ON SUPPORT POINTS IN THE CLASS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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Dedicated to Professor Kurt Strebel on the occasion of his sixtieth birthday

1. Introduction

Let $H(D)$ denote the set of analytic functions in the unit disc $D=\{|z|<1\}$, endowed with the topology of locally uniform convergence, and let $H(D)^*$ be its topological dual. The elements of $H(D)^*$ are the continuous linear functionals L on $H(D)$. Denoted by M the set of real Baire measures μ on the unit circle $C=\partial D$ such that $\int_C d\mu=2$, and by M_k the subset

$$M_k = \left\{\mu: \int_C d\mu = 2, \ \|\mu\| \leq k\right\},$$

where $\|\mu\|$ is the total variation of μ. Obviously we have $2 \leq k < \infty$. The formula

$$(1) \qquad f'(z) = \exp \int_{|\eta|=1} -\log(1-\eta z)\,d\mu, \quad f(0) = 0,$$

for the derivative f' together with the normalization $f(0)=0$ defines a mapping V of M into $H(D)$, which is injective and continuous in the weak* topology of M. The representing measure of f is called μ. Discrete measures of M represent the Schwarz—Christoffel mappings; they are dense in $V(M)$.

Because M_k is compact in this topology, its image $V(M_k)=V_k$ is compact in $H(D)$. V_k is the well-known class of normalized $(f(z)=z+a_2z^2+...)$ conformal mappings of D into C having boundary rotation $\pi\|\mu\|\leq k\pi$. Since the discrete measures μ with $\|\mu\|=k$ are dense in M_k, the Schwarz—Christoffel mappings with boundary rotation $k\pi$ are dense in V_k.

For $2 \leq k \leq 4$ the functions in V_k are univalent. In particular, V_2 consists of the normalized univalent mappings of D onto convex domains.

Since V_k is compact, the real part of each functional L of $H(D)^*$ is maximized within V_k by some function, and such functions are called *support points* if $\mathrm{Re}\,L$ is not constant on V_k. It is known that support points of V_k are Schwarz—Christoffel mappings with boundary rotation $k\pi$, or equivalently, their representing measures are discrete with norm k. This result is in [2], and to provide some framework a

proof of it will be sketched in Lemma 1. The purpose of this paper is to provide additional properties of support points.

Many functionals of $H(D)^*$ have been studied for V_k. One example is the coefficient functional a_n in $f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots$. After early attempts it was finally proved in [1] and [3] that for all n the maximum of $\operatorname{Re} a_n$ over V_k is achieved by the function whose representing measure has its positive and negative part concentrated at $\eta=1$ and at $\eta=-1$, respectively, i.e., for

$$(2) \qquad K(z) = \frac{1}{k}\left(\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right).$$

Another example is obtained by evaluation of f' at a fixed point p of $D\setminus\{0\}$, or more generally, by $\operatorname{Re}\{e^{i\alpha}f'(p)\}$, $\alpha\in R$. The extremals for these functionals give the support points of the region $R(p)=\{f'(p)\colon f\in V_k\}$, which is convex only for p sufficiently small. However, the set $\{\log f'(p)\colon f\in V_k\}$ is convex and is known [7] to have its boundary points furnished by functions f' of the form

$$(3) \qquad f'(z) = (1-\eta z)^{k/2-1}/(1-\zeta z)^{k/2+1}$$

where $|\zeta|=|\eta|=1$ and $\zeta\neq\eta$. By exponentiation, some of these boundary points are taken onto the support points of $R(p)$. This implies that only functions with derivative of the form (3) can maximize the functionals $\operatorname{Re}\{e^{i\alpha}f'(p)\}$.

Similarly, in Section 5 we shall see that extremals for functionals of the form $\operatorname{Re}\{A_2a_2+A_3a_3\}$, where A_2, $A_3\in C$ and a_2, a_3 appear in $f(z)=z+a_2z^2+a_3z^3+\ldots$, have representing measures supported by only two points. It was among these functionals that we were searching for candidates having extremals of a different type. Since there is none, it seems reasonable to conjecture that all support points of V_k must have derivatives of the form (3).

The convex hull of V_k will be denoted by $\operatorname{co} V_k$. It consists of all finite convex combinations of elements of V_k. Its closure will be denoted by $\overline{\operatorname{co}}\,V_k$. A function $f\in V_k$ is an *extreme point* of $\operatorname{co} V_k$ or $\overline{\operatorname{co}}\,V_k$ if f cannot be expressed as a proper convex combination of two distinct elements of $\operatorname{co} V_k$ or $\overline{\operatorname{co}}\,V_k$, respectively.

It is known [4] that the support points of V_2 coincide with the extreme points of $\overline{\operatorname{co}}\,V_2$ and are produced by the one-point measures of M_2. Until now no relations between support points and extreme points were known for $k>2$. For $k\geqq4$ it was shown in [3] that the extreme points of $\overline{\operatorname{co}}\,V_k$ are functions with derivative of the form (3). For these values of k we shall show that support points of V_k have the same form (3). Although we shall give some information for $2<k<4$, a description of the support points of V_k and the extreme points of $\overline{\operatorname{co}}\,V_k$ remains as an intriguing open question for $2<k<4$.

In this paper we shall prove the following:

Theorem 1. *For $k\geqq2$ the representing measure of each support point in V_k is discrete and has its positive part concentrated at a single point.*

Theorem 2. *For $k \geq 4$ each support point of V_k has a derivative of the form* (3).

Theorem 3. *For $2 < k < 4$ each support point of V_k is an extreme point of* co V_k.

Theorem 4. *For $k \geq 2$ each functional* Re $\{A_2 a_2 + A_3 a_3\}$, $A_2, A_3 \in C \setminus \{0\}$, *achieves its maximum within V_k only for functions with f' of the form* (3).

If A_3 is a positive multiple of A_2^2, then the maximum of Re $\{A_2 a_2 + A_3 a_3\}$ over V_k occurs for a rotation of the function $K(z)$ in (2), namely for $(A_2/|A_2|)K(|A_2| z / A_2)$. Some less trivial cases are tabulated in Section 6 where A_3 is a negative multiple of A_2^2.

2. The form of support points

First we sketch a proof of the following important lemma. The framework is basic and will be used later.

Lemma 1. *For each support point of V_k the representing measure μ is discrete and has norm k.*

Proof. Let f_0 be a support point of V_k corresponding to a functional L in $H(\mathbf{D})^*$, and let μ_0 be the representing measure of f_0. Then by setting

$$\mu_\varepsilon = \mu_0 + \varepsilon(\mu - \mu_0), \quad 0 \leq \varepsilon \leq 1,$$

we have a variation of μ_0 within M_k, for each μ in M_k. By substituting μ_ε into (1) we obtain a variation of f_0 within V_k, which has the form

$$f_\varepsilon'(z) = f_0'(z) - \varepsilon \int_C f_0'(z) \log(1 - \eta z) d(\mu - \mu_0) + O(\varepsilon^2)$$

as $\varepsilon \to 0$. Define L' by $L'(f') = L(f)$. Then

$$\operatorname{Re} L(f_\varepsilon) = \operatorname{Re} L(f_0) + \varepsilon \int_C J(\eta) d(\mu - \mu_0) + O(\varepsilon^2),$$

where

$$J(\eta) = -\operatorname{Re} L'(f_0'(z) \log(1 - \eta z))$$

is a harmonic function on $\bar{\mathbf{D}}$. The extremality of f_0 implies that $\int_C J(\eta) d(\mu - \mu_0) \leq 0$ for every $\mu \in M_k$. From this it follows that the positive and negative parts of μ_0 are supported by subsets of C where $J(\eta)$ reaches its maximum and minimum, respectively. Since L is not constant on V_k, the function $J(\eta)$ is not constant either, and we conclude that these two subsets of C are discrete and disjoint. This completes the lemma.

Lemma 1 remains true for solutions to nonlinear, Gâteaux differentiable functionals. An earlier proof of the lemma is given in [2]. The discreteness of μ was shown earlier in [6].

We shall also need the following result from [3].

Lemma 2 (Brannan—Clunie—Kirwan). *If a function g of H(D) is subordinate to a function* $(1+cz)/(1-z)$, *for some complex number c with* $|c| \leqq 1$, *and if* $p \geqq 1$. *then there is a probability measure* λ *on C such that*

$$g(z)^p = \int\limits_{|\xi|=1} \left(\frac{1+c\xi z}{1-\xi z}\right)^p d\lambda.$$

Our basic result on the form of support points is the following.

Theorem 1. *A support point of* V_k *has a discrete representing measure with positive part supported by a single point. Thus the derivative of a support point is of the form*

$$(4) \qquad f'(z) = (1-\zeta z)^{-k/2-1} \prod_{j=1}^{n} (1-\eta_j z)^{-\alpha_j},$$

where $|\zeta| = |\eta_j| = 1$, $\alpha_j > 0$, *and* $\sum_{j=1}^{n} \alpha_j = k/2 - 1$. *Moreover,* ζ *and* $\eta_1, ..., \eta_n$ *are mutually disjoint.*

Remark. Each support point is a Schwarz—Christoffel mapping with excatly one corner at infinity, where the canonical direction turns around a positive angle of $(k/2+1)\pi$, while at all other corners it takes negative turns. For $2 < k \leqq 4$ the image has an infinite convex complement which is not a strip or a halfplane; for $n = 1$ it is a wedge. If $k = 2$, the product is deleted and $f(z) = z/(1-\zeta z)$, $|\zeta| = 1$ (cf. [4]).

Proof of Theorem 1. Let f maximize the real part of a functional L in $H(D)^*$ for which $\operatorname{Re} L$ is nonconstant on V_k. By Lemma 1 f' is of the form

$$(5) \qquad f'(z) = \prod_{i=1}^{m} (1-\zeta_i z)^{-\beta_i} \cdot \prod_{j=1}^{n} (1-\eta_j z)^{\alpha_j}$$

where $|\zeta_i| = |\eta_j| = 1$, $\sum_{j=1}^{n} \alpha_j = k/2 - 1$, $\sum_{i=1}^{m} \beta_i = k/2 + 1$, and where the sets $\{\zeta_i\}$ and $\{\eta_j\}$ are disjoint. Since the function $g(z) = \prod_{i=1}^{m} (1-\zeta_i z)^{-\beta_i/\beta}$, with $\beta = k/2 + 1$, maps D into the halfplane $\operatorname{Re} w > 1/2$, it is subordinate to $1/(1-z)$. As a result, Lemma 2 with $c = 0$ implies that $g(z)^\beta = \int_{|\xi|=1} (1-\xi z)^{-\beta} d\lambda$ for some probability measure λ, and by (5) that

$$(6) \qquad f'(z) = \int\limits_{|\xi|=1} h(z)(1-\xi z)^{-\beta} d\lambda$$

where $h(z) = \prod_{j=1}^{n} (1-\eta_j z)^{\alpha_j}$.

Let L' correspond to L by $L'(f') = L(f)$. Then, by linearity, we have

$$\operatorname{Re} L'(f') = \int\limits_{|\xi|=1} \varphi(\xi) d\lambda$$

where $\varphi(\xi)=\mathrm{Re}\left\{L'\big(h(z)(1-\xi z)^{-\beta}\big)\right\}$ is harmonic on $\overline{D}$. Since $h(z)(1-\xi z)^{-\beta}$ is the derivative of a function in V_k, we also have $\varphi(\xi)\leqq\mathrm{Re}\,L'(f')$ by extremality. As a consequence, the probability measure λ must be supported on a subset of C on which $\varphi(\xi)$ achieves its maximum. Since $\varphi(\xi)$ is harmonic on $\overline{D}$, either λ has finite support or $\varphi(\xi)$ is constant.

If $\varphi(\xi)$ were constant, we would have $L'\big(h(z)z^\nu\big)=0$ for every $\nu=1,2,3,\ldots$. In this case, since $h(z)\neq0$ in D and $(\hat{f}'/h)(0)=1$, we could conclude that

$$L'(\hat{f}') = L'\left(h\frac{f'}{h}\right) = L'(h)$$

for all $\hat{f}$ in V_k. Thus L would be constant on V_k.

Consequently, the support of λ is a finite set $\{\xi_1,\ldots,\xi_N\}$. If λ has mass λ_j at ξ_j $(j=1,\ldots,N)$, then (5) and (6) imply

$$\prod_{i=1}^{m}(1-\zeta_i z)^{-\beta_i} = \lambda_1(1-\xi_1 z)^{-\beta}+\ldots+\lambda_N(1-\xi_N z)^{-\beta}.$$

Since $\beta=\sum_{i=1}^{m}\beta_i$, a comparison of singularities yields $m=N=1$, and the theorem is proved.

3. Support points for $k\geqq4$

For $k\geqq4$ it was shown in [3] that the extreme points of $\overline{\mathrm{co}}\,V_k$ are functions with derivatives of the form (3). In this section we shall show that even the support points of V_k have this form if $k\geqq4$.

Theorem 2. *For $k\geqq4$ each support point of V_k has a derivative of the form*

(3) $$f'(z) = (1-\eta z)^{k/2-1}/(1-\zeta z)^{k/2+1}$$

where $|\zeta|=|\eta|=1$ and $\zeta\neq\eta$.

Proof. Let f be a support point corresponding to a functional L of $H(D)^*$. According to Theorem 1, f' satisfies

$$(1-\zeta z)^2 f'(z) = \prod_{j=1}^{n}\left(\frac{1-\eta_j z}{1-\zeta z}\right)^{\alpha_j}$$

where $\zeta,\ \eta_1,\ldots,\eta_n$ are distinct points of C and $\alpha=\sum_{j=1}^{n}\alpha_j=k/2-1\geqq1$. Each function $\log\big((1-\eta_j z)/(1-\zeta z)\big)$ maps D onto a horizontal strip of width π. Therefore, as in [8, Theorem 2.25], the convex sum

$$s(z) = \sum_{j=1}^{n}\frac{\alpha_j}{\alpha}\log\frac{1-\eta_j z}{1-\zeta z}$$

maps D into some horizontal strip of width π. Thus there exists a constant c, where $|c|=1$ and $c\neq-1$, such that $s(z)$ is subordinate to $\log\big((1+cz)/(1-z)\big)$. Since

$e^{s(z)}$ is subordinate to $(1+cz)/(1-z)$ and $\alpha \geq 1$, Lemma 2 implies that

$$(7) \qquad (1-\zeta z)^2 f'(z) = e^{\alpha s(z)} = \int_{|\xi|=1} \left(\frac{1+c\xi z}{1-\xi z} \right)^\alpha d\lambda$$

for some probability measure λ.

The next part of the proof runs parallel to the proof of Theorem 1. Let L' correspond to L by $L'(f') = L(f)$. Then $\operatorname{Re} L'(f') = \int_{|\xi|=1} \varphi(\xi) d\lambda$ where $\varphi(\xi) = \operatorname{Re}\{L'((1-\zeta z)^{-2}((1+c\xi z)/(1-\xi z))^\alpha)\}$. Since $(1-\zeta z)^{-2}((1+c\xi z)/(1-\xi z))^\alpha$ is the derivative of a function in V_k, we also have $\varphi(\xi) \leq \operatorname{Re} L'(f')$ by extremality. As a consequence, the probability measure λ must be supported on a subset of C on which $\varphi(\xi)$ assumes its maximum. Since $\varphi(\xi)$ is harmonic on $\overline{D}$, either λ has finite support or $\varphi(\xi)$ is constant.

If $\varphi(\xi)$ were constant, we would have $L'((1-\zeta z)^{-2} z^\nu) = 0$ for every $\nu = 1, 2, 3, \ldots$, and so

$$L'(\hat{f}') = L'((1-\zeta z)^{-2}(1-\zeta z)^2 \hat{f}'(z)) = L'((1-\zeta z)^{-2})$$

for all $\hat{f}$ in V_k. Thus L would be constant on V_k. To make the argument conclusive, we have to establish that no coefficient c_ν in the expansion of $((1+cz)/(1-z))^\alpha$ vanishes. But this follows immediately from Lemma 2. If $\varepsilon > 0$ is sufficiently small, the function $1 + \varepsilon z^\nu$ is subordinate to $(1+cz)/(1-z)$, and Lemma 2 implies

$$0 \neq \alpha \varepsilon = c_\nu \int_C \xi^\nu d\tilde{\lambda}.$$

Consequently, the support of λ is a finite set $\{\xi_1, \ldots, \xi_N\}$, and (7) becomes

$$f'(z) = (1-\zeta z)^{-2} \sum_{i=1}^N \lambda_i \left(\frac{1+c\xi_i z}{1-\xi_i z} \right)^\alpha.$$

By comparing the singularities in this representation with that from Theorem 1 it follows that $N=1$ and $\xi_1 = \zeta$. This completes the proof.

4. Support points for $2 < k < 4$

For $k=2$ and $k \geq 4$ we now know that support points of V_k and extreme points of $\overline{\operatorname{co}} V_k$ are of the form (3). A similar relation between support points and extreme points for $2 < k < 4$ would be desirable although nothing is known about extreme points in this case. A slightly weaker statement is given by the following theorem.

Theorem 3. *For $2 < k < 4$ a support point of V_k is an extreme point of* $\operatorname{co} V_k$.

Proof. Let f be a support point of V_k and admit a representation as a proper convex combination of two distinct functions g and h in $\operatorname{co} V_k$. Since g and h are themselves finite convex combinations of functions in V_k, it follows that f itself is

a finite convex combination of distinct functions f_j in V_k:

$$f = \sum_{j=1}^{N} \lambda_j f_j, \quad \lambda_j > 0, \quad \sum_{j=1}^{N} \lambda_j = 1.$$

It is no loss of generality to assume that all f_j differ from f.

Since f maximizes the real part of a functional in $H(\boldsymbol{D})^*$, the same is true of each f_j in the convex sum. Therefore Theorem 1 implies

$$f'(z) = (1 - \zeta z)^{-k/2-1} \prod_{\sigma} (1 - \eta_\sigma z)^{\alpha_\sigma}$$

and

$$f_j'(z) = (1 - \zeta_j z)^{-k/2-1} \prod_{\sigma_j} (1 - \eta_{j\sigma_j} z)^{\alpha_{j\sigma_j}}$$

where ζ, $\eta_1, \ldots, \eta_n$ are distinct, ζ_j, $\eta_{j1}, \ldots, \eta_{jn_j}$ are distinct, $|\zeta| = |\zeta_j| = |\eta_\sigma| = |\eta_{j\sigma_j}| = 1$, the α_σ and $\alpha_{j\sigma_j}$ are positive, and

$$\sum_{\sigma} \alpha_\sigma = \sum_{\sigma_j} \alpha_{j\sigma_j} = k/2 - 1 < 1.$$

From a comparison of singularities, it follows immediately that $\zeta_j = \zeta$ for at least one j. By reindexing, we may assume that $\zeta_1 = \zeta$.

Consider the specific form of the functions f' and f_j' in the identity $f' = \sum_{j=1}^{N} \lambda_j f_j'$. Analytic continuation along a small loop around $\bar{\eta}_\sigma$ leads to an identity

$$e^{2\pi i \alpha_\sigma} f'(z) = \sum_{j=1}^{N} \lambda_j e^{2\pi i \delta_j} f_j'(z)$$

for real constants δ_j. For $z = 0$ this implies

$$e^{2\pi i \alpha_\sigma} = \sum_{j=1}^{N} \lambda_j e^{2\pi i \delta_j}$$

and is possible only if $e^{2\pi i \delta_j} = e^{2\pi i \alpha_\sigma}$, $j = 1, \ldots, N$. In particular, since $\zeta_1 \neq \eta_\sigma$, the product $\prod_{\sigma_1} (1 - \eta_{1\sigma_1} z)^{\alpha_{1\sigma_1}}$ must contain the factor $(1 - \eta_\sigma z)^{\alpha_\sigma}$. At this point we have used the fact that the exponents α_σ and $\alpha_{1\sigma_1}$ are strictly between 0 and 1. Since σ is arbitrary, the functions f and f_1 coincide, contrary to assumption, and this completes the proof of the theorem.

Remark. The proof of Theorem 3 does not extend easily to all values of k. For example, corresponding to $k = 6$ one has $f = f_1/2 + f_2/2$ for the functions

$$f'(z) = \frac{(1-iz)(1+iz)}{(1-\zeta z)^4}, \quad f_1'(z) = \frac{(1-z)^2}{(1-\zeta z)^4},$$

$$f_2'(z) = \frac{(1+z)^2}{(1-\zeta z)^4}.$$

On the other hand, for the values of k not covered by Theorem 3 a stronger conclusion is provided by Theorem 2.

5. Linear combinations of a_1 and a_2

In case $2 < k < 4$ the preceding results do not say whether the product in (4) contains several distinct factors (splitting) for some support points or whether only one factor occurs for all support points. We shall consider this question for a relatively simple class of functionals. Originally, we had hoped to find examples for splitting in this class, but we shall see that splitting does not occur in this situation.

Let B_2 be the second coefficient body of V_k. That is, B_2 is the set of pairs (a_2, a_3) where a_2 and a_3 appear as coefficients in the expansion $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ for some function f in V_k. It is known [7] that B_2 is homeomorphic to a closed ball in C^2 and that to each boundary point of B_2 there corresponds only one function in V_k. The representing measure μ of this function is supported by at most 4 points on C, where some trigonometric polynomial of degree 2, say

$$T(e^{i\theta}) = \mathrm{Re}\,\{c_1 e^{i\theta} + c_2 e^{2i\theta}\},$$

reaches its maximum and minimum. The positive part of μ is concentrated at the maxima and the negative part of μ at the minima. Conversely, each T generates a boundary point of B_2 in this way. A quantitative description of the boundary of B_2 has been given also by H. Haario [5].

Now we focus our interest on the support points of B_2. They maximize over B_2 the real part of a functional

$$L = A_2 a_2 + A_3 a_3, \quad A_2, A_3 \in C.$$

By viewing L as a functional on V_k, it follows from Theorem 1 that functions f corresponding to support points B_2 must be of the form

$$f'(z) = (1 - \eta_1 z)^{\alpha_1} \cdot (1 - \eta_2 z)^{\alpha_2} / (1 - \zeta z)^{k/2 + 1},$$

where $\alpha_j \geqq 0$, $\alpha_1 + \alpha_2 = k/2 - 1$, and $\zeta \neq \eta_j$. That is, the positive part of the representing measure is supported by one point and the negative part by at most two points. We express L as a function on M_k by using the representation (1). It yields

$$2a_2 = \int\limits_{|\eta|=1} \eta \, d\mu \quad \text{and} \quad 6a_3 = (2a_2)^2 + \int\limits_{|\eta|=1} \eta^2 \, d\mu.$$

By setting $A_3 = 6(a + ib)$, $A_2 = 2(c + id)$, $a, b, c, d \in R$, and $\eta = e^{i\theta}$, it follows that

$$(8) \qquad \mathrm{Re}\,L = a \left(\int \cos 2\theta \, d\mu + \left(\int \cos \theta \, d\mu \right)^2 - \left(\int \sin \theta \, d\mu \right)^2 \right)$$

$$- b \left(\int \sin 2\theta \, d\mu + 2 \int \cos \theta \, d\mu \cdot \int \sin \theta \, d\mu \right)$$

$$+ c \int \cos \theta \, d\mu - d \int \sin \theta \, d\mu.$$

Let μ_0 maximize $\mathrm{Re}\,L$ over M_k, and put

$$(9) \qquad \varrho = 2 \int \cos \theta \, d\mu_0, \quad \sigma = 2 \int \sin \theta \, d\mu_0.$$

Applying the μ_ε-variation as considered in the framework of Lemma 1, we obtain

$$\int_0^{2\pi} J(e^{i\theta})\, d(\mu-\mu_0) \leqq 0 \quad \text{for all} \quad \mu \in M_k$$

as an extremality condition, where now

(10) $J(e^{i\theta}) = a(\cos 2\theta + \varrho \cos \theta - \sigma \sin \theta) + c \cos \theta - d \sin \theta$

$\qquad\qquad - b(\sin 2\theta + \varrho \sin \theta + \sigma \cos \theta).$

The same argument as used in the proof of Lemma 1 implies that the positive and negative part of μ_0 is located at the points of C where J attains its maximum and minimum, respectively. By what was said a few lines earlier, the positive part of μ_0 has to be concentrated at a single point while its negative part is supported by at most two points of C.

For the remaining part of this investigation we make the assumption that, for our functional L, *the negative part of μ_0 is supported by two distinct points*. This assumption will lead us to some necessary conditions on L that provide a contradiction. Therefore we will come to the conclusion that also the negative part of μ_0 is supported by a single point.

By assumption the indicator function J must reach its minimum at two distinct points. We shall now characterize such functions J. We set $M = \max J$ and $m = \min J$ and apply a lemma of Fejér and F. Riesz saying that a nonnegative trigonometric polynomial is the square of the modulus of an algebraic polynomial in $e^{i\theta}$. Since $M - J$ has a second or fourth order zero, say at $e^{i\theta}$, and is positive elsewhere on C, it follows that

$$M - J(e^{i\theta}) = p|(e^{i\theta}-e^{i\alpha})(e^{i\theta}-w)|^2,$$

where $w = re^{i\beta}$, $0 < r \leqq 1$, and $p > 0$. Similarly, we can write

$$J(e^{i\theta}) - m = q\,|(e^{i\theta}-e^{i\gamma})(e^{i\theta}-e^{i\delta})|^2$$

with $q > 0$ because $J - m$ is positive on C except for two distinct double zeros, say $e^{i\gamma}$ and $e^{i\delta}$. Taking derivatives, we find

(11) $\dfrac{\partial J}{\partial \theta}(e^{i\theta}) = -2p\big((1+r^2)\sin(\theta-\alpha) + 2r\sin(\theta-\beta)$

$\qquad\qquad\qquad - 2r\sin(2\theta-\alpha-\beta)\big)$

and

(12) $\dfrac{\partial J}{\partial \theta}(e^{i\theta}) = 4q\big(\sin(\theta-\gamma) + \sin(\theta-\delta) - \sin(2\theta-\gamma-\delta)\big).$

Since the right hand sides are identical in θ, they remain so if the parameters α, β, γ, δ are decreased by a constant τ (that corresponds to replacing θ by $\theta+\tau$). For

$\tau=(\gamma+\delta)/2$ we have $\gamma+\delta=0$ without loss of generality. Preserving for the other new parameters the old notations α and β, we find that equation (12) reduces to

$$\frac{\partial J}{\partial \theta}(e^{i\theta}) = -4q\big(\sin 2\theta - \sin(\theta-\gamma) - \sin(\theta+\gamma)\big).$$

By comparing coefficients with the revised (11), it follows that $rp=q$, $\alpha+\beta=\pi$, $p(1+r)^2 \sin\alpha=0$, and $2p(1-r)^2 \cos\alpha=-4q\cdot 2\cos\gamma$. This implies $\sin\alpha=0$, and by a further rotation, we may set $\alpha=0$; hence

$$\cos\gamma = -\frac{(1-r)^2}{4r}.$$

This equation restricts the possible values of r to the interval $(3-2\sqrt{2}, 1]$ and $\cos\gamma$ to $(-1, 0]$. Thus, up to a rotation, a three-point support for μ_0 is of the following form: The point $\eta=1$ supports the positive part of μ_0 and two points $e^{i\gamma}$ and $e^{-i\gamma}$, where $\pi/2 \leq \gamma < \pi$, support the negative part of μ_0. (Observe that $\gamma=\pi/2$ is the limiting case corresponding to $r=1$. Here there are two absolute maxima, at $\theta=0$ and $\theta=\pi$, but only one bears the whole mass $k/2+1$.) The indicator J is then given by

(13) $J(e^{i\theta}) = 2q(\cos 2\theta - 4\cos\gamma \cdot \cos\theta).$

If μ_0 is an extremal measure for the functional $L(f)=A_2a_2+A_3a_3$, then $\tilde{\mu}_0$ defined by $d\tilde{\mu}_0(\theta)=d\mu_0(\theta+\tau)$ is an extremal measure for the similar functional $\tilde{L}(f)=A_2e^{-i\tau}a_2+A_3e^{-2i\tau}a_3$. Therefore we may restrict our attention to a functional L having for its extremal measure μ_0 the support at $\eta=1$ for the positive part and at $\eta=e^{\pm i\gamma}$ for its negative part. Comparison of (13) to (10) implies

(14) $b=0, \quad a\sigma+d=0, \quad a\varrho+c=-4\cos\gamma$

immediately, and since $q>0$ we may further normalize $a=1$. Thus (8) reduces to

$$\operatorname{Re} L = \int \cos 2\theta \, d\mu + \left(\int \cos\theta \, d\mu\right)^2 - \left(\int \sin\theta \, d\mu\right)^2$$
$$+ c \int \cos\theta \, d\mu - d \int \sin\theta \, d\mu.$$

This functional has to be maximized by a measure μ_0 which has its positive part located at $\eta=1$ and its negative part spread over the two points $e^{\pm i\gamma}$ with weights both nonvanishing.

Let $\operatorname{Re} L = \Phi_1 + \Phi_2$ where

$$\Phi_1(\mu) = \int \cos 2\theta \, d\mu + \left(\int \cos\theta \, d\mu\right)^2 + c \int \cos\theta \, d\mu,$$

$$\Phi_2(\mu) = -\left(\int \sin\theta \, d\mu\right)^2 - d \int \sin\theta \, d\mu,$$

and let

$$y(\mu) = \int \sin\theta \, d\mu.$$

By (14) and (9) the choice of γ within $[\pi/2, \pi)$ uniquely determines c, ϱ, and $\Phi_1(\mu_0)$;

this is because these quantities do not depend on how the negative part of μ_0 is spread over $\{e^{i\gamma}, e^{-i\gamma}\}$. Suppose that the negative part of μ_0 has mass $k/2-1-t$ and t at these points, where $0<t<k/2-1$. Within this range we have to determine t so that $\Phi_2(\mu_0)$ is a maximum under the condition $\sigma+d=0$. Obviously, we have $\Phi_2(\mu_0) \leq d^2/4$ and equality holds if and only if $y(\mu_0)=-d/2$, in which case the condition $\sigma=-d$ implies $\Phi_2(\mu_0)=y(\mu_0)^2$. Consequently, because $y(\mu_0)= (2t+1-k/2)\sin\gamma$, the expression $\Phi_2(\mu_0)$ is maximized only for $t=0$ or $t=k/2-1$. This contradicts the assumption that the negative part of μ_0 is located at two distinct points. Thus we have finally proved that a measure corresponding to a support point of B_2 has only one negative mass and one positive mass.

Theorem 4. *For* $k \geq 2$ *each functional* $\mathrm{Re}\,\{A_2 a_2 + A_3 a_3\}$, $A_2, A_3 \in C \setminus \{0\}$, *achieves its maximum within* V_k *only for functions with* f' *of the form* (3).

Of course, if A_2 or A_3 is zero (but not both), the conclusion of Theorem 4 remains true.

Based on the limited information from Theorem 4 it seems reasonable to. conjecture that support points of V_k remain of the form (3) even for $2<k<4$.

6. The functional $\mathrm{Re}\,\{a_3 + i\varrho a_2\}$

Although Theorem 4 does not actually provide the maximum of $\mathrm{Re}\,\{A_2 a_2 + A_3 a_3\}$ in closed form, it reduces the problem to an easy one for a small computer. One needs only to find the maximum of a function of two variables.

If A_3 is a positive multiple of A_2^2, then this maximum occurs for a rotation of the function $K(z)$ in (2). A less trivial example is furnished by the functional

Table for $M = \max_{V_k} \mathrm{Re}\,\{a_3 + i\varrho a_2\}$

	$k=2$		$k=3$			$k=4$			$k=5$		
ϱ	φ	M	φ	ψ	M	φ	ψ	M	φ	ψ	M
0	180	1	180	0	1.833	180	0	3	180	0	4.5
1	194.5	1.125	189.5	33.7	2.012	187.4	20.4	3.186	186.2	14.2	4.687
2	210.0	1.500	200.9	51.0	2.506	195.8	37.1	3.727	192.9	27.3	5.242
3	228.6	2.125	213.9	62.5	3.285	205.4	49.9	4.595	200.3	39.0	6.150
4	270	3	229.8	72.3	4.341	216.5	60.2	5.773	208.7	49.2	7.395
5	270	4	256.4	84.5	5.669	230.2	69.7	7.247	218.5	58.5	8.965
6	270	5	270	90	7.167	250.9	81.0	9.014	230.3	67.4	10.849
7	270	6	270	90	8.667	270	90	11	247.0	77.8	13.040
8	270	7	270	90	10.167	270	90	13	270	90	15.5
>8	270	$\varrho-1$	270	90	$3\varrho/2-11/6$	270	90	$2\varrho-3$	270	90	$5\varrho/2-9/2$

Re $\{a_3 + i\varrho a_2\}$ for ϱ real. We computed the maximum value of this functional over V_k for several values of ϱ and k. There are at most two functions of the form (3) for which the maximum occurs. We have recorded the values φ and ψ in degrees at which the positive and negative parts, respectively, of the representing measure μ are concentrated; that is, $\zeta = \exp i\pi\varphi/180$ and $\eta = \exp i\pi\psi/180$. For the second solution one replaces φ by $180 - \varphi$ and ψ by $180 - \psi$.

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