A NOTE ON LIPSCHITZ COMPACTIFICATIONS

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1. In [1] we introduced Lipschitz compactifications or LIP compactifications as compactifications in the category LIP of metric spaces and locally Lipschitz (LIP) maps. That is, if $f: X \to Y$ is a dense LIP embedding of a metric space $X$ into a compact metric space $Y$, then $Y$ or, more properly, the pair $(Y, f)$ is called a LIP compactification of $X$. Two LIP compactifications of $X$ are called LIP equivalent if they are equivalent through a lipeomorphism. We proved in [1, 1.14] that a metric space has a LIP compactification if and only if it is separable and locally totally bounded.

In this note we consider the problem whether a metrizable compactification $Z$ of a separable locally totally bounded metric space $X$ is equivalent to a LIP compactification of $X$, or equivalently, whether $Z$ can be metrized in such a way that $Z$ becomes a LIP compactification of $X$. Let $K_Z(X)$ denote the set of the LIP equivalence classes of the LIP compactifications of $X$ that are equivalent to $Z$. In Theorem 1 we give characterizations for $K_Z(X) \neq \emptyset$. Our main result, Theorem 2, is that $K_Z(X) \neq \emptyset$ for every $Z$ if and only if $X$ is locally compact. In Theorem 3 we consider the cardinality of $K_Z(X)$. We now give an example where $K_Z(X) = \emptyset$. Let $X$ be the subspace $[0, 1] \setminus \{1/n | n \geq 1\}$ of $\mathbb{R}$ and let $Y \subseteq \mathbb{R}$ be the union of $\{0\}$ and the intervals $(1/(2n+1), 1/2n)$, $n \geq 1$. Then $\bar{X}$ and $\bar{Z} = \bar{Y}$ are compact and there is a homeomorphism $f$ of $X$ onto $Y$ with $f(0) = 0$. However, no neighborhood of 0 in $\bar{X}$ is homeomorphic to any neighborhood of 0 in $\bar{Z}$. Hence the condition (2) of Theorem 1 is not satisfied and thus $K_Z(X) = \emptyset$.

For the undefined LIP terms we refer to [1].

2. A bijection $f$ between uniform spaces is called a locally uniform homeomorphism if both $f$ and $f^{-1}$ are locally uniformly continuous, i.e. uniformly continuous on some neighborhood of every point. We need the following modification of Lavrentiev's theorem [2, 24.9].

Lemma. Let $S$ and $T$ be complete Hausdorff uniform spaces, let $A \subseteq S$ and $B \subseteq T$ be dense subsets, and let $f: A \to B$ be a locally uniform homeomorphism. Then there are open sets $U \supseteq A$ and $V \supseteq B$ and a locally uniform homeomorphism $F: U \to V$ extending $f$.

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Proof. By [1, 2.9.5] \( f \) and \( f^{-1} \) have locally uniformly continuous extensions to open neighborhoods of \( A \) and \( B \), respectively. The proof can now be completed as in [2, 24.9].  

Theorem 1. Let \( X \) be a separable locally totally bounded metric space, let \( f: X \to Z \) be a metrizable compactification of \( X \), and let \( \tilde{X} \) be the completion of \( X \). Then the following conditions are equivalent:

1. \( K_Z(X) \neq \emptyset \).
2. There are neighborhoods \( U \) of \( X \) in \( \tilde{X} \) and \( V \) of \( fX \) in \( Z \) and a homeomorphism \( g: U \to V \) extending \( f \).
3. There is a neighborhood \( U \) of \( X \) in \( \tilde{X} \) and an embedding \( g: U \to Z \) extending \( f \).
4. There is a neighborhood \( V \) of \( fX \) in \( Z \) and an embedding \( h: V \to \tilde{X} \) extending \( f^{-1}: fX \to X \).
5. Consider \( Z \) with its unique compatible uniformity, given by any compatible metric. Then \( f \) defines a locally uniform homeomorphism of \( X \) onto \( fX \).

Proof. (1) \( \Rightarrow \) (5): Trivial.

(5) \( \Rightarrow \) (3) and (5) \( \Rightarrow \) (4): This follows from the Lemma.

(3) \( \Rightarrow \) (2): By [1, 1.13] \( X \) has a locally compact neighborhood in \( \tilde{X} \). Thus we may assume that \( U \) is locally compact. Then \( gU \) is locally compact and hence open in \( Z \).

(4) \( \Rightarrow \) (2): This is proved as (3) \( \Rightarrow \) (2).

(2) \( \Rightarrow \) (1): Let \( e \) be the metric on \( V \) for which \( g: U \to (V, e) \) is an isometry. We may assume that \( V \) is open. Then by [1, 6.4] there is a compatible metric \( r \) on \( Z \) which is LIP equivalent to \( e \) on \( V \). Hence \( f: X \to (Z, r) \) is a LIP embedding.  

Theorem 2. Let \( X \) be a separable locally totally bounded metric space. Then \( K_Z(X) \neq \emptyset \) for every metrizable compactification \( Z \) of \( X \) if and only if \( X \) is locally compact.

Proof. Suppose that \( X \) is locally compact and that \( f: X \to Z \) is a metrizable compactification of \( X \). Then \( X \) is open in \( \tilde{X} \) and \( fX \) in \( Z \). Hence the condition (2) of Theorem 1 is satisfied. Thus \( K_Z(X) \neq \emptyset \).

Suppose now that \( X \) is not locally compact. Then \( X \) is not open in \( \tilde{X} \). Hence by [1, 6.5] there is a compatible totally bounded metric \( e \) on \( X \) having no extension to a compatible metric on a neighborhood of \( X \) in \( \tilde{X} \). Then the completion \( Z \) of \( (X, e) \) is a compactification of \( X \) such that the condition (2) of Theorem 1 is not satisfied. Thus \( K_Z(X) = \emptyset \).  

The sufficiency part of Theorem 2 generalizes the sufficiency part of a similar result [1, 1.6] on one-point compactifications and gives it a new proof.

In the next theorem we consider \( K_Z(X) \) with its partial order which one gets through representatives setting \( (Y, f) \leq (Y', f') \) if there is a LIP map \( g: Y' \to Y \) with \( gf' = f \).
Theorem 3. Let $X$ be a noncompact metric space and let $Z$ be a metrizable compactification of $X$ with $K_Z(X) \neq \emptyset$. Then $K_Z(X)$ has the cardinality of the continuum. In fact, $K_Z(X)$ contains a subset which has the cardinality of the continuum and whose elements are not comparable.

Proof. We may assume that $Z$ is a compact metric space and that $X$ is a subspace of $Z$. Since $X \neq Z$, the proof can now be completed just as for one-point compactifications in the proof of [1, 1.9].

This generalizes [1, 1.9 and 1.10] and improves [1, 1.15.2].

3. Finally we consider the case where we allow the metric of $X$ to vary.

Theorem 4. Let $X$ be a metrizable space which is not locally compact, and let

$f: X \to Z$ be a metrizable compactification of $X$. Then $X$ can be metrized by a totally bounded metric such that $K_Z(X) = \emptyset$.

Proof. Since $fX$ is not open in $Z$, by [1, 6.5] there is a compatible totally bounded metric $e$ on $fX$ such that no compatible metric on $Z$ is LIP equivalent to $e$ on $fX$.

References


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