ON PSEUDO-MONOTONE OPERATORS
AND NONLINEAR PARABOLIC INITIAL-BOUNDARY
VALUE PROBLEMS ON UNBOUNDED DOMAINS

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1. Introduction

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^N$ ($N\geq 1$) and let $Q$ be the cylinder $\Omega \times (0, T)$ with a given $T>0$. We shall consider on $Q$ the quasilinear parabolic partial differential operator of order $2m$ ($m\geq 1$) of the form

\begin{equation}
\frac{\partial u(x, t)}{\partial t} + Au(x, t),
\end{equation}

where $A$ is an elliptic operator given in the divergence form

\begin{equation}
Au(x, t) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u, Du, ..., D^n u).
\end{equation}

The coefficients $A_\alpha$ are regarded as real-valued functions of the point $(x, t)$ in $Q$, of $\eta=\left\{\eta_\beta: |\beta|\leq m-1\right\}$ in $\mathbb{R}^N_1$ and of $\zeta=\left\{\zeta_\gamma: |\gamma|=m\right\}$ in $\mathbb{R}^N_2$, where $\alpha=(\alpha_1, ..., \alpha_N)$ and $\beta=(\beta_1, ..., \beta_N)$ are $N$-tuples of nonnegative integers, $|\beta|=\beta_1+...+\beta_N$ and $D^\alpha=\prod_{i=1}^N (\partial/\partial x_i)^{\alpha_i}$.

If we assume that the functions $A_\alpha$ satisfy the familiar condition

(A1) Each $A_\alpha(x, t, \eta, \zeta)$ is measurable in $(x, t)$ for fixed $\zeta=(\eta, \zeta)$ and continuous in $\zeta$ for fixed $(x, t)$. For a given $p>1$ there exists a constant $c_1>0$ and a function $k_1\in L^p(\Omega)$ with $p'=p/(p-1)$ such that

\[ |A_\alpha(x, t, \eta, \zeta)| \leq c_1(|\zeta|^{p-1}+|\eta|^{p-1}+k_1(x, t)) \]

for all $|\alpha|\leq m$, all $(x, t)\in Q$ and all $\zeta=(\eta, \zeta)\in \mathbb{R}^{N_1+N_2}=\mathbb{R}^N$,

then the operator $A$ gives rise to a bounded map $S$ from the space $\mathfrak{V}^*=L^p(0, T; V)$ to its dual space $\mathfrak{V}^*$, $V$ being a closed subspace of the Sobolev space $W^{m,p}(\Omega)$.

When $\Omega$ is a bounded domain, the operator $\partial/\partial t$ induces a maximal monotone map $L$ from the subset $D(L)=\{v\in \mathfrak{V}: \partial v/\partial t\in \mathfrak{V}^*, v(x, 0)=0 \text{ in } \Omega\}$ to $\mathfrak{V}^*$, and

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from a simple set of additional hypotheses of the Leray—Lions type it can be derived that $S$ is pseudo-monotone on $D(L)$. This result is then applicable to the existence of weak solutions for the parabolic initial-boundary value problems for the operator (1).

When $\Omega$ is unbounded, the situation is different while the compactness part of the Sobolev embedding theorem and Rellich's selection theorem are no more available and the above definition of the set $D(L)$ does not make sense.

The purpose of the present note is to show, for arbitrary domains $\Omega$, that the map $S$ induced by the elliptic operator $A$ is pseudo-monotone as a map from the space $\mathcal{V} = \mathcal{V} \cap L^2(\Omega)$ to $\mathcal{V}^*$ on the set $D(L) = \{v \in \mathcal{V}; \partial v / \partial t \in \mathcal{V}^*, v(x, 0) = 0 \text{ in } \Omega\}$ whenever the coefficients $A_x$ satisfy the following conditions (cf. [5] p. 323, [6]) in addition to (Ar):

A$_3$) For each $(x, t) \in \Omega$, each $\eta \in \mathbb{R}^n$ and any pair of distinct elements $\zeta$ and $\zeta^*$ in $\mathbb{R}^n$, 

$$
\sum_{|\alpha| = m} \{A_x(x, t, \eta, \zeta) - A_x(x, t, \eta, \zeta^*)\}(\zeta_x - \zeta_x^*) > 0.
$$

A$_3$) There exist a constant $c_2 > 0$ and functions $k_2 \in L^1(\Omega)$, $h_2 \in L^p(\Omega)$ for all $|\alpha| \equiv m$, such that 

$$
\sum_{|\alpha| = m} A_x(x, t, \zeta) \zeta_x \equiv - \sum_{|\alpha| = m} h_2(x, t) \zeta_x - k_2(x, t)
$$

for all $(x, t) \in \Omega$ and all $\zeta \in \mathbb{R}^n$.

This result is analogous to the elliptic case studied by F. E. Browder [1]. In fact, our method here is a modification of the method introduced by R. Landes and V. Mustonen [4], which makes it possible to relax one of the classical conditions imposed on the coefficients $A_x$.

The result of pseudo-monotonicity can be applied to the variational problems for the operator (1) involving a domain which is not necessarily bounded. As an example we shall show that the partial differential equation

$$
\frac{\partial u}{\partial t} + Au = f \text{ in } Q
$$

with the initial-boundary conditions

$$
\begin{align*}
u(x, 0) &= 0 \quad \text{in } \Omega \\
D^* u &= 0 \quad \text{on } \partial \Omega \times (0, T) \quad \text{for } |\alpha| \equiv m - 1
\end{align*}
$$

admits a solution $u$ for any given $f$ in $L^p(\Omega)$. Under similar conditions (a condition stronger than our (A$_3$) was needed) this existence theorem was also proved by G. Mahler [6] by an ad hoc approximation method which was originally introduced by P. Hess [2] for elliptic Dirichlet problems.
2. Prerequisites

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The Sobolev space of functions \( u \) such that \( u \) and its distributional derivatives \( D^su \) lie in \( L^p(\Omega) \) for all \( |x| \leq m \) is denoted by \( W^{m,p}(\Omega) \). By \( W^{m,p}_0(\Omega) \) we mean the closure in \( W^{m,p}(\Omega) \) of \( C^\infty_0(\Omega) \), the space of test functions with compact support in \( \Omega \). If \( u \in W^{m,p}(\Omega) \), we shall write \( \eta(u) = \{ D^su: |x| \leq m-1 \} \), \( \zeta(u) = \{ D^su: |x| = m \} \) and \( \xi(u) = \{ D^su: |x| \leq m \} \). When \( T > 0 \) is given and \( V \) is a closed subspace of the Sobolev space \( W^{m,p}(\Omega) \), we denote \( \mathscr{V} = L^p(0, T; V) \), a Banach space equipped with the norm

\[
\| u \|_\mathscr{V} = \left\{ \int_0^T \| u(t) \|_V^p \, dt \right\}^{1/p}.
\]

We let further \( \mathscr{W} \) stand for the Banach space \( \mathscr{V} \cap L^2(\Omega) \) with \( Q = \Omega \times (0, T) \) and with the norm \( \| \cdot \|_\mathscr{W} = \| \cdot \|_\mathscr{V} + \| \cdot \|_{L^2(\Omega)} \).

The duality pairing between the elements \( u \) in a Banach space \( X \) and \( f \) in \( X^* \) is denoted by \( (f, u)_X \), where the subscript \( X \) will be omitted when no confusion is possible. If \( 1 < p < \infty \), \( \mathscr{W} \) is reflexive and its dual space is \( \mathscr{W}^* = L^p(0, T; V^*) \), where \( \mathscr{V}^* = L^p(0, T; V^*) \). Furthermore,

\[
\mathscr{W} \subset L^2(\Omega) \subset \mathscr{W}^* \subset L^1(0, T; V^* + L^2(\Omega));
\]

for each \( u \in \mathscr{W} \) the distribution derivative \( u' = \partial u/\partial t \) can be defined and the condition \( u' \in \mathscr{W}^* \) makes sense. Each \( u \in \mathscr{W} \) with \( u' \in \mathscr{W}^* \) is (after a modification on a set of measure zero) a continuous function, \( [0, T] \to L^2(\Omega) \) and the following integration formula holds (see [6], [7]) for all \( u, v \in \mathscr{W} \) with \( u', v' \in \mathscr{W}^* \):

\[
(u', v)_{\mathscr{W}} + (v', u)_{\mathscr{W}} = (u(T), v(T))_{L^2(\Omega)} - (u(0), v(0))_{L^2(\Omega)}.
\]

Let \( L \) stand for the linear map from \( \mathscr{W} \) to \( \mathscr{W}^* \) which takes \( u \) to \( u' \) having the domain

\[
D(L) = \{ u \in \mathscr{W}: u' \in \mathscr{W}^*, u(x, 0) = 0 \text{ in } \Omega \}.
\]

It follows from (5) that \( (Lu, u) \geq 0 \) for all \( u \in D(L) \). Thus \( L \) is a monotone linear map.

We close this section by recalling the definition of a pseudo-monotone map and an abstract surjectivity result which we will employ in proving the existence theorem in Section 4. Indeed, Theorem 1.2 of [5] p. 319 can be stated as follows:

**Proposition 1.** Let \( X \) be a reflexive Banach space with strictly convex norms in \( X \) and \( X^* \). Let \( L \) be a linear maximal monotone map from \( D(L) \) to \( X^* \) with \( D(L) \) dense in \( X \), let \( T \) be a bounded map from \( X \) to \( X^* \), and suppose that \( T \) is \( D(L) \)-pseudo-monotone, i.e. for any sequence \( (v_n) \subset D(L) \) with \( v_n \rightharpoonup v \) (weak convergence) in \( X \), \( L v_n \rightharpoonup L v \) in \( X^* \) and \( \limsup (T(v_n), v_n - v) \leq 0 \), it follows that \( T(v_n) \rightharpoonup T(v) \) in \( X^* \) and \( (T(v_n), v_n) \rightarrow (T(v), v) \). If \( T \) is coercive on \( X \), i.e. \( (T(u), u) \uparrow \infty \) as \( \| u \| \rightarrow \infty \) in \( X \), then for any \( f \in X^* \) there is \( u \in D(L) \) such that \( Lu + T(u) = f \).
3. Theorem on pseudo-monotonicity

Let us assume that the coefficients $A_x$ of the operator (2) satisfy the conditions $(A_1)$, $(A_2)$ and $(A_3)$ in the given domain $Q = \Omega \times (0, T)$. On account of $(A_1)$ the equation

$$a(u, v) = \sum_{|x| \leq m} \int_Q A_x(x, t, \xi(v)) D^x v \, dx \, dt$$

defines a bounded semilinear form on $\mathcal{V} \times \mathcal{V}$. Hence (6) gives rise to a bounded (nonlinear) map $S$ from $\mathcal{V}$ to $\mathcal{V}^*$ by the rule

$$(S(u), v) = a(u, v), \quad u, v \in \mathcal{V}.$$  

In view of $(A_2)$ and $(A_3)$ it is clear that $\mathcal{V}$ would be the natural space for the mapping $S$ but, on the other hand, the map $S$ is defined on the subset $D(L) \subset \mathcal{V} \subset \mathcal{V}$ only, with values in $\mathcal{V}^*$. Therefore we shall regard $S$ as a map from $\mathcal{V}$ to $\mathcal{V}^*$ and prove Theorem 1.

**Theorem 1.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^N$, $T>0$, $Q=\Omega \times (0, T)$ and let the functions $A_x$ satisfy the conditions $(A_1)$, $(A_2)$ and $(A_3)$. Then the map $S$ from $\mathcal{V}$ to $\mathcal{V}^*$ defined by (7) is $D(L)$-pseudo-monotone.

**Proof.** We can follow the lines of the proof of the elliptic case in [4]. Indeed, let $(v_n) \subset D(L)$ be a sequence such that $v_n \rightharpoonup v$ in $\mathcal{V}$, $L(v_n) \rightharpoonup Lv$ in $\mathcal{V}^*$ and $\lim \sup (S(v_n), v_n - v) \leq 0$. We must verify that $S(v_n) \rightharpoonup S(v)$ in $\mathcal{V}^*$ and that $(S(v_n), v_n) \rightharpoonup (S(v), v)$, at least for an infinite subsequence of $(v_n)$. As $v_n \rightharpoonup v$ in $\mathcal{V}$, $D^x v_n \rightharpoonup D^x v$ in $L^p(Q)$ for all $|x| \leq m$ and $v_n - v$ in $L^2(Q)$. Our aim is to show that $D^x v_n(x, t) \rightharpoonup D^x v(x, t)$ almost everywhere in $Q$ for all $|x| \leq m$ for some subsequence. By $(A_2)$ this implies that $A_x(x, t, \xi(v_n)) \rightharpoonup A_x(x, t, \xi(v))$ a.e. in $Q$ for all $|x| \leq m$. By $(A_3)$ this also means that $A_x(x, \xi(v_n)) \rightharpoonup A_x(x, \xi(v))$ in $L^p(Q)$, and thus $S(v_n) \rightharpoonup S(v)$ in $\mathcal{V}^*$ follows. The a.e. convergence of $D^x v_n(x, t)$ to $D^x v(x, t)$ for all $|x| \leq m - 1$ is established by Aubin’s Lemma ([5] p. 57). Indeed, $W^{m,p}(\Omega)$ is compactly embedded in $W^{m-1,p}(\omega)$ for any subdomain $\omega$ with a compact closure in $\Omega$. Thus $v_n \rightharpoonup v$ in $\mathcal{V}$ and $L(v_n) \rightharpoonup Lv$ in $\mathcal{V}^*$ together imply (cf. [6] p. 205) that $v_n - v$ (strongly) in $L^p(0, T; W^{m-1,p}(\omega))$, i.e. $D^x v_n \rightharpoonup D^x v$ in $L^p(\omega \times (0, T))$ for all $|x| \leq m - 1$, and the a.e. convergence for a subsequence follows.

To verify that $D^x v_n(x, t) \rightharpoonup D^x v(x, t)$ a.e. in $Q$ also for all $|x|=m$ we denote

$$q_n(x, t) = \sum_{|x| = m} \left\{ A_x(x, t, \eta(v_n)) - A_x(x, t, \eta(v_n), \xi(v)) \right\} (D^x v_n - D^x v),$$

$$p_n(x, t) = \sum_{|x| = m} A_x(x, t, \xi(v_n)) (D^x v_n - D^x v),$$

$$r_n(x, t) = \sum_{|x| = m} A_x(x, t, \eta(v_n), \xi(v)) (D^x v - D^x v_n),$$

$$s_n(x, t) = \sum_{|x| = m-1} A_x(x, t, \xi(v_n)) (D^x v - D^x v_n).$$
Then \( q_n = p_n + r_n + s_n \) in \( Q \). If we can show that \( q_n(x, t) \to 0 \) a.e. in \( Q \), then the desired result follows from Lemma 6 due to R. Landes [3]. In fact, as \( q_n(x, t) \equiv 0 \) for almost all \( (x, t) \in Q \) by (A₂), it suffices to show that

(8) \[ \lim \sup_{Q_k} \int q_n(x, t) \, dx \, dt \equiv \varepsilon_k, \]

where \( Q_k = \Omega_k \times (0, T) \), \( (\Omega_k) \) is a growing sequence of bounded subdomains of \( \Omega \) such that \( \mu(\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k) = 0 \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \). For any fixed \( k \) we have

\[
\int_{Q_k} q_n(x, t) = \int_{\Omega} p_n(x, t) - \int_{Q_k} p_n(x, t) + \int_{Q_k} (r_n(x, t) + s_n(x, t)),
\]

where we know by assumption that \( \lim \sup \int_{\Omega} p_n(x, t) \equiv 0 \). Moreover, since \((D^2 v_n)\) is bounded in \( L^p(Q)\) and \((A_z(\cdot, \cdot, \xi(v_n)))\) is bounded in \( L^p(Q)\), we get by (A₃),

\[
- \int_{Q_k} p_n(x, t) = - \sum_{|z| \leq m} \int_{Q_k} A_z(x, t, \xi(v_n)) D^2 v_n.
\]

Moreover, \( A_z(\cdot, \xi(v_n)) \) in \( L^p(Q) \) for all \( |z| = m \) and since \( A_z(\cdot, \xi(v_n), \xi(v)) \to A_z(\cdot, \eta(v), \xi(v)) \) in \( L^p(Q_k) \) for all \( |z| = m \) by (A₁) and the dominated convergence theorem, we can conclude that

\[
\lim \sup_{Q_k} \int (r_n(x, t) + s_n(x, t)) \equiv 0
\]

for any fixed \( k \). Thus (8) has been verified.

We complete the proof by showing that \( (S(v_n), v_n) \to (S(v), v) \). In view of the assumption, \( \lim \sup (S(v_n), v_n) \equiv (S(v), v) \). Hence it suffices to prove that

(9) \[ \lim \inf (S(v_n), v_n) \equiv (S(v), v). \]

By (A₃) we have

\[
\sum_{|z| = m} A_z(x, t, \xi(v_n)) D^2 v_n \equiv \sum_{|z| = m} A_z(x, t, \eta(v_n), \xi(v))(D^2 v_n - D^2 v) + \sum_{|z| = m} A_z(x, t, \xi(v_n)) D^2 v,
\]
and hence we get further for any fixed $k$,
\[
\sum_{|x| \leq m} \int_{Q} A_{x}(x, t, \xi(v_n)) D^2 v_n \equiv \sum_{|x| = m} \int_{Q_h} A_{x}(x, t, \eta(v_n), \zeta(v)) (D^2 v_n - D^2 v) \\
+ \sum_{|x| = m} \int_{Q_h} A_{x}(x, t, \xi(v_n)) D^2 v + \sum_{|x| \leq m-1} \int_{Q_h} A_{x}(x, t, \xi(v_n)) D^2 v_n \\
+ \sum_{|x| \leq m} \int_{Q_h} A_{x}(x, t, \xi(v_n)) D^2 v_n.
\]

By the same arguments as above in proving (8) we obtain
\[
\lim \inf \sum_{|x| \leq m} \int_{Q} A_{x}(x, t, \xi(v_n)) D^2 v_n \equiv \sum_{|x| \leq m} \int_{Q} A_{x}(x, t, \xi(v)) D^2 v - \varepsilon_k,
\]
where $\varepsilon_k \to 0$ as $k \to \infty$, and so the proof is complete.

4. Parabolic initial-boundary value problem

We shall employ Theorem 1 and Proposition 1 for obtaining an existence theorem for the parabolic equation (3) with initial-boundary conditions (4). Indeed, we can choose $V = W^{m,p}_0(\Omega)$ to obtain

**Theorem 2.** Let $\Omega$ be an arbitrary domain in $\mathbb{R}^N$, $T > 0$, $Q = \Omega \times (0, T)$ and let the functions $A_x$ satisfy the conditions $(A_1)$, $(A_2)$ and $(A_3)$. If $a(u, u) \|u\|^{-1} \to \infty$ for all $u \in \mathcal{V}$ with $\|u\|_{\mathcal{V}} \to \infty$, then for any $f \in L^p(Q) + L^q(Q)$ the equation (3) admits a weak solution $u$ in $D(L)$, i.e. there exists $u \in D(L)$ such that

\[
(\frac{\partial u}{\partial t}, w)_{\mathcal{V}} + a(u, w) = (f, w)_{\mathcal{V}} \quad \text{for all } w \in \mathcal{V}.
\]

**Proof.** First we remark that by assumption the map $S$ defined by (6) is coercive on $\mathcal{V}$, although not necessarily on $\mathcal{W}$. Therefore, to invoke Proposition 1 with $X = \mathcal{V}$ we perform the substitution $u = e^{kt} v$ with $k$ a positive constant, as suggested in [6]. Then we obtain from (3) the equation

\[
\frac{\partial v}{\partial t} + \tilde{A} v + kv = \tilde{f} \quad \text{in } Q
\]

with the initial-boundary conditions (4), where

\[
\tilde{A} v(x, t) = \sum_{|x| \leq m} (-1)^{|x|} D^2 \tilde{A}_x(x, t, \xi(v)),
\]

\[
\tilde{A}_x(x, t, \xi) = e^{-kt} A_x(x, t, e^{kt} \xi) \quad \text{and } \tilde{f} = e^{-kt} f.
\]

It is obvious that the functions $\tilde{A}_x$ also satisfy the conditions $(A_1)$ to $(A_3)$ with a new constant $\bar{c}_1$ and new functions $\bar{h}_x \in L^p(\Omega)$. Now we define a map
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\[ S_1 = \mathcal{S} + S_0 : \mathcal{W} \to \mathcal{W}^* \]

by

\[ (\mathcal{S}(v), w) = a(v, w) = \sum_{|z|=m} \int_0^T \tilde{A}_z(x, t, \xi(v)) D^z w \, dx \, dt \]

and

\[ (S_0(v), w) = k \int_0^T vw \, dx \, dt. \]

On account of Theorem 1, \( \mathcal{S} \) is \( D(L) \)-pseudo-monotone. It is easily seen that also the sum \( S_1 = \mathcal{S} + S_0 \) is \( D(L) \)-pseudo-monotone. Moreover, \( S_1 \) is coercive on \( \mathcal{W} \). Indeed, by the assumption of the theorem, \( a(v, v) \| v \|^{-1} \to \infty \) and therefore

\[ \frac{(S_1(v), v)}{\| v \|_{\mathcal{W}}} = \frac{a(v, v) + k \| v \|_{L^2(Q)}}{\| v \|_{\mathcal{W}} + \| v \|_{L^2(Q)}} \to \infty \quad \text{as} \quad \| v \|_{\mathcal{W}} \to \infty. \]

For applying Proposition 1 we must finally verify that the map \( L = \partial \| dt : D(L) \to \mathcal{W}^* \)

is maximal monotone and that \( D(L) \) is dense in \( \mathcal{W} \). By Lemma 1.2 of [5] p. 313 it suffices to show that for any \( v \in \mathcal{W} \) and \( w \in \mathcal{W}^* \) such that

\[ 0 \equiv (w - Lu, v - u) \quad \text{for all} \quad u \in D(L) \]

it follows that \( v \in D(L) \) and \( w = Lv \). To show this let \( v \in \mathcal{W} \) and \( w \in \mathcal{W}^* \) be given, let \( \varphi \in C_0^\infty(0, T) \) and \( \bar{u} \in V \cap L^2(\Omega) \) be arbitrary and let \( u = \varphi \bar{u} \). Then \( u \in \mathcal{W} \), \( u(0) = u(T) = 0 \), \( u' = \varphi' \bar{u} \in L^2(Q) \subset \mathcal{W}^* \) and hence \( u \in D(L) \). Since \( (Lu, u) = 0 \), we get from (11) by (5)

\[ 0 \equiv (w, v) - (\varphi' \bar{u}, v) - (w, \varphi \bar{u}), \]

where

\[ (\varphi' \bar{u}, v) = \left( \int_0^T \varphi'(t) v(t) \, dt, \bar{u} \right)_{L^2(\Omega)} \]

and

\[ (w, \varphi \bar{u}) = \left( \int_0^T \varphi(t) w(t) \, dt, \bar{u} \right)_{L^2(\Omega)}, \]

while

\[ \int_0^T \varphi'(t) v(t) \, dt \in L^2(\Omega) \subset V^* + L^2(\Omega) \quad \text{and} \quad \int_0^T \varphi(t) w(t) \, dt \in V^* + L^2(\Omega). \]

Consequently, for all \( \bar{u} \in V \cap L^2(\Omega) \),

\[ 0 \equiv (w, v) - \left( \int_0^T \varphi'(t) v(t) \, dt + \int_0^T \varphi(t) w(t) \, dt, \bar{u} \right)_{L^2(\Omega)} \]

implying that

\[ \int_0^T \varphi'(t) v(t) \, dt = - \int_0^T \varphi(t) w(t) \, dt \]

for all \( \varphi \in C_0^\infty(0, T) \), i.e. \( w = v' = Lv \). The fact that \( v(0) = 0 \) implying \( v \in D(L) \) follows from (5) and (11) by standard argument (see e.g. [8] p. 176).
Now we are in a position to employ Proposition 1 to establish the existence of an element $v$ in $D(L)$ such that

$$\left( \frac{\partial v}{\partial t}, w \right)_w + a(v, w) + k(v, w)_{L^2(\Omega)} = (f, w)_w$$

for all $w \in \mathcal{W}$, for any given $f \in \mathcal{W}^*$. Reversing the substitution we get (cf. [6])

$$\left( \frac{\partial u}{\partial t}, e^{-kt} w \right)_w + a(u, e^{-kt} w) = (f, e^{-kt} w)_w$$

for all $w \in \mathcal{W}$, which implies that (10) holds. Since also $u \in D(L)$, the proof is complete.

Remark. It is clear from the proof of Theorem 1 that the Dirichlet null boundary condition in (4) can be replaced by any boundary condition associated to $V$ with $W^{m,p}_0(\Omega) \subset V \subset W^{m,p}(\Omega)$. On the other hand, it can be shown that (3) admits also a periodic solution $u(0) = u(T)$ if one chooses

$$D(L) = \{ u \in \mathcal{W}^* : u' \in \mathcal{W}^*, \ u(0) = u(T) \}.$$

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