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ON MAXIMIZING A HOMOGENEOUS FUNCTIONAL IN THE CLASS OF BOUNDED UNIVALENT FUNCTIONS

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1. Introduction

In [4] the functional $\operatorname{Re}\delta = \operatorname{Re}(a_3 - a_2^2)$ is estimated in a_2 and b in the class $S(b)$ of bounded univalent functions f:

$$f(z) = b(z + a_2 z^2 + \ldots),\ |z| < 1,$$
$$|f(z)| < 1, \quad 0 < b \leqq 1.$$

The inequality obtained is sharp and holds for all values of b and a_2 (cf. (9)/V.1.2, (25)/V.1.3 and (46)/V.1.4 of [4]).

By adding to both sides of the estimate a real functional of a_2, we find new combinations of a_3 and a_2 for which the sharp inequality remains valid. Therefore, when maximizing the upper bounds in a_2 we obtain the sharp upper bound for our functionals in b. The maximizing a_2 must lie in the sharpness region of the inequality used. Because the original inequalities for $\operatorname{Re}\delta$ are connected with the totality of boundary functions of the coefficient body (a_2, a_3), we are sure that also the new functional formed reaches its maximum in the range of values of the upper bound available.

In [1] and [4] the above method of direct estimation was applied to $\operatorname{Re}(a_3 + \lambda a_2)$. In the present paper we shall show that the upper bounds to be obtained for $\operatorname{Re}(a_3 + \lambda a_2^2)$ lead to results which are not yet too complicated to handle by this method.

The maxima found hold, of course, also for $|a_3 + \lambda a_2^2|$. The results agree with those obtained by variational method by the first author [2]. In S the corresponding results are derived by Szwankowski [3]. The analysis of the extremal functions in [4] allows completing the sharp estimation for all values of λ.

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2. The equality functions 2:2

Let us start from the condition (9)/V.1.2 of [4]:

$$\mathrm{Re}\,(a_3-a_2^2) \leqq 1-b^2+\frac{(\mathrm{Re}\,a_2)^2}{\log b}. \tag{1}$$

For any $\mu=\mu_1+i\mu_2\in C$, $\mu_1\neq 0$, $a_2=U+iV$, we obtain

$$\begin{aligned}\mathrm{Re}\left(a_3+(\mu-1)a_2^2\right)-(1-b^2) &\leqq \mathrm{Re}\,(\mu a_2^2)+\frac{(\mathrm{Re}\,a_2)^2}{\log b}\\ &= -\frac{1}{\mu_1}\left[(\mu_1 V+\mu_2 U)^2-\left(\mu_1^2+\mu_2^2+\frac{\mu_1}{\log b}\right)U^2\right].\end{aligned} \tag{1$'$}$$

This allows a sharp estimation in two cases. Assume first that

$$\text{1)}\qquad \begin{cases}\mu_1>0,\\ \mu_1^2+\mu_2^2+\dfrac{\mu_1}{\log b}<0 \Leftrightarrow \left(\mu_1+\dfrac{1}{2\log b}\right)^2+\mu_2^2<\dfrac{1}{4\log^2 b}.\end{cases}$$

Under the assumptions 1)

$$\mathrm{Re}\left(a_3+(\mu-1)a_2^2\right) \leqq 1-b^2. \tag{2}$$

Equality holds here only if $U=V=0$. The equality conditions of (1) are given by (11)—(12)/V.1.2 of [4] which in the case $U=0$ imply that

$$\cos\vartheta \equiv 0$$

and, because $V=0$,

$$\int_b^1 \sin\vartheta\, du = 0.$$

In the disc

$$D=\left\{\mu\in C \,\middle|\, |\mu|^2+\frac{\mu_1}{\log b}<0\right\}$$

the estimation (2) thus holds and the extremal domains are the same at every point of D: two-radial-slit domains with slits symmetrically located along the imaginary axis.

Assume next that

$$\text{2)}\qquad \begin{cases}\mu_1>0,\\ \mu_1^2+\mu_2^2+\dfrac{\mu_1}{\log b}=0 \Leftrightarrow \mu\in\partial D.\end{cases}$$

Under these assumptions the estimation (2) remains valid, and the extremal functions consist of a one-parametric family for which

(3) $$V = -\frac{\mu_2}{\mu_1} U, \quad \mu_2 = \sqrt{\frac{\mu_1}{|\log b|} - \mu_1^2}$$

with U as a parameter.

Let us use the normalizations

(4) $$\mu_2 > 0, \quad U \leqq 0.$$

From the Löwner-expression

(5) $$\operatorname{Re}(a_2 + (\mu - 1) a_2^2) = \mu_1 (U^2 - V^2) - 2\mu_2 UV - 2 \int_b^1 u \cos 2\vartheta \, du$$

we then see that in the maximum case

(6) $$V \geqq 0$$

and in the UV-plane we are in the second quadrant. The equality conditions of (1), given in (11)—(12)/V.1.2 of [4], imply that in the case (4) and (6), $a_2 \in \mathrm{I}$ (Figure 1).

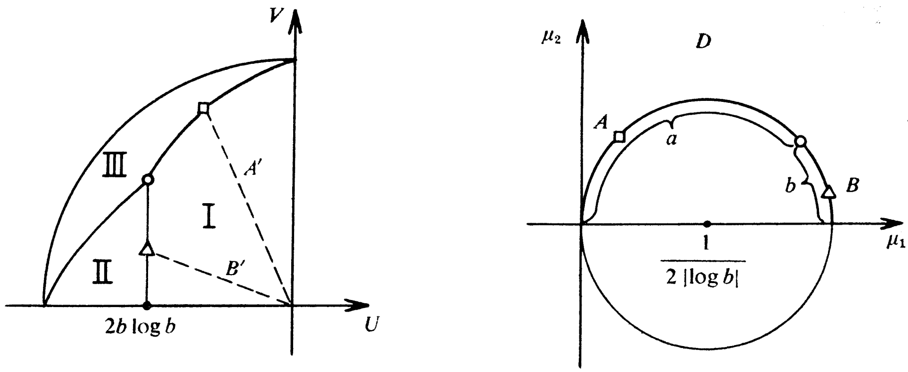


Figure 1.

At the two points $A, B \in \partial D$ there are connected the extremal domains whose $a_2 \in A'$ and B' of I (Figure 1). From Theorem 1/V.1.2 of [4] we see that for $a_2 = U + iV \in A'$ we have

(7) $$\begin{cases} U = 2\sigma \log b, & \sigma \in [0, \sigma_0], \\ V = -2 \dfrac{\mu_2}{\mu_1} \sigma \log b \leqq 2\left(\sqrt{1-\sigma^2} - \sigma \, \overline{\arccos}\, \sigma - \sqrt{b^2 - \sigma^2} + \sigma \, \overline{\arccos} \dfrac{\sigma}{b}\right). \end{cases}$$

Here σ_0 is the equality value of the above inequality. For $a_2 = U + iV \in B'$ we have correspondingly

(8) $$U \geqq 2b \log b, \quad V = -\frac{\mu_2}{\mu_1} U.$$

The types of the extremal domains are schematically presented in Figure 2.

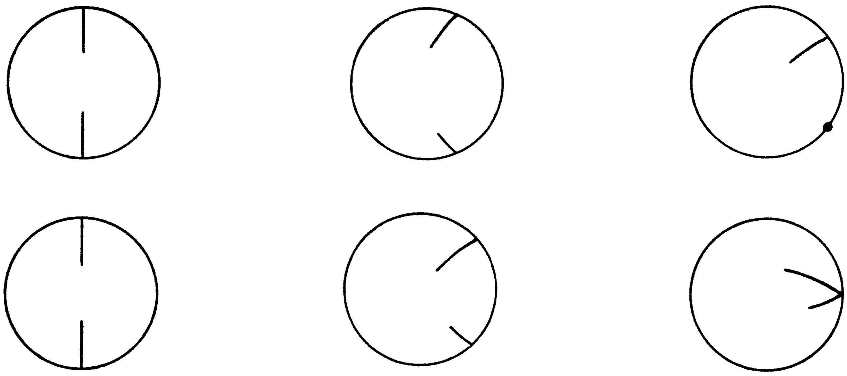


Figure 2.

3. The equality functions 1:2

Next, consider the condition (25)/V.1.3 of [4]:

$$\text{(9)}\qquad \begin{cases} \operatorname{Re}(a_3-a_2^2) \leqq 1-b^2-2|U|\sigma+2(\sigma-b)^2, \\ \sigma\log\sigma-\sigma+b+\dfrac{|U|}{2}=0, \quad \sigma\in[b,1]. \end{cases}$$

This gives

$$\operatorname{Re}(a_3-a_2^2) \leqq 1-b^2+4\sigma(\sigma\log\sigma-\sigma+b)+2(\sigma-b)^2$$

and

$$\text{(10)}\qquad \operatorname{Re}\left(a_3+(\mu-1)a_2^2\right)-(1-b^2) \leqq \mu_1(U^2-V^2)-2\mu_2UV+4\sigma(\sigma\log\sigma-\sigma+b) \\ +2(\sigma-b)^2.$$

Keep the normalization (4), which implies (6) in the maximum case. Thus

$$\text{(11)}\qquad U=2(\sigma\log\sigma-\sigma+b).$$

This defines (Figure 3) a bijective connection

$$\sigma\in[b,1]\rightarrow[-2b\,|\log b|,\ -2(1-b)]\ni U.$$

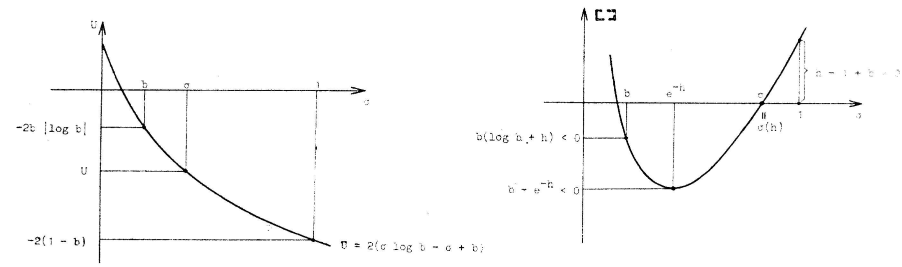


Figure 3.

By aid of (11) we may express the right side of (10) in σ and V:

(12) $$\operatorname{Re}\left(a_3+(\mu-1)a_2^2\right)-(1-b^2) \leqq 4\mu_1(\sigma\log\sigma-\sigma+b)^2$$
$$+4\sigma(\sigma\log\sigma-\sigma+b)+2(\sigma-b)^2-\mu_1V^2-4\mu_2(\sigma\log\sigma-\sigma+b)V.$$

If $\mu_1<0$, the free extremal point defined by $\partial/\partial V=0$ does not give the maximum. Therefore, we have to restrict the use of (12) in the cases where

(13) $$\mu_1 > 0.$$

Write the part depending of V in the form

$$-\mu_1\left[V+2\frac{\mu_2}{\mu_1}(\sigma\log\sigma-\sigma+b)\right]^2.$$

Thus, in the maximum case

(14) $$V=-2\frac{\mu_2}{\mu_1}(\sigma\log\sigma-\sigma+b)=-\frac{\mu_2}{\mu_1}U.$$

This implies

(15) $$\operatorname{Re}\left(a_3+(\mu_1-1)a_2^2\right)-(1-b^2) \leqq 4\mu_1(\sigma\log\sigma-\sigma+b)^2$$
$$+4\sigma(\sigma\log\sigma-\sigma+b)+2(\sigma-b)^2+4\frac{\mu_2^2}{\mu_1^2}(\sigma\log\sigma-\sigma+b)^2$$
$$=4\frac{|\mu|^2}{\mu_1}(\sigma\log\sigma-\sigma+b)^2+4\sigma^2\log\sigma-2\sigma^2+2b^2$$

and

$$\frac{\partial}{\partial\sigma}=8\frac{|\mu|^2}{\mu_1}\log\sigma\left[\sigma\log\sigma+\left(\frac{\mu_1}{|\mu|^2}-1\right)\sigma+b\right].$$

Let us parametrize the points of the μ-plane by requiring that

(16) $$\frac{\mu_1}{|\mu|^2}=h=\text{constant}\Leftrightarrow\left(\mu_1-\frac{1}{2h}\right)^2+\mu_2^2=\frac{1}{4h^2}.$$

If

$$D_h=\left\{\mu\in C\,\middle|\,|\mu|^2-\frac{\mu_1}{h}<0\right\},$$

then (16) implies that $\mu\in\partial D_h$.

Consider the equation

(17) $$[\,]=\sigma\log\sigma+(h-1)\sigma+b=0.$$

The existence of a real root $\sigma=\sigma(h)\geqq e^{-h}$ requires that (Figure 3)

$$b-e^{-h}\leqq 0\Leftrightarrow 0<h\leqq-\log b.$$

Because in this case $1/2h\geqq-1/(2\log b)$ we see that the arc ∂D_h lies in $-\bar{D}$. The limit case $h=-\log b$ gives the circle $\partial D=\partial D_{-\log b}$.

Further, the existence of $\sigma(h)$ requires that $h-1+b\geqq 0$ imply

(18) $$1-b\leqq h\leqq-\log b.$$

The order of the limits $1-b$ and $-\log b$ is valid for the whole interval $b\in(0, 1]$. On the limits we have

$$\sigma(1-b) = 1, \quad \sigma(-\log b) = b.$$

At the point $\sigma(h) > e^{-h}$ the derivative $\partial/\partial\sigma$ changes the sign from positive to negative, i.e. the right side of (15) is maximized at $\sigma(h)$. Because $\sigma(h)=\sigma$ satisfies the condition (17) the maximum assumes the form

$$4\frac{1}{h}(-h\sigma)^2+4\sigma\big(-(h-1)\sigma-b\big)-2\sigma^2+2b^2 = 2(\sigma-b^2).$$

This implies that

$$\text{(19)} \qquad \begin{cases} \operatorname{Re}\big(a_3+(\mu-1)a_2^2\big)-(1-b^2) \leqq 2(\sigma-b)^2; \\ \sigma\log\sigma+(h-1)\sigma+b = 0, \\ h = \dfrac{\mu_1}{|\mu|^2}\in[1-b, \ -\log b]. \end{cases}$$

In the extremal case we have

$$U = 2(\sigma\log\sigma-\sigma+b) = -2h\sigma,$$

so that

$$\text{(20)} \qquad U = -2h\sigma = -2\frac{\mu_1}{|\mu|^2}\sigma, \quad V = -\frac{\mu_2}{\mu_1}U = 2\frac{\mu_2}{\mu_1}h\sigma = 2\frac{\mu_2}{|\mu|^2}\sigma.$$

On the arc $(\partial D_h)_+$ lying in the upper μ-half-plane h, and thus $\sigma(h)$, is a constant which implies that U in (20) is a constant. V increases with the ratio μ_2/μ_1. The extremal point (U, V) must lie in the sharpness region II (Figure 1) of the inequality (9) defined by (28)/V.1.3 of [4]. The upper boundary arc of II is thus

$$\text{(21)} \qquad \begin{cases} U = 2(\sigma\log\sigma-\sigma+b) \quad (=-2h\sigma), \\ V = 2\big(\sqrt{1-\sigma^2}-\sigma\,\overline{\operatorname{arc}}\cos\sigma\big). \end{cases}$$

For the points $\mu\in(\partial D_h)_+$ we thus have the limitation

$$V = 2\frac{\mu_2}{\mu_1}h\sigma \leqq 2\big(\sqrt{1-\sigma^2}-\sigma\,\overline{\operatorname{arc}}\cos\sigma\big)$$

and hence

$$\text{(22)} \qquad \frac{\mu_2}{\mu_1} \leqq \frac{\sqrt{1-\sigma^2}-\sigma\,\overline{\operatorname{arc}}\cos\sigma}{h\sigma} = m.$$

This means that the end point μ of the are $(\partial D_h)_+$ in the upper half-plane satisfies

$$\mu_2 = m\mu_1, \quad \mu_1^2+\mu_2^2-\frac{\mu_1}{h} = 0,$$

and so

$$\text{(23)} \qquad \mu_1 = \frac{1}{(1+m^2)h}, \quad \mu_2 = \frac{m}{(1+m^2)h}.$$

Observe that if $h \to 1-b$ then $\sigma \to 1$, $m \to 0$ and

$$\mu = (\mu_1, \mu_2) \to \left(\frac{1}{1-b}, 0\right).$$

This means that the sharpness region of (19) lying in the ring

$$D_{1-b} - D_{-\log b}$$

becomes narrower with increasing h and touches ∂D_{1-b} at the point on the μ_1-axis.

The sharpness conditions of (9) are given by (27)/V.1.2 of [4]. For $U<0$ the extremizing ϑ satisfies the conditions

$$\cos \vartheta = \begin{cases} 1, & b \leqq u \leqq \sigma, \\ \sigma/u, & \sigma \leqq u \leqq 1. \end{cases}$$

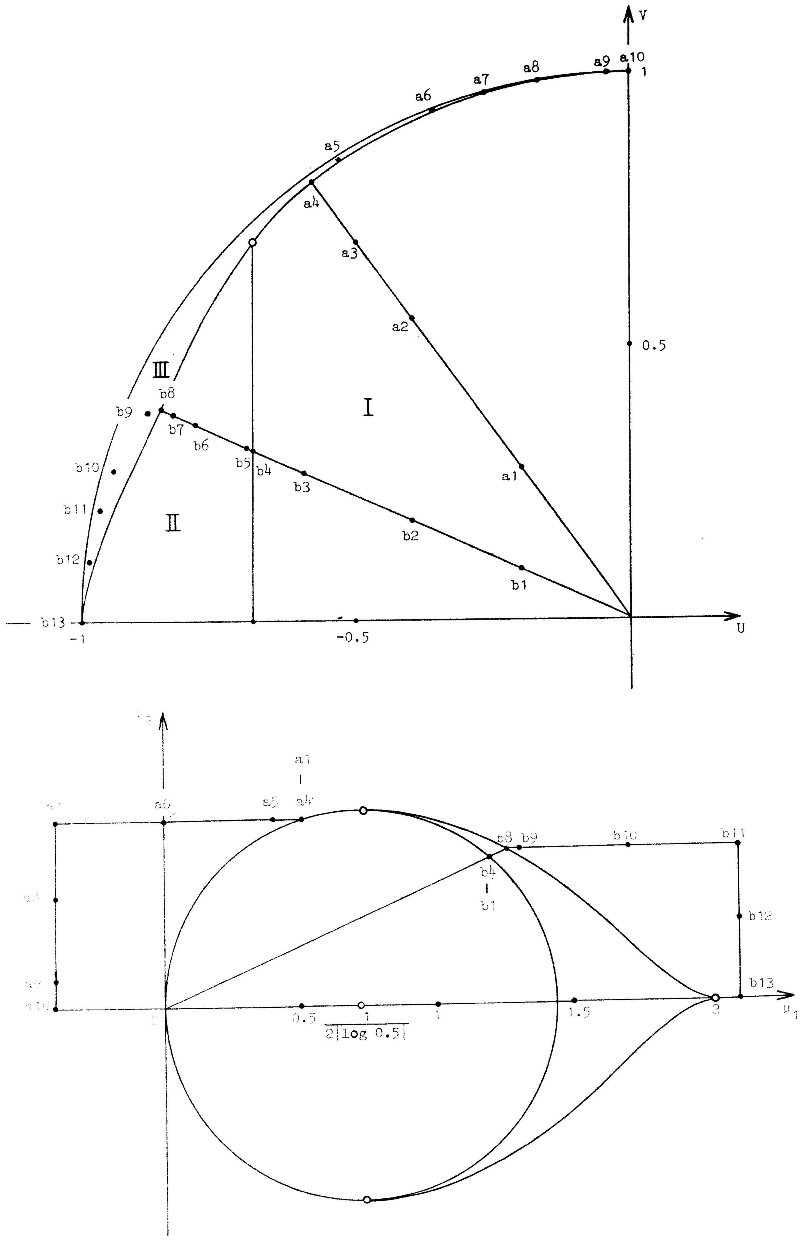
This determines U of (11). Thus, the sharpness conditions of (9) are in agreement with the extremal conditions (19)—(20) and the estimation (19) is sharp in the region of μ defined by (22)—(23).

Figure 4 illustrates the regions I and II in the case $b=0.5$. The upper boundary arc ∂I is obtained from the equality case of (7) and that of ∂II from (21).

In Table 1 some points $\mu=(\mu_1, \mu_2)$ corresponding to ∂II are determined in the case $b=0.5$. This is done by solving $\sigma=\sigma(h)\in[e^{-h}, 1]$ from (19) for $h\in[1-b, -\log b]$. The number m obtained from (22) determines then (μ_1, μ_2) according to (23). The corresponding graphs are in the lower part of Figure 4.

Table 1.

h	σ	μ_1	μ_2
0.5	1	2.	0.
0.52	0.959 977	1.922·635	0.029·133
0.54	0.919·805	1.848·389	0.080·003
0.56	0.879·290	1.774·133	0.143·341
0.58	0.838·163	1.696·637	0.216·067
0.6	0.796·033	1.612·228	0.296·255
0.62	0.752·281	1.516·276	0.382·771
0.64	0.705·818	1.402·150	0.474·167
0.65	0.680·945	1.335·043	0.521·126
0.66	0.654·338	1.258·307	0.568·497
0.67	0.625·011	1.167·798	0.615·817
0.68	0.590·675	1.054·458	0.662·414
0.69	0.541·860	0.882·540	0.707·225
0.693	0.508·677	0.761·716	0.720·379
0.693·147	0.5	0.730·031	0.721·295


v
a9 a10
1
a8
a7
a6
a5
a4
a3
a2
0.5
III
I
b8
b9
b7
b6
b5
b4
b3
b10
a1
b11
II
b2
b12
b1
b13
-1
-0.5
u
a1
a4
a6
a5
b8 b9
b10
b11
b4
b1
b12
b13
0.5
1
1.5
2
$\frac{1}{2|\log 0.5|}$

$b = 0.5$

Figure 4.

From (5) we see by conjugation that the triple (ϑ, U, V) changes to $(-\vartheta, U, -V)$, where

$$U = -2 \int_b^1 \cos \vartheta \, du, \quad V = -2 \int_b^1 \sin \vartheta \, du.$$

With the normalization $\mu_2 < 0$, the maximum belongs to the case $U < 0$, $V < 0$, i.e. we are led to the third quadrant in the UV-plane and in the lower μ-half-plane. The rotation $\vartheta \Rightarrow \vartheta + \pi$ changes the signs of U and V. Thus the first and fourth quadrants of the UV-plane give the previous maxima for $\operatorname{Re}(a_3 + (\mu - 1) a_2^2)$.

In Figure 5 the sharpness regions of the μ-plane are numbered according to the corresponding numbers in the UV-half-plane with $U \leqq 0$.

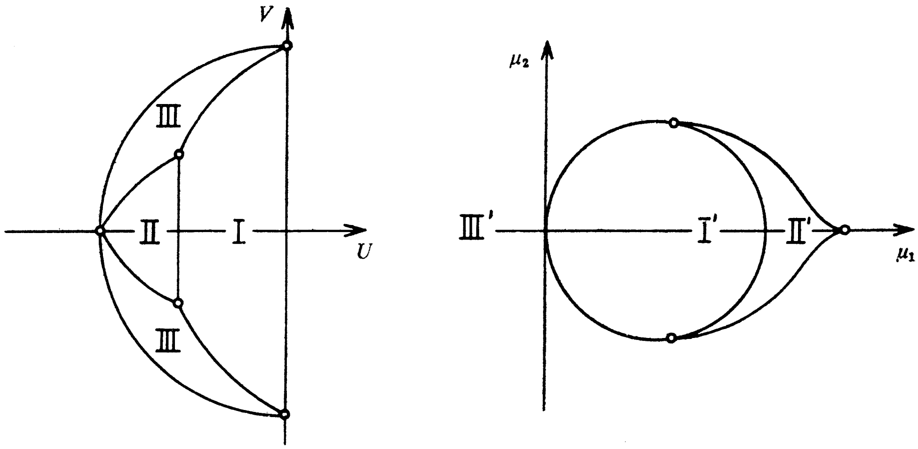


Figure 5.

4. The special case $\mu \in R$

We consider separately the case $\mu_2 = 0$. Then (5) yields

(24) $$\operatorname{Re}(a_3 + (\mu_1 - 1) a_2^2) = \mu_1 (U^2 - V^2) - 2 \int_b^1 u \cos 2\vartheta \, du.$$

1°. $\mu_1 \geqq 0$

In this case

(25) $$\operatorname{Re}(a_3 + (\mu_1 - 1) a_2^2) \leqq \mu_1 U^2 - 2 \int_b^1 u \cos 2\vartheta \, du$$

with the equality for $V = 0$. This upper bound is the value of our functional in the subclass $S_R(b) \subset S(b)$ in which all coefficients are real, because in $S_R(b)$

$$
(26) \qquad \begin{cases} a_3+(\mu_1-1)a_2^2 = \mu_1 U^2 - 2\int_b^1 u \cos 2\vartheta \, du, \\ a_2 = -2\int_b^1 \cos \vartheta \, du, \\ a_3 = a_2^2 - 2\int_b^1 u \cos 2\vartheta \, du. \end{cases}
$$

The estimation (9) leads in the present case to the upper bound

$$
(27) \quad \mathrm{Re}\left(a_3+(\mu_1-1)a_2^2\right)-(1-b^2) \leqq 4\mu_1(\sigma \log \sigma - \sigma + b)^2 + 4\sigma^2 \log \sigma - 2\sigma^2 + 2b^2.
$$

For the derivative of the right side we have

$$
(28) \qquad \frac{\partial}{\partial \sigma} = 8\mu_1 \log \sigma \, [\sigma \log \sigma + (h-1)\sigma + b]; \quad h = \frac{1}{\mu_1}.
$$

If $\mu_1 \geqq (1-b)^{-1}$, i.e. $h-1+b \leqq 0$, then $\partial/\partial\sigma \geqq 0$ (Figure 3). This implies that the maximum of the right side of (27) is reached for $\sigma = 1$. The extremal function f is the right radial-slit-mapping having $\cos \vartheta \equiv 1$. The equality conditions of (9) and the conditions (20) under which the right side of (10) or (27) is maximized thus agree in the present case also. Hence the following estimation, obtainable from (27), is sharp:

$$
(29) \qquad \mathrm{Re}\left(a_3+(\mu_1-1)a_2^2\right) \leqq b^2 - 1 + 4\mu_1(1-b)^2.
$$

Using the notation

$$
\lambda = \mu_1 - 1 \geqq \frac{b}{1-b},
$$

we may rewrite (29) in form where the coefficients a_3 and a_2 of the radial-slit-mapping reappear:

$$
\mathrm{Re}\,(a_3+\lambda a_2^2) \leqq 3 - 8b + 5b^2 + \lambda(2(1-b))^2, \quad \lambda \in R.
$$

If

$$
(30) \qquad -\frac{1}{\log b} < \mu_1 < \frac{1}{1-b},
$$

we can check from (27) that the estimation (19) and the type 1:2 of extremal domains hold on the interval (30).

In the remaining case

$$
0 \leqq \mu_1 \leqq -\frac{1}{\log b}
$$

we write the initial condition (1)′ in the form

$$
\mathrm{Re}\left(a_3+(\mu_1-1)a_2^2\right)-(1-b^2) \leqq \mu_1 U^2 - \mu_1 V^2 + \frac{U^2}{\log b} \leqq \left(\mu_1 + \frac{1}{\log b}\right) U^2.
$$

From this we read out immediately that the estimation (2) together with the extremal domain 2:2, having two symmetric radial slits, are extended to the diameter of the disc D.

2°. $\mu_1 = -|\mu_1| < 0$

From (24) we obtain now

$$\mathrm{Re}\,(a_3-(|\mu_1|+1)\,a_2^2) \leqq |\mu_1|\,V^2-2\int_b^1 u\cos 2\vartheta\,du, \tag{31}$$

with the equality for $U=0$. The right side is maximized for

$$\sin\vartheta \equiv 1 \quad \text{and} \quad \sin\vartheta \equiv -1,$$

which define the radial-slit-mappings having the slit along the positive and negative real axis, respectively. These are obtained from the right radial-slit-mapping

$$f(z) = b(z+a_2z^2+\ldots);\quad a_2 = -2(1-b),\quad a_3 = 3-8b+5b^2,$$

by applying the notation

$$\tau^{-1}f(\tau z) = b(z+\tau a_2 z^2+\tau^2 a_3 z^2+\ldots)$$

with $\tau=i$ and $\tau=i^{-1}$. The inequality (31) yields the estimate

$$\mathrm{Re}\,(a_3-(|\mu_1|+1)\,a_2^2) \leqq 1-b^2+4|\mu_1|(1-b)^2. \tag{32}$$

The choice $\tau=i$ gives the coefficients

$$a_2 = -2(1-b)\,i,\quad a_3 = -(3-8b+5b^2)$$

for which the sharpness of (32) can be checked.

Observe that for $\mu_1<0$ the functional *is not* maximized in the class $S_R(b)$. From (26) we see that in $S_R(b)$,

$$a_3-(|\mu_1|+1)\,a_2^2 = -|\mu_1|\,a_2^2-2\int_b^1 u\cos 2\vartheta\,du \tag{33}$$

$$= 1-b^2-|\mu_1|\,a_2^2-4\int_b^1 u\cos^2\vartheta\,du \leqq 1-b^2.$$

Equality here requires that

$$\cos\vartheta \equiv 0.$$

Because we are in the class $S_R(b)$ the extremal domain is a two-radial-slit mapping with symmetric slits along the positive and negative imaginary axis.

5. The equality functions 1:1

The regions I and II of Figure 1 are completely covered by the points a_2 defined by the sharpness conditions (3) and (20). Thus, only the region III connected with the boundary functions 1:1 remains. As in [4] (V.3.4) we may check, by using variation of ϑ, that the only maximizing radial-slit-mappings are those where $\cos\vartheta\equiv 1$ and $\cos\vartheta\equiv 0$, i.e. they are connected with the corners of III lying on the U- and V-axis correspondingly.

Let us now apply the variation mentioned in the expression (5). The result follows by aid of formal differentiation $(1/4)(\partial/\partial\vartheta)$ which yields

$$(\mu_1 U-\mu_2 V)\sin\vartheta+(\mu_1 V+\mu_2 U)\cos\vartheta = -u\sin 2\vartheta. \tag{34}$$

The solution of this condition is a boundary function of the type 1:1. For these Theorem 5/V.1.4 of [4] is available. This implies that for the generating function ϑ of the boundary function

$$C_1\sin\vartheta+C_2\cos\vartheta = u\sin 2\vartheta \tag{35}$$

(cf. (39)/V.1.4 of [4]). We may now proceed as in V.3 of [4], when maximizing $\operatorname{Re}(a_3+\lambda a_2)$. Comparison of (34) and (35) yields the connection between the extremal points (U, V) and (μ_1, μ_2):

$$\mu_1 U-\mu_2 V=-C_1,\quad \mu_2 U+\mu_1 V=-C_2;$$

on

$$U=-\frac{\mu_1 C_1+\mu_2 C_2}{|\mu|^2},\quad V=-\frac{\mu_1 C_2-\mu_2 C_1}{|\mu|^2}\quad (\mu\neq 0). \tag{36}$$

Theorem 5/V.1.4 of [4] defines the bijective connection $(\mu_1, \mu_2)\Rightarrow(\alpha, \omega)$, where α and ω are the parameters used for the boundary functions 1:1:

$$\begin{cases} C_1\log\dfrac{\cos\alpha}{\cos\omega}+C_2(\cot\alpha-\cot\omega+\alpha-\omega)+\dfrac{\mu_1C_1+\mu_2C_2}{|\mu|^2}=0,\\[2mm] C_1(\tan\alpha-\tan\omega-\alpha+\omega)+C_2\log\dfrac{\sin\alpha}{\sin\omega}+\dfrac{\mu_1C_2-\mu_2C_1}{|\mu|^2}=0; \end{cases} \tag{37}$$

$$\begin{cases} C_1=2\dfrac{\sin\alpha-b\sin\omega}{\sin(\alpha-\omega)}\cos\alpha\cos\omega,\\[2mm] C_2=2\dfrac{\cos\alpha-b\cos\omega}{\sin(\alpha-\omega)}\sin\alpha\sin\omega. \end{cases} \tag{38}$$

The corresponding bijective connection, i.e. the uniqueness of the solution (α, ω), was considered in detail by the first author in [2].

The sharp upper bound of the functional is written by aid of Theorem 8/V.1.4 of [4] in the form

$$\text{(39)} \qquad \operatorname{Re}\left(a_3+(\mu-1)a_2^2\right) \leqq 1-b^2+C_1C_2(\tan\alpha-\tan\omega)-\frac{C_2^2}{2}\left(\frac{1}{\sin^2\alpha}-\frac{1}{\sin^2\omega}\right).$$

6. Some extremal domains in the class $S(1/2)$

In [4] the boundary functions of (a_2, a_3) are determined by integration of the corresponding Löwner's differential equation. In the cases 2:2 and 1:2 the result can be expressed in terms of U and V, whereas in the case 1:1 the parameters α and ω determine the solution. Thus, in the present case we will pick up some extremal points (U, V), (μ_1, μ_2) for determining the corresponding extremal domains. Let us fix $b=1/2$.

We start from a defined boundary point $A\in a$ (Figure 1):

$$\text{(40)} \qquad \mu_1 = 0.5, \quad \mu_2 = \sqrt{\frac{\mu_1}{|\log b|}-\mu_1^2} = 0.686{\cdot}548.$$

Keep the ratio

$$t = \frac{\mu_2}{\mu_1} = 1.373{\cdot}095$$

fixed and choose some points (U, V) lying on the line segment A' (Figure 1). The extremal domains all belong to the same point (40). The number σ_0 determining the limit point (7) on the upper boundary of $\partial\mathrm{I}$ is the root of

$$\sqrt{1-\sigma^2}-\sigma\,\overline{\arccos}\,\sigma-\sqrt{b^2-\sigma^2}+\sigma\,\overline{\arccos}\,\frac{\sigma}{b}+t\sigma\log b = 0,$$

which in the present case is

$$\sigma_0 = 0.419{\cdot}790.$$

Thus we end up to the points

$$a1, \ldots, a4$$

of Table 2 and Figure 4. From $a4$ we proceed first in the direction of the μ_1-axis and then in the direction of the μ_2-axis through the points

$$a5, \ldots, a10$$

of Table 2 and Figure 4.

Next, start from the point

$$b8\colon\ \mu_1 = 1.258{\cdot}307, \quad \mu_2 = 0.568{\cdot}497$$

lying on the boundary of II′ (Table 1). This defines the ratio

$$t = \frac{\mu_2}{\mu_1} = 0.451{\cdot}795$$

Table 2.

	U	V	μ_1	μ_2	$t=\dfrac{\mu_1}{\mu_2}=-\dfrac{V}{U}$		$\mathrm{Re}\,(a_3+(\mu-1)a_2^2)$
a1	−0.2	0.274·619	0.5	0.686·548	1.373·095		0.75
a2	−0.4	0.549·238	0.5	0.686·548	1.373·095	2:2	0.75
a3	−0.5	0.686·548	0.5	0.686·548	1.373·095		0.75
a4	−0.581·952	0.799·075	0.5	0.686·548	1.373·095		0.75
a5	−0.531·716	0.837·641	0.4	0.686·548			0.786·084
a6	−0.368·530	0.927·332	0.	0.686·548			1.023·040
a7	−0.269·261	0.962·295	−0.4	0.686·548		1:1	1.342·178
a8	−0.169·400	0.985·213	−0.4	0.4			1.218·787
a9	−0.044·206	0.998·998	−0.4	0.1			1.154·423
a10	0	1	−0.4	0.			1.150·000
b1	−0.2	0.090·359	1.198·133	0.541·311	0.541·795		0.75
b2	−0.4	0.180·718	1.198·133	0.541·311	0.451·795	2:2	0.75
b3	−0.6	0.271·077	1.198·133	0.541·311	0.451·795		0.75
b4	−0.693·147	0.313·161	1.198·133	0.541·311	0.451·795		0.75
b5	−0.725·220	0.327·655	1.20	0.542·154	0.451·797		0.751·148
b6	−0.800·342	0.361·591	1.22	0.551·190	0.451·795	1:2	0.765·439
b7	−0.838·264	0.378·724	1.24	0.560·226	0.451·795		0.781·656
b8	−0.863·727	0.390·228	1.258·307	0.568·497	0.451·795		0.797·641
b9	−0.878·302	0.379·212	1.3	0.568·497			0.823·126
b10	−0.949·233	0.273·675	1.7	0.568·497			1.119·334
b11	−0.975·921	0.203·285	2.1	0.568·497		1:1	1.469·224
b12	−0.991·560	0.114·265	2.1	0.3			1.384·759
b13	−1	0	2.1	0.			1.350·000

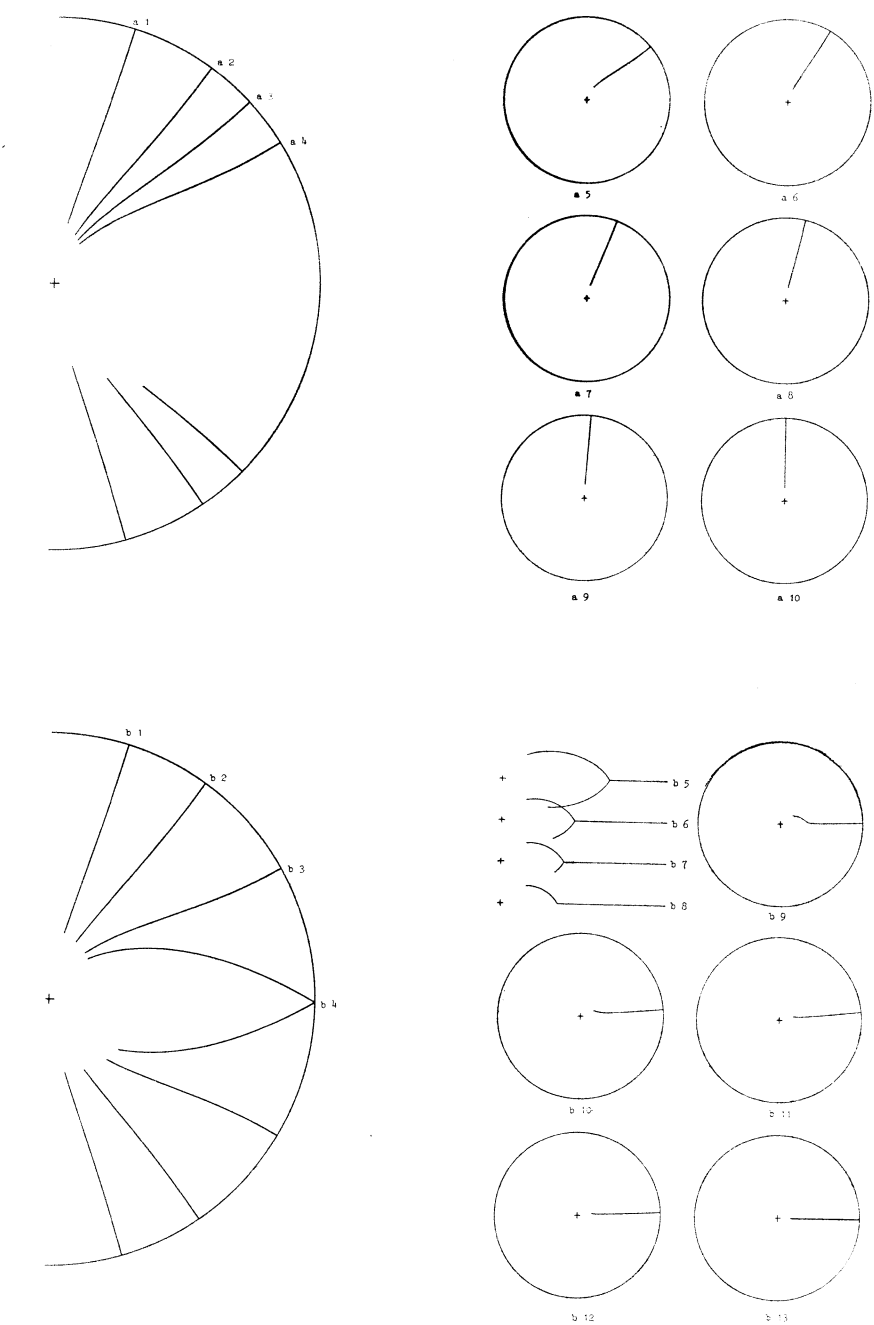

a 1
a 2
a 3
a 4
a 5
a 6
a 7
a 8
a 9
a 10
b 1
b 2
b 3
b 4
b 5
b 6
b 7
b 8
b 9
b 10
b 11
b 12
b 13

Figure 6.

determining a line segment through the origin of the μ-plane. This meets the boundary ∂D of II$'$ at the second point

$$b4\colon \begin{cases} \mu_1 = -\dfrac{1}{(1+t^2)\log b} = 1.198\cdot 133, \\ \mu_2 = -\dfrac{t}{(1+t^2)\log b} = 0.541\cdot 311. \end{cases}$$

Between $b4$ and $b8$ we pick up the points (Table 2)

$$b5, \ldots, b7.$$

The corresponding points (U, V) are determined by (20), and they lie on the line segment (Figure 4)

$$V = -tU,$$

on which we further choose the points (Table 2)

$$b1, b2, b3.$$

Finally, we proceed in the μ-plane along two line segments, through the points (Table 2 and Figure 4):

$$b9, \ldots, b13.$$

The extremal domains corresponding to the points $a1, \ldots, a10$ and $b1, \ldots, b13$ are computer-drawn in Figure 6.

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