ENTIRE FUNCTIONS WITH TWO LINEARLY DISTRIBUTED VALUES

I. N. BAKER

In a series of papers [2, 3, 4] T. Kobayashi has given some interesting characterisations of the exponential function by its property of having its \( a \)-points collinear on a line \( \lambda(a) \) for several values of \( a \). In [4] he proved

**Theorem A.** Let \( G \) be a transcendental entire function of finite lower order. Assume that the zero points of \( G \) lie on the line \( \Re z = 0 \) and that the one-points lie on \( \Re z = 1 \). Then

\[
G(z) = P(\exp Cz)
\]

where \( P \) is a polynomial and \( C \) a non-zero real constant.

Of course there are restrictions on the polynomials which can occur in Theorem A and the possible forms are given in [4].

Kobayashi asks whether Theorem A still holds if the assumption that \( G \) has finite order be omitted from the hypotheses. We prove the

**Theorem.** Let \( f \) be a transcendental entire function such that all the zeros of \( f \) lie on \( \Re z = 0 \) and all the one-points on \( \Re z = 1 \). Then \( f \) has finite order, so by Theorem A there exist a polynomial \( P \) and a non-zero real constant \( C \) such that

\[
f(z) = P(\exp Cz).
\]

The proof depends on a recent result of J. Miles (Lemma 6) which states that if an entire function has both infinite order and real zeros then the zeros are in a certain sense scarce. Applying this to \( f(-iz) \) and \( f(1-iz) - 1 \) and using Nevanlinna's second fundamental theorem gives the result after a number of subsidiary points have been checked.

The proof follows in six lemmas and a concluding section.

**Lemma 1.** Let \( g \) be analytic in \( H: \Im z > 0 \) and omit the values 0 and 1 in \( H \). Then there exists a constant \( K = K(g) \) such that

\[
\log |g(re^{i\theta})| < Kr/(\sin \theta), \quad r > 1, \quad 0 < \theta < \pi.
\]

**Proof.** The map \( z = \varphi(t) = i(1+t)/(1-t) \) maps the disc \( D: |t| < 1 \) to the half-plane \( H \). Applying Schottky's theorem to \( g(\varphi(t)) \) we obtain

\[
|g(\varphi(t))| < \exp \{K/(1-|t|)\}.
\]
Using
\[1 - |t|^2 = 4r \sin \theta/r^2 + 2r \sin \theta + 1, \quad z = re^{i\theta}\]
we have
\[1 - |t| > 2r \sin \theta/r^2 + 2r \sin \theta + 1, \quad |t| < 1\]
and
\[\log |g(re^{i\theta})| < K(1 + r)^2/(2r \sin \theta) < Kr/(\sin \theta)\]

**Lemma 2.** Let \(g\) be analytic in the strip \(S: 0 \leq \text{Im } z < 1\) and omit the values 0 and 1 there. Then there exists a constant \(K = K(g)\) such that
\[\log |g(x + iy)| < Ke^{\pi|x|}/(\sin \pi y), \quad 0 < y < 1, -\infty < x < \infty.\]

**Proof.** Putting \(w = e^{\pi z}\) which maps \(S\) onto the half-plane \(H: \text{Im } w > 0\), the result follows from Lemma 1 for \(x > 0\). For \(x < 0\) the result follows by symmetry.

**Lemma 3.** Suppose \(f\) is entire and that all the zeros of \(f\) are real and all the one-points have imaginary part one. Then there is a constant \(A\) such that
\[T(r) = T(r, f) < Ar^{-4} e^{4\pi r}\]
for all sufficiently large \(r\).

**Remark.** We assume without explanation the standard notations of Nevanlinna theory.

**Proof.** The result of Lemma 1 shows that \(f\) is of order one and exponential type in any angle which is either strictly interior to \(\text{Im } z > 0\) or to \(\text{Im } z < 0\). If the lower order \(\mu = \lim_{r \to \infty} (\log T(r))/(\log r) < \infty\) it follows from the Phragmén–Lindelöf principle that \(f\) has at most order one in the plane and the assertion of the lemma holds.

Thus we may assume that \(\mu = \infty\) so that \(T(r) \to \infty\) faster than any power of \(r\). Put \(\eta = \sin^{-1}(1/r)\) and \(\delta = \{T(r)\}^{-1/2}\). Split the range of integration in
\[T(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta\]
at \(\pm \delta, \eta \pm \delta, \pi - \eta \pm \delta\) and \(\pi \pm \delta\). In the intervals \([- \delta, \delta], [\eta - \delta, \eta + \delta], [\pi - \eta - \delta, \pi - \eta + \delta]\) and \([\pi - \delta, \pi + \delta]\) we put
\[\log^+ |f(re^{i\theta})| \equiv \log M(r, f),\]
and in the remaining intervals use the estimates from Lemmas 1 and 2. It follows that there are constants \(K, K', P\) and \(Q\) such that for sufficiently large \(r\)
\[T(r) < \frac{Kr}{\sin \delta} + \frac{K'e^{\pi \eta}}{\sin (\pi r \delta)} + \frac{4\delta}{\pi} \log M(r, f)\]
\[< \frac{Pr}{\delta} + \frac{Qe^{\pi r}}{r^2 \delta} + \frac{4\delta}{\pi} \log M(r, f).\]
Entire functions with two linearly distributed values

Taking \( R = r + \varphi \) where \( \varphi = r/\log T(r) \) we have

\[
\log M(r, f) \leq \frac{R + r}{R - r} \cdot T(R) > \frac{3r}{\varphi} T \left( r + \frac{r}{T(r)} \right).
\]

A lemma of Borel [1] states that for any increasing function \( V(r) \) which is continuous in \( r > r_0 \) and such that \( V(r) \to \infty \) as \( r \to \infty \) and for any \( \varepsilon > 0 \)

\[
\left( r + \frac{r}{\log V(r)} \right) \leq V(r)^{1 + \varepsilon}
\]

holds outside a set of finite logarithmic measure. Taking \( V = T \) and \( 1/2 > \varepsilon > 0 \), (4) and (5) show that outside a set \( E \) of finite logarithmic measure in \( r > 1 \)

\[
T(r) < Pr \delta^{-1} + Qr^{-2} e^{\varepsilon r} + 12 \delta r T(r)^{1 + \varepsilon}/(\pi \varphi),
\]

or putting in the values of \( \delta \) and \( \varphi \)

\[
T(r) < PrT(r)^{1/2} + Qr^{-2} e^{\varepsilon r} T(r)^{1/2} + (12/\pi)(\log T(r)) T(r)^{1/2 + \varepsilon}.
\]

For \( 0 < \varepsilon < \varepsilon' < 1/2 \) we have

\[
\log T(r) = o ((T(r))^{\varepsilon' - \varepsilon}) \quad (r \to \infty)
\]

and so

\[
(1 - o(1)) T(r)^{1/2} < 2Qr^{-2} e^{\varepsilon r}
\]

as \( r \to \infty \) outside \( E \). Hence for large \( r \) outside \( E \)

\[
(6)
\]

\[
T(r) < 5Q^2 r^{-4} e^{2\varepsilon r}.
\]

Since \( E \) has finite logarithmic measure there exists \( r_0 \) such that for any \( r > r_0 \)

there are \( s \notin E, t \notin E, s < r < t < 2s \), while (6) holds for \( r = s \) and \( r = t \). Thus

\[
T(r) < T(t) < 5Q^2 t^{-4} e^{2\varepsilon t} < 5Q^2 r^{-4} e^{4\varepsilon r},
\]

so that (2) holds with \( A = 5Q^2 \).

Lemma 4. Suppose the increasing continuous function \( V(r) \) satisfies \( V(r) < e^{Ar} \)

for some constant \( A \), at least for all \( r \geq r_0 > 0 \). Suppose also that \( V(r) \to \infty \) as \( r \to \infty \).

Then

\[
(7)
\]

\[
V(r + r^{-1}) < 2V(r)
\]

holds outside a set of finite logarithmic measure in \( [r_0, \infty) \).

Proof. Suppose the assertion is not true. Then there is a first \( r_1 \geq r_0 \) where (7) fails. Define \( r_1^* = r_1 + r_1^{-1} \). Then

\[
V(r_1^*) \equiv 2V(r_1).
\]

Denote the interval \([r_1, r_1^*]\) by \( I_1 \).

Now proceed inductively. Assuming \( r_n, r_n^* = r_n + r_n^{-1} \) and \( I_n = [r_n, r_n^*] \) have been constructed, let \( r_{n+1} \) be the first \( r \geq r_n^* \) at which (7) fails, i.e. \( V(r_{n+1}) \equiv 2V(r_{n+1}) \).
Such an \( r_{n+1} \) exists since we are supposing that (7) fails for a set of infinite logarithmic measure. Thus there is an infinite sequence \( r_n, n=1, 2, \ldots \), and we have
\[
r_n < r_n^* \equiv r_{n+1} < r_{n+1}^*.
\]
From this it follows that \( r_n \to \infty \) for otherwise we would have \( \lim r_n = \lim r_{n+1} = \lim r_n^* = \text{finite} \) and hence \( \lim r_n^{-1} = 0 \) which gives a contradiction. Further, the inequality (7) holds in the complement of \( F = \bigcup_{n=1}^\infty I_n \).

The logarithmic measure of \( I_n \) is \( \log (1+r_n^{-2}) \leq r_n^{-2} \). Since the logarithmic measure of \( F \) is infinite \( \sum r_n^{-2} = \infty \). Take any constant \( B \) such that \( B \log 2 > A \). There must be arbitrarily large \( n \) such that \( r_n^{-2} \geq B^2 n^{-2} \), that is such that \( n \geq B r_n \).

For any \( n \)
\[
V(r_n) \equiv V(r_{n-1}^*) \equiv 2V(r_{n-1}) \equiv \lambda 2^n,
\]
where \( \lambda = V(r_1)/2 \). For infinitely many \( n \) we have in addition that \( n \geq B r_n \) so that
\[
V(r_n) > \lambda \exp (r_n B \log 2).
\]
Since \( r_n \to \infty \) and \( B \log 2 > A \) this contradicts the hypothesis that \( V(r) < e^A r, r > r_0 \). Thus the lemma is established.

**Lemma 5.** Suppose \( f \) satisfies the assumptions of Lemma 3 and let \( g \) be defined by \( g(z) = f(i+z) \).

Then
\[
T(r, g) < 8T(r, f)
\]
outside a set \( E_1 \subset [1, \infty) \) of finite logarithmic measure.

**Proof:** We can assume \( f \) transcendental. By Nevanlinna’s second fundamental theorem
\[
T(r, g) \equiv N(r, 0, g) + N(r, 1, g) + S(r)
\]
where \( S(r) = O \{ \log r + \log T(r, g) \} \) as \( r \to \infty \) outside a set \( H \) of finite measure. Now
\[
n(r, 0, g) \equiv n(r, 0, f),
\]
\[
n(r, 1, g) \equiv n((1+r^2)^{1/2}, 1, f),
\]
so that
\[
N(r, 0, g) \equiv N(r, 0, f) + O(\log r)
\]
and
\[
N(r, 1, g) = O(\log r) + \int_1^r n(t, 1, g) \ dt
\]
\[
\equiv O(\log r) + \int_{\sqrt{2}}^{\sqrt{1-r^2}} n(u, 1, f) \ du
\]
\[
\equiv O(\log r) + 2N((1+r^2)^{1/2}, 1, f).
\]
Entire functions with two linearly distributed values

Thus from (8), (9) and (10) we have outside $H$ that

$$T(r, g) < 3T((1+r^2)^{1/2}, f) + O(\log r + \log T(r, g))$$

whence

$$T(r, g) < 4T(r+r^{-1}, f)$$

outside a set $H'$ of finite measure.

By Lemma 3 $T(r, f) = \exp(4\pi r)$ for large $r$, so applying Lemma 4 with $V(r) = T(r, f)$ (11) gives

$$T(r, g) < 8T(r, f)$$

outside a set $E_1$ in $[1, \infty)$, of finite logarithmic measure.

Lemma 6. (Cf. [5].) Suppose $h$ is entire of infinite order with zeros restricted to a finite number of rays through the origin. Then there exists a set $G \subset [1, \infty)$ having logarithmic density zero and such that $\lim N(r, 0)/T(r, h) = 0$ as $r \to \infty$ outside $G$.

**Final section of the proof of the theorem.** Let $f$ satisfy the assumptions of the theorem and consider $h(z) = f(-iz)$ which has the same characteristic as $f$, while the zeros and ones of $h$ lie on the lines $\text{Im } z = 0$ and $\text{Im } z = 1$ respectively.

Suppose that the order of $f$ (and hence of $h$) is infinite. Then by Lemma 6 there is a set $G_1 \subset [1, \infty)$ of logarithmic density zero and such that

$$\lim N(r, 0, h)/T(r, h) = 0 \quad \text{as } r \to \infty, \quad r \notin G_1.$$

Now consider $g(z) = h(i+z) - 1$ which is also of infinite order with real zeros. By Lemma 5 there is a set $G_2 \subset [1, \infty)$ of finite logarithmic measure such that $T(r, g) < 8T(r, h), r \notin G_2$.

Since $n(r, 1, h) \equiv n(r, 0, g)$ holds we have

$$N(r, 1, h) \equiv N(r, 0, g) + O(\log r)$$

and for $r \notin G_2$

$$\frac{N(r, 1, h)}{T(r, h)} \equiv \frac{8N(r, 0, g)}{T(r, g)} + o(1).$$

Applying Lemma 6 to $g$ shows that there is a set $G_3$ of logarithmic density zero such that as $r \to \infty$ outside $G_2$ the right hand side of (13) tends to zero.

The second fundamental theorem shows that

$$\frac{1+o(1)}{T(r, h)} \equiv N(r, 0, h) + N(r, 1, h)$$

outside a set $G_4$ of finite measure. Thus $G = G_1 \cup G_2 \cup G_3 \cup G_4$ has zero logarithmic density and as $r \to \infty$ outside $G$ we have by (12), (13) and (14) that

$$(1+o(1))T(r, h) = o(T(r, h))$$

which is a contradiction. The proof is now complete.
References


Imperial College  
Department of Mathematics  
London S. W. 7  
England

Received 25 February 1980