Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 6, 1981, 189—196

THE CANONICAL POTENTIALS IN $\mathbf{R}^n$

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1. Introduction

This article contains some complements to the Nevanlinna theory of subharmonic functions and specially of certain (canonical) potentials in $\boldsymbol{R}^n$.

A Radon measure $\mu\geqq 0$ in $\boldsymbol{R}^n$, $n\geqq 2$, is said to be of finite genus if $\int^\infty |y|^{-\alpha} d\mu(y)$ is finite for some $\alpha\geqq 0$. Using the integral representation of subharmonic functions with suitable kernels, we can associate a unique subharmonic function in $\boldsymbol{R}^n$ with every μ of finite genus, called the canonical potential associated with μ.

We obtain here some properties of the canonical potentials analogous to those of the canonical products in the analytic function theory in the complex plane and note that every subharmonic function $u(x)$ of finite genus can be represented uniquely as the sum of a canonical potential and a harmonic polynomial.

It leads us to the problem of finding the relation between the growth of $u(x)$ and the growth of its potential part and harmonic part. With this in view, we determine the order and the type of $u(x)=u_1(x)+u_2(x)$ where $u_1(x)$ and $u_2(x)$ are subharmonic functions of known order and type. This, in part, helps us to connect the growth of a subharmonic function with the distribution of its associated measure.

We thank the referee for his very useful comments.

2. Preliminaries

The classical notions of the order, the genus and the canonical product associated with an entire function in the complex plane (see R. Nevanlinna [5], G. Valiron [8]) were reviewed by M. Arsove [1] in the context of subharmonic functions in $\boldsymbol{R}^2$. In this section we summarize these definitions and results and give some complements with reference to subharmonic functions in $\boldsymbol{R}^3$ (see also M. Brelot [2] and W. K. Hayman and P. B. Kennedy [4]).

We start with the expansion, for $|y|>|x|$,

$$|x-y|^{-1} = |y|^{-1}+\sum_{m=1}^{\infty} H_m |x|^m |y|^{-m-1}.$$

doi:10.5186/aasfm.1981.0615

Here $H_m = P_m(\cos\theta)$, where P_m is the Legendre polynomial of degree m and θ is the angle between Ox and Oy; further $\left|\sum_{m=n}^{\infty} H_m |x|^m |y|^{-m-1}\right| \leqq K|x|^n|y|^{-n-1}$ for some constant K when $|x| < \alpha|y|$, $0<\alpha<1$.

Define

$$B_n'(x, y) = \begin{cases} -|x-y|^{-1} & \text{if} \quad |y|<1 \\ -|x-y|^{-1}+|y|^{-1}+\sum_{m=1}^{n-1} H_m|x|^m|y|^{-m-1} & \text{if} \quad |y| \geqq 1. \end{cases}$$

If $u(x) = \int B_n'(x, y)\,d\mu(y)$ is subharmonic for some Radon measure $\mu \geqq 0$, then we say that $u(x)$ is a B_n'-potential. M. Brelot [2] showed that if $\int^{\infty} |y|^{-n-1} d\mu(y)$ is finite, then $u = \int B_n' d\mu$ is a B_n'-potential and $u^+(x) = o(|x|^n)$. Conversely, it is proved in [6] that if $u = \int B_n' d\mu$ is a subharmonic function, then $\int^{\infty} |y|^{-n-1} d\mu(y)$ is finite.

Let $S(r)$ be an increasing function of r such that $S(r) \geqq 1$ for large r. We define the order-type-class of $S(r)$ in the standard way (see, for example, the book of W. K. Hayman and P. B. Kennedy [4]). The function $S(r)$ is said to be of regular growth if $\lim (\log S(r)/\log r)$ exists. We say that two functions are of similar type if they are of minimal, mean or maximal type simultaneously.

For a subharmonic function $u(x)$ not bounded above in $\boldsymbol{R}^3$, we take $S(r) = B(r, u) = \max_{|x|=r} u(x)$, and the order-type-class of $u(x)$ is defined accordingly. Since a subharmonic function bounded above in $\boldsymbol{R}^3$ is a potential up to an additive constant, for the discussion below we always consider subharmonic functions that are not bounded above.

Given a Radon measure $\mu \geqq 0$, let $n(r) = \mu(B_0^r)$, and for a subharmonic function $u(x)$, let $M(r, u)$ denote the mean of $u(x)$ on $|x| = r$. Then for $\alpha > 1$, $\int^{\infty} r^{-\alpha} M(r, u)\,dr$ is finite if and only if

$$(*) \qquad \int^{\infty} r^{-\alpha-1} n(r)\,dr$$

is finite (M. Brelot [2]).

Given a measure $\mu \geqq 0$, the order-type-class-growth of $n(r)$ is referred to as the order-type-class-growth of μ. It is useful to remark that if ϱ is the order of μ, then $\varrho = \inf \alpha$, where $\alpha > 0$ satisfies the condition $\int^{\infty} r^{-\alpha-1} n(r)\,dr$ is finite. Consequently, if $\varrho < 1$ and if $u(x)$ is a subharmonic function with associated measure μ (recall that given a measure μ in $\boldsymbol{R}^n$, $n \geqq 2$, there always exists a subharmonic function with associated measure μ, M. Brelot [3]), then $u(x)$ is admissible; that is, $u(x)$ has a harmonic majorant in $\boldsymbol{R}^3$. On the other hand, when $\varrho \geqq 1$, we have the following result:

Theorem 2.1. *Let $u(x)$ be a subharmonic function with associated measure μ of order $\varrho \geqq 1$. Then $M(r, u)$ is of order $\varrho - 1$. Also when $1 < \varrho < \infty$, $M(r, u)$ and $n(r)$ are of the same class and of similar type.*

Proof. Let the order of $M(r, u)=\sigma<\infty$. Since $M(r, u)\leqq r^{\sigma+\varepsilon}$ for large r, $\int^{\infty} r^{-\sigma-1-2\varepsilon}M(r, u)\,dr$ is finite, and hence $\int^{\infty} r^{-\sigma-2-2\varepsilon}n(r)\,dr$ is finite, which implies that $\varrho\leqq\sigma+1$. By a similar argument we see that $\sigma+1\leqq\varrho$. If $\sigma=\infty$, it is clear that $\varrho=\infty$.

Now suppose that $0<\sigma<\infty$. From $(*)$ it follows that $n(r)$ is of convergence class if and only if $M(r, u)$ is of convergence class.

To prove the statement about the type, note that if $n(r)\leqq cr^{\varrho}$, then $M(r, u)\leqq (c/(\varrho-1))(r^{\varrho-1}-1)$. This together with the inequality $M(r, u)\geqq\int_{r/2}^{r} t^{-2}n(t)\,dt\geqq n(r/2)/r$ implies that $M(r, u)$ and $n(r)$ are of similar type.

Corollary 2.2. *Let $u(x)$ be a subharmonic function with associated measure μ of order $\varrho\geqq 1$. Assume that one of the following conditions is satisfied:*

(i) *$u(x)$ is lower bounded,*
(ii) *the order of $u(x)$ is not an integer.*

Then $u(x)$ is of order $\varrho-1$, and when $1<\varrho<\infty$, $u(x)$ and $n(r)$ are of the same class and similar type.

For, the assumption (i) implies that $M(r, u)$ and $u(x)$ have the same order-type-class, which is true under the assumption (ii) also by Theorem 4.4 of W. K. Hayman and P. B. Kennedy [4]. Now the corollary follows immediately.

The *genus of a measure* μ is defined as the smallest integer $g\geqq 0$ for which $\int^{\infty} r^{-g-1}dn(r)$ is finite or equivalently $\int^{\infty} r^{-g-2}n(r)\,dr$ is finite.

Then it follows that a subharmonic function $u(x)$ with associated measure μ is a B_n'-potential up to a harmonic function if and only if the genus of μ is not greater than n; $u(x)$ is admissible if and only if the genus of μ is 0.

Further, as in M. Arsove [1], it follows that if $u(x)$ is a subharmonic function in $\boldsymbol{R}^3$ of order λ with associated measure μ of order ϱ and genus g, then $\varrho-1\leqq g\leqq\varrho\leqq\lambda+1$.

Definition 2.3. Let $\mu\geqq 0$ be a Radon measure of genus $g>0$. Then the B_g'-potential $\int B_g'(x, y)\,d\mu(y)$ is called the *canonical potential* associated with μ.

Let $u(x)$ be a subharmonic function with associated measure μ of finite genus $g>0$. Then $u(x)$ is the sum of the canonical potential associated with μ and a harmonic function $h(x)$.

In this case, the *genus of $u(x)$* is defined as the $\max(g, \text{order of } h(x))$. Thus a subharmonic function $u(x)$ of finite genus n has a unique representation as the sum of a canonical potential and a harmonic polynomial of degree $\leqq n$.

Here is a natural and useful theorem extending a classical result:

Theorem 2.4. *Let $\mu\geqq 0$ be a Radon measure of genus $g>0$ and order ϱ. If λ is the order of the canonical potential $u(x)=\int B_g'(x, y)\,d\mu(y)$ associated with μ, then $\varrho=\lambda+1$.*

Proof. Since $u^+(x)=o(|x|^g)$, $\lambda\leqq g$, and hence $g\leqq\varrho\leqq\lambda+1\leqq g+1$. If $\varrho=g+1$, then clearly $\varrho=\lambda+1$. Let $\varrho<g+1$. By the definition of ϱ, $n(r)=o(r^{\varrho+\varepsilon})$ for $\varepsilon>0$. Choose ε small so that for some δ $(0<\delta<1)$ $\varrho+\varepsilon=g+\delta$.

Consequently, using the inequality (a variation of Lemma 4.4 in W. K. Hayman and P. B. Kennedy [4]; see also [6]), where we suppose that $n(1)=0$ by taking balayage, $|x|=r$, $u(x)\leqq c\left[gr^{g-1}\int_1^r t^{-g-1}n(t)\,dt+(g+1)r^g\int_r^\infty t^{-g-2}n(t)\,dt\right]$, we deduce that $u^+(x)=o(r^{\varrho+\varepsilon-1})$, which implies that $\lambda\leqq\varrho-1$.

This shows that $\varrho=\lambda+1$.

Corollary 2.5. *Let $u(x)$ be the canonical potential associated with a measure μ of genus $g>0$. Then $u(x)$ is of regular growth if $n(r)$ is.*

In fact, since $n(r/2)/r\leqq M(r,u)\leqq B(r,u)$, $\varrho=$the lower order of $n(r)\leqq 1+$the lower order of $u(x)\leqq 1+\lambda=\varrho$, by the above theorem. Hence $u(x)$ is of regular growth.

Remark. In the context of the above corollary, Rajeswari Thyagarajan remarks: If the order λ of a subharmonic function $u(x)$ is not an integer, then $u(x)$ is of regular growth if and only if $n(r)$ is of regular growth.

To prove this, she makes use of an upper bound for the lower order of a canonical potential obtained as in S. M. Shah [7] for the case of entire functions.

3. The growth of the sum of two subharmonic functions

In this section we consider only subharmonic functions that are not bounded above in $\boldsymbol{R}^3$. Let $u(x)$ and $v(x)$ be two such functions. Then

(A) $B(r,u+v)\leqq B(r,u)+B(r,v)$.

Note that for large r, $M(r,v)=M(r,v^+)-M(r,v^-)$ is either a bounded or an unbounded positive function depending on whether $v(x)$ has a harmonic majorant or not in $\boldsymbol{R}^3$.

As a consequence, for large r,

$$\begin{aligned} M(r,u^+) &\leqq M(r,(u+v)^+)+M(r,v^-)\\ &\leqq M(r,(u+v)^+)+M(r,v^+)+a \text{ constant}\\ &\leqq B(r,u+v)+B(r,v)+a \text{ constant}. \end{aligned}$$

But $M(r,u^+)\geqq((1-\alpha)^2/(1+\alpha))B(\alpha r,u)$, $0<\alpha<1$, for large r (see p. 127, W. K. Hayman and P. B. Kennedy [4]). Hence for large r, $0<\alpha<1$,

(B) $((1-\alpha)^2/(1+\alpha))B(\alpha r,u)\leqq B(r,u+v)+B(r,v)+a$ constant.

From (A) and (B) we obtain the following useful result:

Theorem 3.1. *Let $u_1(x)$ and $u_2(x)$ be subharmonic functions of orders λ_1 and λ_2, respectively, with $\lambda_1 > \lambda_2$. Then $u_1(x)+u_2(x)$ is of order λ_1 and of type similar to $u_1(x)$. Further, $u_1(x)+u_2(x)$ is of regular growth if and only if $u_1(x)$ is.*

We have already seen in the last section that if $u(x)$ has non-integral order λ, then $n(r)$ has order $\lambda+1$ and is of type similar to $u(x)$. Now we consider the case when $u(x)$ has integral order.

Lemma 3.2. *Let p be the canonical potential associated with the measure μ of genus $g>0$. Suppose $n(r)=O(r^g)$. Then $B(r, p)=O(r^{g-1}\log r)$.*

Proof. The canonical potential $p(x)=\int B'_g(x, y)d\mu(y)$ is such that for $|x|=r$, $p(x)\leqq c[gr^{g-1}\int_1^r t^{-g-1}n(t)dt+(g+1)r^g\int_r^\infty t^{-g-2}n(t)dt]$. Since $n(r)=O(r^g)$, $r^{g-1}\int_1^r t^{-g-1}n(t)dt=O(r^{g-1}\log r)$ and $r^g\int_r^\infty t^{-g-2}n(t)dt=O(r^{g-1})$. Hence $B(r, p)=O(r^{g-1}\log r)$.

Theorem 3.3. *Let $u(x)$ be a subharmonic function of integral order λ and non-mean type. Then the order of $n(r)$ is $\lambda+1$. If $u(x)$ is of minimal type, then $n(r)$ is of minimal type; if $u(x)$ is of maximal type and if* $\limsup B(r, u)/(r^\lambda \log r)=\infty$, *then $n(r)$ also is of maximal type.*

Proof. Let $u(x)=p(x)+h(x)$, where $p(x)$ is the canonical potential and $h(x)$ is a harmonic polynomial. Let ϱ be the order of $n(r)$. Then the order of $p(x)$ is $\varrho-1\leqq\lambda$. If $\varrho-1<\lambda$, then $h(x)$ has order λ and $u(x)$ is of type similar to $h(x)$, a contradiction since $h(x)$ is of mean type. Hence the order of $n(r)$ is $\lambda+1$.

If $u(x)$ is of minimal type, then it follows from the inequality $n(r)\leqq 2rB(2r, u)$, that $n(r)$ also is of minimal type.

Suppose now that $u(x)$ is of maximal type. Then the genus of μ is $\lambda+1$. For, if the genus of $\mu<\lambda+1$, then $p(x)=\int B'_\lambda(x, y)d\mu(y)$ is such that $p^+(x)=o(|x|^\lambda)$ so that $p(x)$ is of minimal type, a contradiction.

Now, if $n(r)$ is not of maximal type, then $n(r)=O(r^{\lambda+1})$. Hence by Lemma 3.2, $B(r, p)=O(r^\lambda\log r)$, and consequently $B(r, u)=O(r^\lambda\log r)$, a contradiction. This proves the theorem.

The case when the order of $u_1(x)$=the order of $u_2(x)$.

Theorem 3.4. *Let $u_1(x)$ and $u_2(x)$ be subharmonic functions both having the same non-integral order λ. Then $u_1(x)+u_2(x)$ also has order λ. If τ, τ_1 and τ_2 denote the types of $u_1(x)+u_2(x)$, $u_1(x)$ and $u_2(x)$, respectively, then for a constant $\beta>0$,* $\max(\beta\tau_1-\tau_2, \beta\tau_2-\tau_1)\leqq\tau\leqq\tau_1+\tau_2$. *Moreover, if $u_1(x)$ and $u_2(x)$ are of regular growth, then so is $u_1(x)+u_2(x)$.*

Proof. Let μ_1 and μ_2 be the measures associated with $u_1(x)$ and $u_2(x)$, respectively. Then the genus of μ_1=the genus of $\mu_2=g=[\lambda+1]$. Now $u_1(x)=\int B'_g(x, y)d\mu_1(y)+h_1(x)$ and $u_2(x)=\int B'_g(x, y)d\mu_2(y)+h_2(x)$, where h_1 and h_2 are harmonic polynomials of degree at most $g-1$. Hence

$$u_1(x)+u_2(x)=\int B'_g(x, y)d\mu(y)+h(x),$$

where $\mu=\mu_1+\mu_2$ and $h(x)=h_1(x)+h_2(x)$.

Since the order of $n_1(r)(=\mu_1(B_0^r))$=the order of $n_2(r)(=\mu_2(B_0^r))=\lambda+1$, the order of $n(r)=n_1(r)+n_2(r)$ is $\leqq\lambda+1$. Also, for a sequence of r tending to ∞, $n(r)>n_1(r)>r^{\lambda+1-\varepsilon}$, which implies that the order of $n(r)$ is $\lambda+1$. Hence $u_1(x)+u_2(x)$ has order $\geqq\lambda$. Consequently, using (A) we see that the order of $u_1(x)+u_2(x)$ is λ and $\tau\leqq\tau_1+\tau_2$.

Let $\beta=\sup_{0<\alpha<1}((1-\alpha)^2/(1+\alpha))\alpha^\lambda$. Then using (B) we find that $\beta\tau_1\leqq\tau+\tau_2$. Similarly we also have $\beta\tau_2\leqq\tau+\tau_1$. Consequently $\tau\geqq\max(\beta\tau_1-\tau_2, \beta\tau_2-\tau_1)$.

Finally, to prove the statement about the regular growth, we make use of the remark following Corollary 2.5. Since $n_1(r)$ and $n_2(r)$ are of regular growth, so is $n(r)$. Consequently, $u_1(x)+u_2(x)$ has lower order $\geqq\lambda$. This completes the proof of the theorem.

Theorem 3.5. *Let $u_1(x)$ and $u_2(x)$ be subharmonic functions both having the same integral order λ. Then $u_1(x)+u_2(x)$ has order λ when any one of the following conditions is satisfied:*

(i) *If $\beta\tau_1>\tau_2$, where $\beta=\sup_{0<\alpha<1}((1-\alpha)^2/(1+\alpha))\alpha^\lambda$, in which case $u_1(x)+u_2(x)$ is of type similar to $u_1(x)$.*

(ii) *If $\tau_1=\tau_2=0$, in which case $\tau=0$.*

(iii) *If $\tau_1=\tau_2=\infty$. In this case, if* $\limsup B(r, u_1)/(r^\lambda \log r)=\infty$, *then $\tau=\infty$.*

Proof. (i) Suppose that the order of $u_1(x)+u_2(x)$ is $<\lambda$. Then it follows from (B) that for $0<\alpha<1$, $((1-\alpha)^2/(1+\alpha))\alpha^\lambda\tau_1\leqq\tau_2$, which gives $\beta\tau_1\leqq\tau_2$, a contradiction. Hence $u_1(x)+u_2(x)$ has order λ.

Also from (A) and (B) we get $\tau\leqq\tau_1+\tau_2$ and $\beta\tau_1\leqq\tau+\tau_2$, which shows that $u_1(x)+u_2(x)$ is of type similar to $u_1(x)$.

(ii) By Theorem 3.3, $n_1(r)$ and $n_2(r)$ are of order $\lambda+1$ and, consequently, the order of $u_1(x)+u_2(x)$ is λ. In this case, $\tau=0$ since $\tau\leqq\tau_1+\tau_2$.

(iii) Again by Theorem 3.3, the order of $u_1(x)+u_2(x)$ is λ and $n_1(r)$, and hence $n(r)$ is of maximal type. But $n(r)\leqq 2r\ B(2r, u_1+u_2)$, which implies that $\tau=\infty$. This completes the theorem.

Corollary 3.6. *Let $u(x)=p(x)+h(x)$ be of finite genus, where $p(x)$ is its potential part with order ϱ and $h(x)$ its harmonic part with order q. Then the order of $u(x)=\max(\varrho, q)$.*

For, if $\varrho\neq q$, then by Theorem 3.1, the order of $u(x)=\max(\varrho, q)$. Suppose that $\varrho=q$. In this case, by (i) above $u(x)$ is of order ϱ except possibly when $p(x)$ is of mean type. But, in this exceptional case, the genus of μ has to be $\varrho+1$, and hence the order of $u(x)$ is ϱ.

4. The plane case

Finally, we note that the above discussion can be carried out easily in $\boldsymbol{R}^n$, $n>3$, with very little modifications, starting with the kernel $|x-y|^{-n+2}$.

In $\boldsymbol{R}^2$, as indicated earlier, most of the results corresponding to those given in the preliminaries are due to M. Arsove [1]. We give here certain complementary results.

For $|y|>|x|$, $\log|x-y|=\log|y|-\sum_{m=1}^{\infty} H_m|x|^m|y|^{-m}$, where $H_m=(1/m)\cos m\theta$, θ being the angle between Ox and Oy. Let

$$\Lambda'_m(x, y)=\begin{cases}\log|x-y| & \text{if} \quad |y|<1,\\ \log|x-y|-\log|y|+\sum\limits_{n=1}^{m-1} H_n|x|^n|y|^{-n} & \text{if} \quad |y|\geqq 1.\end{cases}$$

It can be proved (see [6]) that a Radon measure $\mu\geqq 0$ is of finite genus m if and only if $v(x)=\int \Lambda'_{m+1}(x, y)\,d\mu(y)$ is subharmonic. In this case $v(x)$ is the canonical potential associated with μ and $v^+(x)=o(|x|^{m+1})$.

Further, let $u(x)$ be a subharmonic function in $\boldsymbol{R}^2$ of order λ and associated measure μ. Then $M(r, u)$ and $n(r)=\mu(B_0^r)$ have the same order-type-class; when λ is not an integer, $u(x)$ and μ have the same order-type-class.

We conclude with a general remark concerning the regular growth of a subharmonic function of finite genus in $\boldsymbol{R}^2$.

Theorem 4.1. *Let $u(x)=p(x)+h(x)$ be a subharmonic function of finite genus in $\boldsymbol{R}^2$, where $p(x)$ is the canonical potential of order ϱ and type τ and $h(x)$ is a harmonic polynomial of degree q and type σ. Then*

(i) *the order of* $u(x)=\max(\varrho, q)$;

(ii) *suppose* $\varrho=q$ *and* $\tau\notin[\sigma/8, \sigma]$. *Then $u(x)$ is of maximal type if and only if $p(x)$ is of maximal type; otherwise $u(x)$ is of mean type;*

(iii) *$u(x)$ is of regular growth if any of the following conditions is satisfied:*

a) $q>\varrho$,

b) *$p(x)$ is of regular growth and* $\varrho>q$,

c) $\varrho=q=m$ *and for some* α, $0<\alpha<1$, $((1-\alpha)/(1+\alpha))\alpha^m\sigma>\tau$.

Corollary 4.2. *Let $u(x)$ be a subharmonic function of finite order in $\boldsymbol{R}^2$. Let $M(r, u)=o((\log r)^2)$. Then $u(x)$ is of regular growth with integral order. In particular, the conclusion is valid if the measure μ associated with $u(x)$ is such that $\int d\mu$ is finite.*

In fact, if $M(r, u)=o((\log r)^2)$, from the inequalities $M(r^2, u)\geqq\int_r^{r^2} t^{-1}n(t)\,dt\geqq n(r)\log r$, we note that $n(r)=o(\log r)$. Hence $u(x)$ is of the form $u(x)=\int \Lambda'_1(x, y)\,d\mu(y)+h(x)$, where $\int \Lambda'_1 d\mu$ is of order 0 and $h(x)$ is a harmonic polynomial. Then, by the above theorem, $u(x)$ is of regular growth with integral order.

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Received 21 April 1980
Revision received 19 January 1981