ a rectangular operator part of $V$. The fundamental decompositions corresponding to (2.5.3) and (2.5.5) have the following form in $\mathfrak{H}'$:

$$\mathfrak{D}(V_s)^\perp = \{(\mathfrak{D}_+[\dot{+}]\mathfrak{K}_+)[\dot{+}]\mathfrak{D}(K)\}[\dot{+}]\{\mathfrak{K}_-[\dot{+}]\mathfrak{R}(K)\},$$

$$\mathfrak{R}(V_s)^\perp = \{\mathfrak{K}_+[\dot{+}]\mathfrak{D}(L)\}[\dot{+}]\{(\mathfrak{R}_-[\dot{+}]\mathfrak{K}_-)[\dot{+}]\mathfrak{R}(L)\}.$$

Consequently, for the defect numbers $\alpha'_\pm$, $\beta'_\pm$ of $V$ with respect to $\mathfrak{H}'$ we get

$$\alpha'_+ + \beta'_- = (\alpha_+ + \delta_+) + (\beta_- + \delta_-) = \delta_+ + \delta_- = \beta'_+ + \alpha'_-.$$

Theorem 3.1.3 implies now the desired result. □

**3.2. Characterizations.** In this section we describe the rectangular isometric extensions of a given rectangular isometric linear relation in a Krein space. For the corresponding operator versions in a Hilbert space, see [4], § 5.

Theorem 3.2.1. *Let* $V$ *be a rectangular isometric linear relation in a Krein space* $\mathfrak{H}$.

*If* $V'$ *is a rectangular isometric extension of* $V$ *in* $\mathfrak{H}$, *then*

$$V' = V[\dot{+}]W, \tag{3.2.1}$$

*where* $W$ *is a rectangular isometric linear relation in* $\mathfrak{H}$ *such that* $\mathfrak{D}(W)\subset\mathfrak{D}(V)^\perp$, $\mathfrak{R}(W)\subset\mathfrak{R}(V)^\perp$, $V\cap W=\{0\}$ *and* $V^0\dot{+}W^0$ *is closed.*

*Conversely, let* $W$ *be as above, then* $V'$ *defined by* (3.2.1) *is a rectangular isometric extension of* $V$ *in* $\mathfrak{H}$.

*Proof.* 1° Suppose first that $V'$ is a rectangular isometric extension of $V$. Using Lemma 2.5.3 and its notation we get

$$V' = V_s[\dot{+}]W_s[\dot{+}]V'^0.$$

As $V^0$ and $V'^0$ are closed and $V^0 \subset V'^0$, there exists a closed subspace $W_0$ such that $V'^0 = V^0 \dot{+} W_0$. Furthermore, this $W_0$ is neutral and orthogonal to $V^0$, and it has a representation $W_0 = \mathfrak{N}_0 \oplus \mathfrak{N}_\infty$, where $\mathfrak{N}_0$ and $\mathfrak{N}_\infty$ are closed complementary subspaces of $\mathfrak{N}(V)$ and $V(0)$, resp., with respect to $\mathfrak{N}(V')$ and $V'(0)$, resp. This is easily established on the equalities $V^0 = \mathfrak{N}(V) \oplus V(0)$ and $V'^0 = \mathfrak{N}(V') \oplus V'(0)$. Consequently, a straightforward calculation shows that the linear relation $W := W_s[\dot{+}]W_0$ satisfies the required conditions.

2° Conversely, let the linear relation $W$ be given, and define $V'$ by (3.2.1). Then $V'$ is isometric and it has a representation

$$V' = (V_s[\dot{+}]W_s)[\dot{+}](V^0[\dot{+}]W^0),$$

which implies, by Lemma 2.4.1 and Proposition 2.4.2, that $V'$ is rectangular. □

By combining the previous result with Theorem 2.5.5 and Proposition 2.3.1 we get

Corollary 3.2.2. *Let $V$, $V'$ and $W$ be as in Theorem* 3.2.1. *Then*

(i) *$V'$ is maximal rectangular if and only if the spaces $\mathfrak{D}(V)^\perp \cap \mathfrak{D}(W)^\perp$ and $\mathfrak{R}(V)^\perp \cap \mathfrak{R}(W)^\perp$ are semi-definite and of opposite signs;*

(ii) *$V'$ is formally unitary if and only if*

$$V^0 \dot{+} W^0 = \{\mathfrak{D}(V) \oplus \mathfrak{R}(V) \dot{+} \mathfrak{D}(W) \oplus \mathfrak{R}(W)\}^\perp$$

*or equivalently*

$$\mathfrak{N}(V) + \mathfrak{N}(W) = \mathfrak{D}(V)^\perp \cap \mathfrak{D}(W)^\perp,\ V(0) + W(0) = \mathfrak{R}(V)^\perp \cap \mathfrak{R}(W)^\perp.$$

## 4. Symmetric linear relations

**4.1. Rectangular symmetric linear relations.** We first recall some basic facts about symmetric linear relations; see [6]. Let $\mathfrak{H}$ be an inner product space. A linear relation $S$ in $\mathfrak{H}$ is called *symmetric* if $S \subset S^+$, i.e.,

$$[g|h] = [f|k] \quad \text{for all} \quad (f, g), (h, k) \in S.$$

Let $z$ be a non-real complex number, and let $S$ be a symmetric linear relation in $\mathfrak{H}$. Then the *Cayley transform*

$$C_z(S) := \{(g - zf,\ g - \bar{z}f) | (f, g) \in S\}$$

of $S$ is an isometric linear relation. Conversely, if $V$ is an isometric linear relation, then the inverse transform

$$F_z(V) := \{(g - f,\ zg - \bar{z}f) | (f, g) \in V\}$$

of $V$ is a symmetric linear relation. Furthermore, a symmetric linear relation $S$ is *self-adjoint*, i.e., $S=S^+$, if and only if its Cayley transform is formally unitary. As the Cayley transformation $C_z$ preserves the orthogonality only in case $z=\pm\mathrm{i}$, we suppose in the sequel that $z=\mathrm{i}$.

For the following auxiliary result, recall that the *deficiency spaces* $M_{\pm\mathrm{i}}(S):=\{(h,k)\in S^+ \mid k=\pm\mathrm{i}h\}$ of a symmetric linear relation $S$ are operators with the domains $\mathfrak{D}(M_{\pm\mathrm{i}}(S))=\mathfrak{R}(S^+\mp\mathrm{i}I)$.

Lemma 4.1.1. *Let $S$ be a symmetric linear relation in $\mathfrak{H}$ and set $V:=C_\mathrm{i}(S)$. Then*

(i) $V^0=C_\mathrm{i}(S^0)$;

(ii) $\mathfrak{R}(V)=\mathfrak{D}(M_{-\mathrm{i}}(S)\cap S)$, $V(0)=\mathfrak{D}(M_\mathrm{i}(S)\cap S)$;

(iii) $F_\mathrm{i}(\mathfrak{R}(V)\oplus V(0))=M_\mathrm{i}(S)\cap S[\dotplus]M_{-\mathrm{i}}(S)\cap S$;

(iv) $\mathfrak{D}(M_{\pm\mathrm{i}}(S))^0=\mathfrak{D}(M_{\pm\mathrm{i}}(S)^0)$.

The proof is a direct verification.

We call a symmetric linear relation $S$ in $\mathfrak{H}$ *rectangular* if its Cayley transform $C_\mathrm{i}(S)$ is rectangular. A more direct definition is included in the following result.

Theorem 4.1.2. *A closed symmetric linear relation $S$ in $\mathfrak{H}$ is rectangular if and only if*

$$N(S+\mathrm{i}I)=\mathfrak{R}(S-\mathrm{i}I)^0, \tag{4.1.1}$$

*and there exists a linear relation $S_s\subset S$ such that the subspaces $\mathfrak{R}(S_s\pm\mathrm{i}I)$ are ortho-complemented and*

$$S=S_s[\dotplus]S^0. \tag{4.1.2}$$

For the proof, use the Cayley transformation together with Lemma 4.1.1, Theorem 2.2.3 and [6], Lemma 2.12.

Remarks. 1° Theorem 2.2.3 and Lemma 4.1.1 imply that (4.1.1) is equivalent e.g. to the equation

$$S^0=M_\mathrm{i}(S)\cap S[\dotplus]M_{-\mathrm{i}}(S)\cap S.$$

2° In spite of the notation, $S_s$ is not necessarily an operator. In fact, $S_s$ is an operator if and only if the Cayley transform of $S$ has a rectangular operator part for which the number one is not an eigenvalue.

3° Let $S$ be a closed symmetric linear relation in a Pontrjagin space with the following properties: $\mathfrak{D}(S)$ includes the negative component of a fundamental decomposition of that space, and the constant $c_s$ introduced in [6] is smaller than one. Then one can prove that this $S$ is rectangular. Thus the study of rectangular symmetric linear relations extends at least partially the considerations of [6].

Let us list some useful properties of rectangular symmetric linear relations.

Theorem 4.1.3. *Let $S$ be a rectangular symmetric linear relation in $\mathfrak{H}$, and let* (4.1.2) *be its decomposition. Then*

(i) $\mathfrak{R}(S \pm iI)$ *are closed and*

$$\mathfrak{R}(S \pm iI) = \mathfrak{R}(S_s \pm iI)[\dot{+}]\mathfrak{R}(S^0 \pm iI);$$

(ii) $\mathfrak{D}(M_{\pm i}(S))^0 = \mathfrak{D}(M_{\pm i}(S) \cap S) = \mathfrak{N}(S \mp iI) = \mathfrak{R}(S \pm iI)^0 = \mathfrak{R}(S^0 \pm iI);$

(iii) $M_{\pm i}(S) \cap S = M_{\pm i}(S) \cap S^0;$

(iv) $\mathfrak{N}(S^+ \pm iI) = \mathfrak{N}(S_s^+ \pm iI) \cap \mathfrak{R}(S^+ \mp iI).$

*Proof.* The formula in (i) follows from Theorem 4.1.2. The closedness of $\mathfrak{R}(S \pm iI)$ follows if we use the Cayley transformation, Theorem 2.2.2, Lemma 2.4.1 and the formula already proved.

To prove (ii), note that, by (i), $\mathfrak{R}(S \pm iI)^0$ are closed and

$$\mathfrak{R}(S \pm iI)^0 = \mathfrak{N}(S^+ \mp iI) = \mathfrak{D}(\mathfrak{M}_{\pm i}(S))^0.$$

Furthermore, by Theorem 4.1.2, we have

$$\mathfrak{R}(S \pm iI)^0 = \mathfrak{N}(S \mp iI) = \mathfrak{D}(\mathfrak{M}_{\pm i}(S) \cap S).$$

Using the Cayley transform $V$ of $S$ and the equation $V^0 = \mathfrak{N}(V) \oplus V(0)$ we get the missing link $\mathfrak{N}(S \mp iI) = \mathfrak{R}(S^0 \pm iI)$.

The verification of (iii) is a straightforward calculation, in which one can use (ii). (iv) follows from (i) and (ii). □

From Theorem 4.1.3 and from similar considerations as in the operator case we get the following extension of the von Neumann formula; cf [4], Theorem 6.1 and [6], Theorem 4.10.

Theorem 4.1.4. *Let $S$ be a rectangular symmetric linear relation in $\mathfrak{H}$, and let* (4.1.2) *be its decomposition. Then*

$$S^+ = S_s[\dot{+}]M_i(S)[\dot{+}]M_{-i}(S).$$

Corollary 4.1.5. *Let $S$ be a rectangular symmetric linear relation in $\mathfrak{H}$. Then the following assertions are equivalent:*

(i) *$S$ is self-adjoint;*

(ii) $S^0 = M_i(S)[\dot{+}]M_{-i}(S);$

(iii) $\mathfrak{N}(S^+ \pm iI) = \mathfrak{N}(S \pm iI).$

**4.2. Extensions.** Let $S$ be a rectangular symmetric linear relation in a Krein space $\mathfrak{H}$. Using the Cayley transformation and the results of Section 2.5 we can find the fundamental decompositions

$$\mathfrak{R}(S - iI)^{\perp} = \mathfrak{D}_+[\dot{+}]\mathfrak{D}_-[\dot{+}]\mathfrak{N}(S + iI),$$

$$\mathfrak{R}(S + iI)^{\perp} = \mathfrak{R}_+[\dot{+}]\mathfrak{R}_-[\dot{+}]\mathfrak{N}(S - iI)$$

such that the subspaces $\mathfrak{D}_\pm$ and $\mathfrak{R}_\pm$ are ortho-complemented and the dimensions of these spaces are independent of the particular decomposition of this type. Define $\alpha_\pm(S):=\dim\mathfrak{D}_\pm$, $\alpha_0(S):=\dim\mathfrak{N}(S+\mathrm{i}I)$, $\beta_\pm(S):=\dim\mathfrak{R}_\pm$ and $\beta_0(S):=\dim\mathfrak{N}(S-\mathrm{i}I)$. Then $\alpha_\lambda(S)=\alpha_\lambda(C_\mathrm{i}(S))$ and $\beta_\lambda(S)=\beta_\lambda(C_\mathrm{i}(S))$ for $\lambda\in\{+,-,0\}$. Thus we call $\alpha_\pm(S)$ and $\beta_\pm(S)$ the *defect numbers* of $S$.

Using the Cayley transformation we can deduce the following result from the corresponding results concerning rectangular isometric linear relations; see Sections 2.5 and 3.1.

Theorem 4.2.1. *Let $S$ be a rectangular symmetric linear relation in a Krein space $\mathfrak{H}$. Then*

(i) *$S$ is maximal rectangular if and only if*

$$\min\{\alpha_+(S),\beta_+(S)\}=\min\{\alpha_-(S),\beta_-(S)\}$$
$$=\min\{\alpha_+(S),\alpha_-(S)\}=\min\{\beta_+(S),\beta_-(S)\}=0;$$

(ii) *$S$ is self-adjoint if and only if $\alpha_\pm(S)=\beta_\pm(S)=0$;*

(iii) *$S$ has always maximal rectangular extensions in $\mathfrak{H}$;*

(iv) *$S$ has rectangular self-adjoint extensions in $\mathfrak{H}$ if and only if*

$$\alpha_+(S)+\beta_-(S)=\beta_+(S)+\alpha_-(S);$$

(v) *$S$ has always rectangular self-adjoint extensions in a possibly larger Krein space.*

Characterizations of rectangular extensions of a rectangular symmetric linear relation follow also with the help of the Cayley transformation; for the Hilbert space case, see [3], II.2 and [4], § 6.

Theorem 4.2.2. *Let $S$ be a rectangular symmetric linear relation in a Krein space $\mathfrak{H}$.*

*If $S'$ is a rectangular symmetric extension of $S$ in $\mathfrak{H}$, then*

$$S'=S[\dotplus](I-V')\mathfrak{D}(V'),\tag{4.2.1}$$

*where $V'$ is a rectangular isometric linear relation such that $\mathfrak{D}(V')\subset M_\mathrm{i}(S)$, $\mathfrak{R}(V')\subset M_{-\mathrm{i}}(S)$ and $S^0\dotplus((I-V')\mathfrak{D}(V'))^0$ is closed.*

*Conversely, if $V'$ is as above and if the sum $S\dotplus(I-V')\mathfrak{D}(V')$ is direct, then the formula* (4.2.1) *defines a rectangular symmetric extension $S'$ of $S$ in $\mathfrak{H}$.*

*Proof.* Let $S'$ be a rectangular symmetric extension of $S$. Then $V_1:=C_\mathrm{i}(S')$ is a rectangular isometric extension of the rectangular isometric linear relation $V:=C_\mathrm{i}(S)$. Define

$$V':=\{((k,\mathrm{i}k),(h,-\mathrm{i}h))|(h,k)\in W\},$$

where $W$ is the rectangular isometric linear relation attached to $V_1$ by Theorem 3.2.1. Then a boring but straightforward calculation shows that this $V'$ has the desired properties.

The converse follows similarly. We remark only that in this case one can define the $W$ to be used in applying Theorem 3.2.1 by the equation

$$W := \{(g, f) \in \mathfrak{H}^2 | ((f, \mathrm{i}f), (g, -\mathrm{i}g)) \in V'\}. \quad \square$$

Corollary 4.2.3. *Let $S$, $S'$ and $V'$ be as in Theorem 4.2.2. Then $S'$ is self-adjoint if and only if*

$$S^0[\dotplus]((I-V')\mathfrak{D}(V'))^0 = M_{\mathrm{i}}(S) \cap \mathfrak{D}(V')^{\perp}[\dotplus]M_{-\mathrm{i}}(S) \cap \mathfrak{R}(V')^{\perp}. \tag{4.2.2}$$

This result can be deduced from Corollary 3.2.2 in a similar way as Theorem 4.2.2 was deduced from Theorem 3.2.1. We only write down two formulae, which can be used in the course of the proof:

$$S^0[\dotplus]((I-V')\mathfrak{D}(V'))^0 = F_{\mathrm{i}}(V^0[\dotplus]W^0),$$

$$F_{\mathrm{i}}((\mathfrak{D}(V) \oplus \mathfrak{R}(V) \dotplus \mathfrak{D}(W) \oplus \mathfrak{R}(W))^{\perp}) = M_{\mathrm{i}}(S) \cap \mathfrak{D}(V')^{\perp} \dotplus M_{-\mathrm{i}}(S) \cap \mathfrak{R}(V')^{\perp}.$$

Note that although the relation (4.2.2) looks quite cumbersome, it gives us the known criteria for self-adjointness in the Hilbert space case. In that case the left side of (4.2.2) is automatically zero, and then Corollary 4.2.3 says that $S'$ is self-adjoint if and only if the corresponding isometry $V'$ maps $M_{\mathrm{i}}(S)$ onto $M_{-\mathrm{i}}(S)$; cf [4], Corollary 6.4.

## References


[1] Arens, R.: Operational calculus of linear relations. - Pacific J. Math. 11, 1961, 9—23.

[2] Bognár, J.: Indefinite inner product spaces. - Ergebnisse der Mathematik und ihrer Grenzgebiete 78, Springer-Verlag, Berlin—Heidelberg—New York, 1974.

[3] Coddington, E. A.: Extension theory of formally normal and symmetric subspaces. - Mem. Amer. Math. Soc. 134, 1973, 1—80.

[4] Dijksma, A., and H. S. V. de Snoo: Self-adjoint extensions of symmetric subspaces. - Pacific J. Math. 54, 1974, 71—100.

[5] Kreĭn, M. G., and Ju. L. Šmul'jan: $\mathfrak{J}$-polar representation of plus-operators. - Amer. Math. Soc. Transl. (2) 85, 1969, 115—143.

[6] Sorjonen, P.: On linear relations in an indefinite inner product space. - Ann. Acad. Sci. Fenn. Ser. A I 4, 1978/1979, 169—192.



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