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QUASISYMMETRIC AND LIPSCHITZ APPROXIMATION OF EMBEDDINGS

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Introduction

This paper is concerned with the following concepts. Let (X, d) and (Y, d') be metric spaces and let $f\colon X \to Y$ be an embedding. Then f is said to be *bilipschitz* (or *L-bilipschitz*) if, for some $L \geqq 1$,

$$d(x, y)/L \leqq d'(f(x), f(y)) \leqq L d(x, y)$$

for all $x, y \in X$. It is called *quasisymmetric* (or *η-quasisymmetric*) if, for some homeomorphism $\eta\colon R^1_+ \to R^1_+$,

$$d'(f(a), f(x)) \leqq \eta(t) d'(f(b), f(x))$$

whenever $a, b, x \in X$, $t \geqq 0$, and $d(a, x) \leqq t d(b, x)$. The above condition is motivated by the properties of quasiconformal maps: For instance, a homeomorphism of R^n is quasiconformal if and only if it is quasisymmetric. Like bilipschitz embeddings, quasisymmetric embeddings form a category: The composite of two quasisymmetric embeddings is quasisymmetric, and so is the inverse of a quasisymmetric homeomorphism (cf. [38]). Every bilipschitz embedding is quasisymmetric.

A map $f\colon X \to Y$ is called an LQS *immersion* if every point of X has a neighborhood on which f is quasisymmetric. It is called a LIP *immersion* if every point of X has a neighborhood on which f is bilipschitz. We let CAT denote either LQS or LIP. A CAT *embedding* or a CAT *homeomorphism* is a CAT immersion which is an embedding or a homeomorphism, respectively. It is obvious that the inverse of a CAT homeomorphism is a CAT homeomorphism. Every LQS embedding of a compact space is quasisymmetric by [38, Theorem 2.23], and every LIP embedding of a compact space is bilipschitz. A piecewise linear (PL) embedding between polyhedra in Euclidean spaces is a CAT embedding. We call a separable metric space a *metric* CAT *n-manifold*, $n \geqq 0$, if every point has a closed neighborhood CAT homeomorphic to the cube $[-1, 1]^n$ in R^n.

There is a familiar and apparently more general alternative way to define CAT manifolds and CAT immersions based on atlases; see 1.3. As LQS atlases are the

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same as locally quasiconformal atlases, LQS manifolds are also called *quasiconformal manifolds*. The two definitions of LIP or *Lipschitz manifolds* were proved to be (essentially) equivalent in [26, Theorems 3.5 and 4.2]. J. Väisälä raised the question whether this also holds for LQS manifolds. Our main result implies that such is really the case.

Section 1 is preliminary except for Theorem 1.14, where it is proved that every LIP embedding between LIP manifolds is locally flat if the codimension is at least three. Lemma 1.9 recapitulates known results about PL approximation of embeddings from dimension n into dimension q. It is valid if either $q \geqq n=1$ or $n \geqq 2$, $q \geqq n+3$ or $(n, q) \in \{(2, 2), (2, 3), (3, 3)\}$. We call these pairs (n, q) *admissible*.

In Section 2 we prove that if (n, q) is admissible, $Y \subset X \subset R^n$, and Y is open in R^n, then every quasisymmetric embedding $f\colon X \to R^q$ can be approximated by quasisymmetric embeddings which coincide with f on $X \setminus Y$ and are PL on Y. In fact, we consider here more general, if quite special, polyhedra Y; these are open in X and have a certain decomposition into n-cubes. In the case $n=q \leqq 3$ similar but simpler problems for bilipschitz or quasiconformal embeddings are studied in [41, Theorem 2.4] and [18, Theorems 2.1 and 3.1]. Our proof is similar to the ones in [41] and [18]. It is based on Lemma 1.9 and the finiteness idea of Carleson in [10], which can be used by virtue of a compactness property of quasisymmetric embeddings.

In Section 3 we apply results of Section 2 and obtain analogous results for bilipschitz embeddings.

Using these theorems we prove in Section 4 the main result of this paper, Theorem 4.4. In a simplified form it states (cf. 4.14.4) that if M and N are CAT manifolds of dimensions n and q, respectively, such that (n, q) is admissible, then every embedding $f\colon M \to N$ can be approximated by CAT embeddings in the source majorant topology. A special case of Theorem 4.4 for CAT=LIP and $n=q \leqq 3$ is given in [37, Theorem 2], and our proof is a modification of the proof in [37]. As a corollary we get the result that M can be CAT embedded into R^{2n+1} $(n \geqq 0)$, from which it follows that the two definitions of CAT manifolds are equivalent. Further, by recent results about topological embeddings, R^{2n+1} can here be replaced by R^{2n} $(n \geqq 1)$ and, if the target majorant topology is used and $q \geqq 2n+1$, it suffices to assume that f is only a continuous map.

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1. Terminology and preliminary results

1.1. *Notation.* The letters n, q denote non-negative integers. Let R^n be the Euclidean n-space, $R^n_+=\{x\in R^n | x_n \geqq 0\}$, $R^{n-1}\equiv \partial R^n_+$, $I^n=[-1,1]^n$, $J^n=(-1,1)^n$, $J^n_+=J^n\cap R^n_+$, $I^n(r)=rI^n$ and $J^n(r)=rJ^n$ for $r>0$, and $I=[0,1]$. Let $e_1=(1,0,\dots,0)\in R^n$. If S is a topological space, let $C_+(S)=\{f | f\colon S\to(0,\infty)$ continuous$\}$. If S is a set and $f,g\colon S\to R^n$, we write $d(f,g)=\sup\{|f(x)-g(x)| \,|\, x\in S\}$.

1.2. *Weakly quasisymmetric embeddings.* An embedding f of a metric space (X,d) into a metric space (Y,d') is called *weakly quasisymmetric* if there is $H\geqq 1$ such that $d'(f(a),f(x))\leqq Hd'(f(b),f(x))$ whenever $a,b,x\in X$ and $d(a,x)\leqq d(b,x)$; then f is also said to be *weakly H-quasisymmetric.* This concept, too, was considered in [38]. Every quasisymmetric embedding is weakly quasisymmetric. Every weakly quasisymmetric embedding $f\colon S\to R^q$, $S\subset R^n$, has a unique extension to a closed embedding $\bar f\colon \bar S\to R^q$. If f is L-bilipschitz, this is well-known, and $\bar f$ is L-bilipschitz. If f is η-quasisymmetric, this follows from [38, Theorems 2.24 and 2.25], and $\bar f$ is, η-quasisymmetric. The general case follows from [33, Lemmas 2, 4, and 5] (the case $n=q=2$ considered in [33] can be generalized in the obvious way). Moreover, in [33] one does not assume that f is an embedding but only that f is injective. (For a similar definition for quasisymmetric embeddings, see [38, Theorem 2.21].) We do not know whether $\bar f$ is always weakly quasisymmetric. (However, it is easy to see that if S is open and convex and if f is weakly H-quasisymmetric, then $\bar f$ is weakly H-quasisymmetric.)

1.3. *Atlases.* We give the definition of CAT manifolds in the atlas sense. Let CAT (n) be the category whose objects are open subsets of R^n and of R^n_+ and whose morphisms are CAT homeomorphisms. Then CAT (n) is a pseudogroup of transformations in a slightly more general sense than in [20, p. 1]. Consider a homeomorphism $f\colon U\to V$, where U, V are open either in R^n or in R^n_+. By [38, Theorem 2.16], [43, Theorems 2.3 and 2.4], and [40, Theorem 35.2], the following conditions are equivalent for a point $x\in U$: (1) f is quasisymmetric on a neighborhood of x in U; (2) f is weakly quasisymmetric on a neighborhood of x in U; (3) (for $n\geqq 2$) there is an open neighborhood W of x in R^n such that $f|W\cap \operatorname{int} U$ is a quasiconformal embedding; (4) (for $n\geqq 2$) there is an open neighborhood W of x in R^n such that $f|W\cap U$ extends to a quasiconformal embedding of W into R^n.

A CAT (n) *atlas* $\mathscr{A}$ on a topological space M is a family of pairs (U,h), called *charts*, such that the U's are open sets of M covering M, h is a homeomorphism of U onto an open subset of R^n or of R^n_+, and for charts (U,h), (U',h'), the homeomorphism $h'h^{-1}\colon h(U\cap U')\to h'(U\cap U')$ belongs to CAT (n). If M is a separable metrizable space and if $\mathscr{A}$ is a CAT (n) atlas on M which is maximal with respect to inclusion, we call the pair $(M,\mathscr{A})$ a CAT *n-manifold* and $\mathscr{A}$ a CAT *structure* on M. The terms *quasiconformal structure* and *Lipschitz structure* are also used.

(We could also define a CAT structure as an equivalence class of CAT (n) atlases, two atlases being equivalent if their union is a CAT (n) atlas.)

Let $(M, \mathscr{A})$ be a CAT manifold. The underlying space M is a topological manifold, whose interior $\operatorname{int} M$ and boundary ∂M inherit a CAT structure from $\mathscr{A}$ in a natural way. If A and B are subsets of CAT manifolds and if X is a metric space, we can define CAT immersions $A \to B$, $A \to X$, and $X \to A$ in a familiar way using charts (cf. [42, 1.8]). One defines similarly CAT embeddings and CAT homeomorphisms. If $(N, \mathscr{B})$ is a CAT manifold such that $N \subset M$ and that the inclusion of $(N, \mathscr{B})$ into $(M, \mathscr{A})$ is a CAT embedding, we call $(N, \mathscr{B})$ a CAT *submanifold* of M. Suppose that $\partial M = \emptyset$. Then a CAT q-submanifold N of M is said to be *locally* CAT *flat* if for each $x \in N$ there is $(U, h) \in \mathscr{A}$ such that $x \in U$ and $h(U \cap N)$ equals $hU \cap R^q$ or $hU \cap R^q_+$. A CAT embedding is called locally CAT flat if its image is locally CAT flat.

Every metric CAT n-manifold M (in the sense of the Introduction) has a natural CAT structure consisting of all pairs (U, h) where U is open in M and h is a CAT homeomorphism of U onto an open set in R^n or in R^n_+. Moreover, if $A \subset M$, the two definitions of CAT immersions of A or into A coincide. We consistently define a subset N of M to be a CAT submanifold of M if it is a metric CAT manifold in the induced metric.

In 4.7 we will see that every CAT manifold has a metric which induces the original CAT structure.

In 4.11 we will need the fact that every CAT n-manifold can be CAT embedded into a CAT n-manifold without boundary. For this reason we construct the double of a CAT manifold.

1.4. Lemma. *Let* $(M, \mathscr{A})$ *be a* CAT *n-manifold. Then there exists a* CAT *n-manifold* $(DM, \mathscr{B})$, *called the double of* M, *with the following properties:* DM *contains* CAT *submanifolds* M_1, M_2 *such that* $DM = M_1 \cup M_2$, $M_1 \cap M_2 = \partial M_1 = \partial M_2$, *and there are* CAT *homeomorphisms* $f_i \colon M \to M_i$ *such that* $f_1 | \partial M = f_2 | \partial M$. *The triple* (DM, f_1, f_2) *is unique up to a* CAT *homeomorphism except possibly when* CAT$=$LQS, $n=1$, *and* $\partial M \neq \emptyset$, *in which case, however,* (DM, M_1, M_2) *is unique up to a* CAT *homeomorphism. Moreover,* $\partial DM = \emptyset$, *and the submanifolds* M_1, M_2, $M_1 \cap M_2$ *are locally* CAT *flat in* DM.

Proof. We only consider CAT$=$LQS, because for CAT$=$LIP one can give a similar but slightly simpler proof and because one has already proved this case by another method in [26, Theorem 3.13]. It is well-known that the lemma holds for topological manifolds and homeomorphisms (i.e., if CAT is replaced by TOP). This gives us the manifold DM and the homeomorphisms f_i. We construct an LQS structure on DM as follows. Define $p \colon R^n \to R^n$ by $p(x) = (x_1, \ldots, x_{n-1}, -x_n)$. For each chart $(U, h) \in \mathscr{A}$ with $U \cap \partial M = \emptyset$ we have the charts $(f_i U, h f_i^{-1} | f_i U)$, $i = 1, 2$, of DM. For each chart $(U, h) \in \mathscr{A}$ with $U \cap \partial M \neq \emptyset$ we define a chart (U^*, h^*) of DM as follows. Since hU is open in R^n_+, the set $V = hU \cup phU$ is open in R^n. Let $U^*=f_1U\cup f_2U$; then U^* is open in DM. We define a homeomorphism $h^*\colon U^*\to V$ by $h^*(x)=hf_1^{-1}(x)$ if $x\in f_1U$ and $h^*(x)=phf_2^{-1}(x)$ if $x\in f_2U$. These charts form an atlas $\mathscr{B}_0$ on DM. One can use [40, Theorem 35.2] for $n\geqq 2$ and [24, II, Lemma 7.1 and (7.2)] for $n=1$ to see that $\mathscr{B}_0$ is an LQS (n) atlas. The LQS structure $\mathscr{B}$ determined by $\mathscr{B}_0$ depends only on $\mathscr{A}$. The sets M_1, M_2, and $M_1\cap M_2$ are locally LQS flat LQS submanifolds of $(DM, \mathscr{B})$, and the homeomorphisms f_i are LQS.

To show the uniqueness, let $(D'M, M_1', M_2', f_1', f_2')$ have the properties of (DM, M_1, M_2, f_1, f_2) listed in the first part of the lemma. Let $g\colon DM\to D'M$ be the unique homeomorphism with $gf_i=f_i'$, $i=1, 2$. Since $M_1\cap M_2$ is locally LQS flat in DM, it follows from [40, Theorem 35.1] that g is LQS if $n\geqq 2$. If $n=1$ and $\partial M=\emptyset$, then trivially g is LQS. If $n=1$ and $\partial M\neq\emptyset$, the classification of LQS 1-manifolds in 4.8 implies that there exists an LQS homeomorphism $h\colon DM\to D'M$ with $hM_i=M_i'$. The existence of an LQS homeomorphism of (DM, M_1, M_2) onto $(D'M, M_1', M_2')$ implies that the submanifolds M_1', M_2', $M_1'\cap M_2'$ of $D'M$ are locally LQS flat (this can also be proved directly). □

1.5. Example. Define a homeomorphism $g\colon J^1\to J^1$ by $g(x)=x$ if $x\leqq 0$ and $g(x)=x^2$ if $x\geqq 0$. Then $g|(-1, 0]$ and $g|[0, 1)$ are quasisymmetric, but g is not LQS. This implies that in 1.4 the uniqueness of (DM, f_1, f_2) does not hold for $n=1$.

1.6. *Function spaces.* Let $C(X, Y)$ denote the set of all continuous maps of a metrizable space X into a metrizable space Y. Let $f\in C(X, Y)$. We call *f proper* if the inverse image of every compact set is compact. An embedding of X into Y is closed if and only if it is proper. Let d be a metric for Y. The sets

$$U_d(f, \varepsilon)=\{g\in C(X, Y)\mid \forall x\in X,\ d(f(x), g(x))<\varepsilon(x)\}$$

for $\varepsilon\in C_+(X)$ form a neighborhood basis of f in the *source majorant topology* of $C(X, Y)$, which is independent of d. If X and Y are locally compact and f is proper, there is a neighborhood $U_d(f, \varepsilon)$ whose elements are proper. The sets

$$N(f, \mathscr{U})=\{g\in C(X, Y)\mid \forall x\in X\ \exists U\in\mathscr{U},\ f(x), g(x)\in U\},$$

where $\mathscr{U}$ is an open cover of Y, form a neighborhood basis of f in the *target majorant topology* of $C(X, Y)$. For every $N(f, \mathscr{U})$ there is $U_d(f, \varepsilon)\subset N(f, \mathscr{U})$. Conversely, if f is proper, every $U_d(f, \varepsilon)$ contains an $N(f, \mathscr{U})$. It is easy to prove that if X, Y, and Z are metrizable spaces, then the map $C(X, Y)\times C(Y, Z)\to C(X, Z)$, $(f, g)\mapsto gf$, is continuous whenever each function space has the target majorant topology.

These facts are well-known; some references are given in [25, 1.2].

In the next three lemmas we have collected known PL approximation results.

1.7. Lemma. *Let (n, q) be admissible, M a* PL *n-manifold, N a* PL *q-manifold with $\partial N=\emptyset$, $f\colon M\to N$ an embedding, d a metric for N, and $\varepsilon\in C_+(M)$. Then there is a* PL *embedding $g\colon M\to N$ in $U_d(f, \varepsilon)$.*

Proof. The set fM is locally compact and thus closed in an open set N_0 of N; hence replacing N by N_0 we may assume that f is closed. One can give an elementary proof in the case $q \geqq n = 1$. The case $n \geqq 2$, $q \geqq n+3$ follows both from [28, Theorem 3] and [6, Theorem 1] (or [13, Theorem 8.1]). For the case $n=q=2$, see [30, Theorem 6.4]. The case $n=q=3$ is proved in [4, Theorem 9]. Consider finally the case $(n, q)=(2, 3)$. By each of [3, Theorem 7], [4, Theorem 5], and [4, Theorem 10], there is an embedding $h\colon fM \to N$ such that $hf \in U_d(f, \varepsilon/2)$ and such that 1
$M' = hfM$ is a subpolyhedron and thus, by [30, Theorem 4.9], a PL submanifold of N. Hence there is a PL homeomorphism $g_0\colon M \to M'$ in $U_d(hf, \varepsilon/2)$; cf. [30, Theorem 6.4] or [5, Theorem 4.6]. Then the PL embedding $g\colon M \to N$ defined by g_0 is in $U_d(f, \varepsilon)$. □

1.8. Lemma. *Let n, q, M, N, and d be as in* 1.7 *with* $n \geqq 2$, *let* $f\colon M \to N$ *be a closed embedding, and* $\varepsilon \in C_+(N)$. *Then there is* $\delta \in C_+(M)$ *with the following property: If* $g_i\colon M \to N$, $i=0, 1$, *is a* PL *embedding in* $U_d(f, \delta)$, *there is a* PL *homeomorphism* $h\colon N \to N$ *in* $U_d(\mathrm{id}_N, \varepsilon)$ *such that* $hg_0 = g_1$.

Proof. The case $n \geqq 2$, $q \geqq n+3$ follows from each of [7, Theorem 1], [28, Theorem 2], and [13, Corollary 6.1], the case $2 \leqq n \leqq q = 3$ from [11, Theorem 7.1], and the case $n=q=2$ from [11, Theorem 7.2]. (These results give, moreover, a PL ambient ε-isotopy from id_N to h.) □

1.9. Lemma. *Let n, q, M, N, f, d, and ε be as in* 1.7. *Then for each* PL *n-submanifold M_1 of M which is closed in M, there exists* $\delta \in C_+(M_1)$ *with the following property: If* $g\colon M_1 \to N$ *is a* PL *embedding in* $U_d(f|M_1, \delta)$, *there is a* PL *embedding* $g^*\colon M \to N$ *in* $U_d(f, \varepsilon)$ *which extends* g.

Proof. If $M_1 = \emptyset$, the lemma reduces to 1.7. We may assume that f is closed; cf. the proof of 1.7. An elementary proof can be given if $q \geqq n = 1$. Suppose that $n \geqq 2$. By 1.6 there are $\delta_0 \in C_+(M)$ and $\varepsilon_0 \in C_+(N)$ such that if $f_0 \in U_d(f, \delta_0)$ and $h_0 \in U_d(\mathrm{id}_N, \varepsilon_0)$, then $h_0 f_0 \in U_d(f, \varepsilon)$. Let $\delta \in C_+(M_1)$ be the function which 1.8 gives if we substitute $M \mapsto M_1$, $f \mapsto f|M_1$, $\varepsilon \mapsto \varepsilon_0$. Now let $g \in U_d(f|M_1, \delta)$ be a PL embedding. Choose $\delta_1 \in C_+(M)$ with $\delta_1|M_1 = \delta$ and set $\delta'(x) = \min(\delta_0(x), \delta_1(x))$ for $x \in M$. By 1.7 we can choose a PL embedding $g' \in U_d(f, \delta')$. Then there is a PL homeomorphism $h \in U_d(\mathrm{id}_N, \varepsilon_0)$ with $h(g'|M_1) = g$. Hence $g^* = hg'\colon M \to N$ is a PL embedding in $U_d(f, \varepsilon)$ with $g^*|M_1 = g$. □

1.10. Remarks. 1. For $n \geqq 1$, $q \geqq 2n+1$, Lemma 1.9 also follows from PL general position results ([16, Lemma 4.8, p. 102]; cf. [25, Lemma 3.5]). Moreover, the manifold pair (M, M_1) can now be replaced by a pair (X, X_1), where X is a polyhedron with $\dim X \leqq n$ and X_1 is a closed subpolyhedron of X.

2. We will only need 1.9 in the case where $M_2 = \overline{M \setminus M_1}$ is a PL n-manifold and $M_1 \cap M_2$ a PL $(n-1)$-manifold. If $n=q=2$ or 3, there is the following proof for this special case of 1.9. Suppose first that M is compact. Then, if $n=3$, the

proof is given in [29, Lemma 4] (cf. [2, Theorem 1′]), and for $n=2$ the result certainly also holds with a simpler proof. The general case can easily be deduced from this special case; for example, one can proceed as in the proof of [2, Theorem 3].

3. For $n\geqq 2$, $q\geqq n+3$, Lemmas 1.7, 1.8, and 1.9 also hold if M and (M, M_1) are replaced, respectively, by a polyhedron X or a polyhedral pair (X, X_1) as in 1.10.1 above; this follows from the results quoted in the proofs. However, we do not know whether this holds for 1.9 if $2\leqq n\leqq q\leqq 3$ (or if $n=1$, $q=2$).

4. Lemma 1.9 holds trivially if $q\geqq n=0$. For $n=q\geqq 5$, Lemma 1.7 fails by [19], and 1.9 is not true even if $N=R^q$ (cf. [41, 2.3]). By [27], 1.7 does not hold if $q=n+2\geqq 4$, but if $n=2$, $q=4$, and $M=I^2$, it holds by [45, Theorem 1]. The case $q=n+1\geqq 4$ seems to be unsettled.

1.11. We need the rest of this section only for applications in Section 4. We now give some definitions. A set A in a metric space X is called a *Z^n-set* in X if every continuous map of I^n into X can be uniformly approximated by continuous maps into $X\setminus A$. An embedding of a space into X is called a *Z^n-embedding* if its image is a Z^n-set in X. Let (X, d) and (Y, d') be metric spaces. A map $f\colon X\to Y$ is said to be LIP if for every point p of X there is a neighborhood U of p and $L\geqq 0$ such that $d'(f(x), f(y))\leqq Ld(x, y)$ for all $x, y\in U$. If A, B are subsets of LIP manifolds, the definitions of LIP maps $A\to B$, $A\to X$, and $X\to A$ are obvious; cf. 1.3.

1.12. Lemma. *Let $n\geqq 0$, $q\geqq 2n+1$, X a separable metrizable space with $\dim X\leqq n$, N a topological q-manifold, $f\colon X\to N$ continuous, C a locally compact closed subset of X, $f|C$ a closed Z^n-embedding, and $\mathcal{U}$ an open cover of N. Then $N(f, \mathcal{U})$ contains an embedding g with $g|C=f|C$.*

Proof. The case $C=\emptyset$ is proved in [25, Theorem 5.6]; if, in addition, N can be embedded into R^q, a simpler proof is given in [25, Theorem 2.1]. The general case is due to Heisey and Toruńczyk; see [25, Theorem E of the Introduction]. □

1.13. Lemma. *Let X be a separable metric space, N a metric* LIP *q-manifold, $f\colon X\to N$* LIP, *and the q-dimensional Hausdorff measure $\mathcal{H}^q(X\times I^n)=0$, where $n<q$. Then fX is a Z^n-set in N.*

Proof. Let $g\colon I^n\to N$ be continuous and U a uniform neighborhood of g. Using a topological collar of ∂N in N we may assume that $gI^n\subset \operatorname{int} N$ ([25, Lemma 2.3]). By [26, Corollary 5.18] we may further assume that g is LIP. Then $\mathcal{H}^q(fX\times gI^n)=0$. Hence by a slight generalization of [26, Theorem 6.9] there is $h\in U$ with $hI^n\cap fX=\emptyset$. □

The following theorem is related to Theorem 4.4. For $q\geqq 5$ it is an observation of L. Siebenmann.

1.14. Theorem. *Let* $n \geqq 0$, $q \geqq n+3$, *M a* LIP *n-manifold, N a* LIP *q-manifold,* $\partial N = \emptyset$, *and* $f\colon M \to N$ *an embedding which is a* LIP *map. Then f is locally flat.*

Proof. We may assume that $M = I^n$ and $N = R^q$. The case $n=0$ is trivial. For $q \geqq 3n+1$ the theorem follows from [44, Theorem 3.8]. This implies the case $n=1$. Let $n \geqq 2$. By [7, Theorem 2], 1.7, and [35, Theorem 1.7.2, p. 34], it suffices to prove that if $x \in fI^n$, $\varepsilon > 0$, and $B = \{y \in R^q \mid |x-y| < \varepsilon\}$, then $U = B \setminus fI^n$ is simply connected. By [17, Corollary 1, p. 48], U is connected. Let $g\colon \partial I^2 \to U$ be continuous. There is a continuous extension $g_1\colon I^2 \to B$ of g. Choose a LIP approximation $h\colon I^2 \to B$ of g_1. By [26, Theorem 6.5], $(hI^2 + y) \cap fI^n = \emptyset$ for almost all $y \in R^q$. If $d(g_1, h)$ and $|y|$ are small enough, $g' = h + y$ is a map $I^2 \to U$ such that g and $g'|\partial I^2$ are homotopic in U. (The existence of g' also follows from 1.13.) Hence g is null-homotopic in U. □

1.15. Lemma. *Let X be a locally compact separable metrizable space with* $\dim X \leqq n \geqq 0$. *Then there is a closed embedding of X into* R^{2n+1}.

Proof. This follows easily from 1.12. We can also reduce it to a classical special case of 1.12. Choose a compact metric space Y containing X as a subspace with $Y \setminus X = \{p\}$. Then $\dim Y = \dim X$ by [17, Corollary 2, p. 32]. Hence [17, Theorem V 2] gives an embedding $f\colon Y \to I^{2n+1} \times \{1\}$. Then fX is closed in $\partial I^{2n+2} \setminus \{f(p)\} \approx R^{2n+1}$. □

The following lemma is due to J. Väisälä.

1.16. Lemma. *Let* $n \geqq 4$, *and let X be a locally connected locally compact separable metrizable space with* $\dim X \leqq n-2$ *such that for each component X_j of X there is a closed embedding* $f_j\colon X_j \to R^n$. *Then there is a closed embedding* $f\colon X \to R^n$.

In addition, if X is a topological manifold and each f_j is locally flat, f can also be chosen to be locally flat.

Proof. We show first that if X is connected, there is a closed embedding $g\colon X \to J^n$ with $\partial J^n \not\subset \overline{gX}$. Choose a closed embedding $f\colon X \to R^n$. There is a closed PL embedding $\alpha\colon R^1_+ \to R^n$ with $\alpha R^1_+ \cap fX = \emptyset$. By [35, Theorem 3.4.3, p. 109], α is flat. Thus we get a closed embedding $h\colon X \to J^n$ with $hX \cap J^1_+ = \emptyset$. Obviously, there is a homeomorphism $\varphi\colon J^n \to J^n$ with $e_1 \notin \overline{\varphi h X}$. Then $g = \varphi h$ is the required map.

In the general case each component X_j of X is open and we may assume that $j \in \{1, 2, \ldots\}$. Let $H^n = \operatorname{int} R^n_+ \approx R^n$ and $U_j = J^n \cap H^n + 3je_1$. The first part of the proof implies that there is a closed embedding $g_j\colon X_j \to U_j$ with $\overline{g_j X_j} \cap \partial U_j \subset \partial H^n$. Then $f = \bigcup_j g_j\colon X \to H^n$ is a closed embedding.

The assertion concerning local flatness can be proved similarly. □

1.17. Lemma. *Let M be a topological n-manifold,* $n \geqq 1$. *Then there exists a closed locally flat embedding of M into* R^{2n}.

Proof. Replacing M by its double, we may assume that $\partial M=\emptyset$. The case $n=1$ is trivial. Suppose $n=2$ or 3. By the tubular neighborhood theorem it suffices to prove that M is homeomorphic to a closed C^∞-differentiable submanifold of R^{2n}. By [30, Theorems 4.8, 8.3, 23.1, and 35.3], M is homeomorphic to a PL manifold. Hence, by [8, Theorem III], M is homeomorphic to a C^1-differentiable manifold N. Thus, by [31, Theorem 4.8], it suffices to show that if N is connected, there exists a closed C^1-embedding $f\colon N\to R^{2n}$ with $fN\subset J^{2n-1}\times R^1$. If N is compact, this follows from [46, Theorem 5]. If N is non-compact, there is a C^1-embedding $f_0\colon N\to J^{2n-1}$ by [15, Theorem 4.6]. Choose a proper C^1-differentiable function $f_1\colon N\to R^1$; then $f=(f_0, f_1)$ is the required map.

Suppose now $n\geqq 4$. By 1.16 we may assume that M is connected. Only the case of compact M can be found in the literature: If M is orientable, the lemma follows from [23, Theorem 1, p. 11]. If the orientability is not supposed, define $f\colon M\to R^{2n}$ by $f(x)=0$. Then f is simply connected, i.e., $\pi_0(f)=\pi_1(f)=0$, where $\pi_i(f)=\pi_i(C_f, M)$ with C_f the mapping cylinder of f. By [32, Theorem 7, p. 445], f is homotopic to a locally flat embedding (even for $n\geqq 3$). Cf. also [12, Embedding Theorem 3] (for $n\geqq 2$). For the case where M is non-compact a proof has been sketched by J. Dancis (written personal communication, 1980) and L. Siebenmann (oral personal communication, 1981). $\square$

2. PL approximation of quasisymmetric embeddings in Euclidean spaces

2.1. In this section we prove first the main result of the section, Theorem 2.16. Then we give two corollaries of it. These results concern PL approximation of quasisymmetric embeddings in Euclidean spaces. We close the section by a similar theorem about weakly quasisymmetric embeddings.

We begin by defining cubical decompositions.

2.2. Let $n\geqq 1$. For $k\in \mathbf{Z}$ let $\mathscr{L}_n(k)$ denote the family of closed n-cubes in R^n with side length 2^k and with vertices in $2^k\mathbf{Z}^n$. Let $\mathscr{L}_n=\cup\{\mathscr{L}_n(k)|k\in \mathbf{Z}\}$. For $Q\in\mathscr{L}_n$ let z_Q denote the center and $2\lambda_Q$ the side length of Q. Define $\alpha_Q\colon R^n\to R^n$ by $\alpha_Q(x)= z_Q+\lambda_Q x$; then $Q=\alpha_Q I^n$.

2.3. Lemma. *Let $X\subset R^n$ and let Y be an open subset of X such that Y is the union of a subfamily of $\mathscr{L}_n$ which is locally finite in Y. Then there exists a subfamily χ of $\mathscr{L}_n$ with the following properties:*

(1) $Y=\cup\chi$.

(2) *If* $Q, R\in\chi$, $Q\cap R\neq\emptyset$, *and* $Q\neq R$, *then* $\operatorname{int} Q\cap\operatorname{int} R=\emptyset$ *and*

$$\lambda_Q/\lambda_R\in\{1/2, 1, 2\}.$$

(3) $\alpha_Q I^n(3)\cap X=\cup\{R\in\mathscr{L}_n|R\cap Q\neq\emptyset,\ \lambda_R=\lambda_Q,\ R\subset Y\}$ *for* $Q\in\chi$.

Moreover, if φ is a continuous map of Y into a metric space (Z, d) and if $\varepsilon \in C_+(Y)$, then χ can be chosen in such a way that

(4) $d(\varphi Q) < \varepsilon(x)$ *if* $x \in Q \in \chi$.

Proof. We construct χ which satisfies (1), ..., (4). For $Q \in \mathscr{L}_n$ let Q' denote the right side of (3). Let

$$\mathscr{L} = \{Q \in \mathscr{L}_n | Q \subset Y,\ \alpha_Q I^n(3) \cap X = Q',\ d(\varphi Q') < \min \varepsilon Q'\}.$$

Let $\chi_0 = \mathscr{L} \cap \mathscr{L}_n(0)$. For $i \geqq 1$ we define χ_i inductively as the family of all cubes $Q \in \mathscr{L} \cap \mathscr{L}_n(-i)$ such that for no $j < i$ there is $R \in \chi_j$ containing Q. Define $\chi = \cup \{\chi_i | i \geqq 0\}$. To prove (1), let $x \in Y$. There are $i \geqq 0$, $k \geqq 1$, and $P_1, \ldots, P_k \in \mathscr{L}_n(-i)$ such that $P = P_1 \cup \ldots \cup P_k$ is a neighborhood of x in Y. For each $j > i$ there is $Q_j \in \mathscr{L}_n(-j)$ with $x \in Q_j \subset Y$. If j is large enough, then $\alpha_{Q_j} I^n(3) \cap X \subset P$, which implies $\alpha_{Q_j} I^n(3) \cap X = Q'_j$, and $d(\varphi Q'_j) < \min \varepsilon Q'_j$. Thus $Q_j \in \mathscr{L}$, whence $x \in \cup_{l=0}^{j} (\cup \chi_l)$. The first assertion in (2) is clear. In the second we may assume that $Q \in \chi_i$ and $R \in \chi_j$ with $j \geqq i$. Let $\mathscr{S} = \{S \in \mathscr{L}_n(-i-1) | S \cap Q \neq \emptyset,\ S \not\subset Q,\ S \subset Y\}$. Then $\alpha_Q(2I^n \setminus J^n) \cap X = \cup \mathscr{S}$. If $S \in \mathscr{S}$, we have $S' = \alpha_S I^n(3) \cap X \subset Q'$ and, hence, $d(\varphi S') < \min \varepsilon S'$, which implies $S \in \mathscr{L}$. It follows that $j = i$ or $i+1$; consequently, $\lambda_Q / \lambda_R \in \{1, 2\}$. The conditions (3) and (4) are satisfied by construction. □

2.4. Let X and Y be as in 2.3 and let χ be any subfamily of $\mathscr{L}_n$ satisfying (1), ..., (3). Dividing each cube of χ into 2^n cubes by bisecting the sides we get a new cube family $\bar{\chi}$. Obviously, the conditions (1), ..., (3) are satisfied also by $\bar{\chi}$. If χ satisfies (4), $\bar{\chi}$ satisfies it, too. A cube family $\bar{\chi}$ obtained in this manner is called a *cubical decomposition of Y with respect to X.*

2.5. From now on and up to the end of the proof of 2.16, we suppose that X and Y are as in 2.3 and that $\mathscr{K}$ is a cubical decomposition of Y with respect to X. Let $\varepsilon > 0$ and let $f\colon X \to R^q$ be an η-quasisymmetric embedding, where (n, q) is admissible. Finally, we assume that Y is a manifold and thus a PL manifold.

For $Q \in \mathscr{K}$ we set $\varrho_Q = |f(z_Q) - f(z_Q + \lambda_Q e_1)|$ and $Q^* = \alpha_Q I^n(9/8) \cap X$. Then $Q^* \cap \alpha_R J^n(15/8) = \emptyset$ if $Q, R \in \mathscr{K}$ and $Q \cap R = \emptyset$.

2.6. Lemma. *There are constants $c_1 \geqq 1$ and $c_2 \geqq 1$ depending only on n and η with the following properties:*

(1) *If* $Q, R \in \mathscr{K}$, $Q \cap R \neq \emptyset$, *and* $Q \neq R$, *then* $\varrho_Q / c_1 \leqq |f(z_Q) - f(z_R)| \leqq c_1 \varrho_Q$.

(2) *If* $Q, R \in \mathscr{K}$ *and* $Q \cap R \neq \emptyset$, *then* $\varrho_Q / \varrho_R \leqq c_1^2$.

(3) *If* $Q \in \mathscr{K}$, $x \in Q^*$, *and* $y \in X \setminus \alpha_Q J^n(15/8)$, *then* $|f(x) - f(y)| \geqq \varrho_Q / c_2$.

Proof. (1): We have $(3/2)\lambda_Q < |z_Q - z_R| \leqq 3\sqrt{n}\,\lambda_Q$. Hence

$$\varrho_Q / \eta(2/3) \leqq |f(z_Q) - f(z_R)| \leqq \eta(3\sqrt{n})\varrho_Q.$$

(2): This follows from (1).

(3): Since $(15/8)\lambda_Q \leqq |z_Q - y|$, we have $\varrho_Q \leqq \eta(8/15)|f(z_Q)-f(y)|$. Since $|z_Q - x| \leqq (9/8)\sqrt{n}\lambda_Q \leqq (3/2)\sqrt{n}|x-y|$ and thus $|z_Q - y| \leqq t|x-y|$, where $t=(3/2)\sqrt{n}+1$, we have $|f(z_Q)-f(y)| \leqq \eta(t)|f(x)-f(y)|$. Hence $\varrho_Q \leqq \eta(8/15)\eta(t)|f(x)-f(y)|$. □

2.7. Let $\varkappa_1$, $\varkappa_2$, and $\varkappa$ be the least natural numbers such that

$$2^{\varkappa_1} \geqq 2\sqrt{q}, \quad 2^{\varkappa_2} \geqq 2c_1^2, \quad \text{and} \quad \varkappa \geqq 2^{\varkappa_1+\varkappa_2}(2c_1 + \sqrt{q}\, 2^{\varkappa_2-\varkappa_1}).$$

These numbers depend only on n, q, and η. Define

$$E_1 = \{2^k \mid k \in \mathbf{Z},\ |k| \leqq \varkappa_2\},$$

$$E_2 = \{2^{-\varkappa_1-\varkappa_2}(k_1, \ldots, k_q) \mid k_i \in \mathbf{Z},\ |k_i| \leqq \varkappa\}.$$

For each $Q \in \mathscr{K}$ let μ_Q be the integer with $2^{\mu_Q} \leqq \varrho_Q < 2^{\mu_Q+1}$, and define $s_Q = 2^{\mu_Q}$. Then $\varrho_Q/2 < s_Q \leqq \varrho_Q$. We choose $b_Q \in 2^{\mu_Q-\varkappa_1}\mathbf{Z}^q$ such that $|b_Q - f(z_Q)| \leqq \sqrt{q}2^{\mu_Q-\varkappa_1-1}$ ($\leqq s_Q/4$). The following lemma can now be proved by the aid of 2.6(1) and 2.6(2) exactly as [18, Lemma 2.4].

2.8. Lemma. *If $Q, R \in \mathscr{K}$, $Q \cap R \neq \emptyset$, then*
(1) $s_Q/s_R \in E_1$,
(2) $(b_Q - b_R)/s_Q \in E_2$. □

2.9. We express $\mathscr{K}$ as a disjoint union $\mathscr{K} = \mathscr{K}_1 \cup \ldots \cup \mathscr{K}_M$, where each family $\mathscr{K}_i$ is disjoint and where $M = M(n)$. In fact, this can be done with $M(n) = 2^n$ as follows. Number the cubes of $\mathscr{L}_n(0)$ in I^n by $S_1, \ldots, S_{M(n)}$. Let $\mathscr{K} = \bar{\chi}$. If $Q \in \mathscr{K}$, let R be the unique cube in χ with $\lambda_R = 2\lambda_Q$ and $Q \subset R$; then set $S_Q = \alpha_R^{-1}Q$. Define $\mathscr{K}_i = \{Q \in \mathscr{K} \mid S_Q = S_i\}$. It is obvious that the families $\mathscr{K}_1, \ldots, \mathscr{K}_{M(n)}$ satisfy the requirements. We set $\mathscr{K}_i^* = \mathscr{K}_1 \cup \ldots \cup \mathscr{K}_i$ and $F_i = \cup\{Q^* \mid Q \in \mathscr{K}_i^*\}$.

2.10. Let $\mathscr{I}_n = \{Q \in \mathscr{L}_n(0) \mid Q \subset 2I^n \setminus J^n\}$. If $Q \in \mathscr{I}_n$ and $t \in \{1, 2, 3\}$, we set $Q(t) = \alpha_Q I^n(1+2^{-t})$. Let $\mathscr{T}_n$ be the finite set consisting of manifolds which can be expressed as a union $I^n \cup (\cup \mathscr{I})$, $\mathscr{I} \subset \mathscr{I}_n$. If $T \in \mathscr{T}_n$, let $\mathscr{P}(T)$ denote the finite set of pairs (P', P) of the form

$$P' = (I^n(9/8) \cup Q_1(t_1) \cup \ldots \cup Q_k(t_k)) \cap T,$$

$$P\ = (Q_1(t_1) \cup \ldots \cup Q_k(t_k)) \cap T,$$

where $Q_i \in \mathscr{I}_n$, $Q_i \subset T$, and $t_i \in \{1, 2, 3\}$ for $i \leqq k \geqq 0$. It is obvious that the sets P', P, and $\overline{P' \setminus P}$ are PL n-manifolds and that $P \cap \overline{P' \setminus P}$ is a PL $(n-1)$-manifold.

Let $1 \leqq i \leqq M(n)$ and $Q \in \mathscr{K}_i$. Since Y is a manifold, we have $T_Q = I^n(2) \cap \alpha_Q^{-1}X \in \mathscr{T}_n$. We set

$$P_Q = (\cup\{\alpha_Q^{-1}R^* \mid R \in \mathscr{K}_{i-1}^*,\ R \cap Q \neq \emptyset\}) \cap T_Q,$$

$$P'_Q = \alpha_Q^{-1}Q^* \cup P_Q.$$

Then $(P'_Q, P_Q) \in \mathscr{P}(T_Q)$.

2.11. Let $T\in\mathcal{T}_n$, let $g\colon T\to R^q$ be an embedding, and $\varepsilon_0>0$. Let

$$(P', P)\in\mathcal{P}(T).$$

Then by 1.9 (or also by 1.10.2 if $n=q=2$ or 3) there exists $\delta\in(0, \varepsilon_0]$ such that if $g_1\colon P\to R^q$ is a PL embedding with $d(g_1, g|P)<\delta$, there is a PL embedding $g_1^*\colon P'\to R^q$ with $g_1^*|P=g_1$ and $d(g_1^*, g|P')<\varepsilon_0$. We let $\delta(g, \varepsilon_0)$ denote the greatest δ satisfying this for all $(P', P)\in\mathcal{P}(T)$.

2.12. For $T\in\mathcal{T}_n$ let $H_\eta(T)$ be the set of all η-quasisymmetric embeddings $g\colon T\to R^q$ with $|g(0)|\leqq 1/4$ and $3/4\leqq|g(e_1)|\leqq 3$. In the topology of uniform convergence $H_\eta(T)$ is compact by [38, Remark 3.6 and Theorem 3.7]. Let $H_\eta=\cup\{H_\eta(T)|T\in\mathcal{T}_n\}$. We set $\delta^*(\varepsilon_0)=\inf\{\delta(g, \varepsilon_0)|g\in H_\eta\}$ $(\leqq\varepsilon_0)$ for $\varepsilon_0>0$. Then as in [41, Lemma 2.6], we get $\delta^*(\varepsilon_0)>0$, and $\delta^*(\varepsilon_0)$ depends only on n, q, η, and ε_0.

For each $Q\in\mathcal{K}$ we define $\beta_Q\colon R^q\to R^q$ by $\beta_Q(x)=(x-b_Q)/s_Q$ and set $f_Q=\beta_Q f\alpha_Q|T_Q$. Then $f_Q\in H_\eta$, because α_Q and β_Q are similarities and because $|f_Q(0)|=|f(z_Q)-b_Q|/s_Q\leqq 1/4$, $|f_Q(e_1)|\leqq\varrho_Q/s_Q+1/4<9/4$, and $|f_Q(e_1)|\geqq\varrho_Q/s_Q-1/4\geqq 3/4$.

2.13. We choose positive numbers $\delta_1\leqq\ldots\leqq\delta_{M(n)}$ such that

$$\delta_{M(n)}<\min\left(\varepsilon/2,\ 1/4c_1^2c_2,\ 1/4c_1^2\eta(2)\eta\left(2\sqrt{n}\right)\right)$$

and $\delta_i<\delta^*(\delta_{i+1})/(2^{\varkappa_2+1}+1)$. The numbers δ_i depend only on n, q, η, and ε.

2.14. Lemma. *For every* $i\in\{1, \ldots, M(n)\}$ *there exist a finite set* A_i *of* PL *embeddings* $g\colon I^n(9/8)\cap T\to R^q$, *where* $T\in\mathcal{T}_n$, *such that* A_i *depends only on* n, q, η, *and* ε, *and an injective* PL *map* $\varphi_i\colon F_i\to R^q$ *with*

(1) $\beta_Q\varphi_i\alpha_Q|\alpha_Q^{-1}Q^*\in A_i$ *if* $Q\in\mathcal{K}_i^*$,

(2) $d(f|Q^*, \varphi_i|Q^*)\leqq 2\delta_i s_Q$ *if* $Q\in\mathcal{K}_i^*$.

Proof. The proof is by induction on i. Let $T\in\mathcal{T}_n$. We choose a finite subset $H(T, \delta_1)\subset H_\eta(T)$ such that for every $h\in H_\eta(T)$ there is $h'\in H(T, \delta_1)$ with $d(h', h)\leqq\delta_1$. By 1.7 we choose for every $h\in H(T, \delta_1)$ a PL embedding $g_h\colon I^n(9/8)\cap T\to R^q$ with $d(g_h, h|I^n(9/8)\cap T)\leqq\delta_1$. We define $A_1=\{g_h|h\in H(T, \delta_1),\ T\in\mathcal{T}_n\}$. If $Q\in\mathcal{K}$, we choose $h_Q\in H(T_Q, \delta_1)$ with $d(h_Q, f_Q)\leqq\delta_1$ and set $g_Q=g_{h_Q}$.

We define a PL map $\varphi_1\colon F_1\to R^q$ by $\varphi_1|Q^*=\beta_Q^{-1}g_Q\alpha_Q^{-1}|Q^*$ for $Q\in\mathcal{K}_1$. Then (1) is satisfied. We have $d(f|Q^*, \varphi_1|Q^*)=s_Q d(f_Q|\alpha_Q^{-1}Q^*, g_Q)\leqq 2\delta_1 s_Q$ if $Q\in\mathcal{K}_1$. If $Q, R\in\mathcal{K}_1$, $Q\neq R$, $s_Q\geqq s_R$, $x\in Q^*$, and $y\in R^*$, then, since $y\notin\alpha_Q J^n(15/8)$, we get by 2.6(3)

$$\begin{aligned}(2.15)\qquad |\varphi_1(x)-\varphi_1(y)| &\geqq |f(x)-f(y)|-|f(x)-\varphi_1(x)|-|f(y)-\varphi_1(y)|\\ &\geqq \varrho_Q/c_2-2\delta_1(s_Q+s_R)\geqq s_Q(1/c_2-4\delta_1)>0.\end{aligned}$$

Hence φ_1 is injective.

Suppose now that A_{i-1} and φ_{i-1} satisfying (1) and (2) have been constructed. Let $Q\in\mathcal{K}_i$. Consider the PL embedding $\gamma_Q=\beta_Q\varphi_{i-1}\alpha_Q|P_Q$. If $x\in P_Q$, there is $R\in\mathcal{K}_{i-1}^*$ such that $\alpha_Q(x)\in R^*$ and $R\cap Q\neq\emptyset$. Then $\gamma_Q(x)=\beta_Q\beta_R^{-1}\psi\alpha_R^{-1}\alpha_Q(x)$, where $\psi=\beta_R\varphi_{i-1}\alpha_R|\alpha_R^{-1}R^*\in A_{i-1}$. It is easy to see that the maps $\alpha_R^{-1}\alpha_Q$ belong to a finite set depending only on n. The sets $I^n(2)\cap\alpha_Q^{-1}R^*$ are among the sets P considered in 2.10. Thus the sets $P_Q\cap\alpha_Q^{-1}R^*$ belong to a finite set depending only on n. Since $\beta_Q\beta_R^{-1}(y)=s_Ry/s_Q+(b_R-b_Q)/s_Q$, by 2.8 the maps $\beta_Q\beta_R^{-1}$ belong to a finite set depending only on n, q, and η. Hence the maps γ_Q belong to a finite set C_i depending only on n, q, η, and ε. If x and R are as above, we have

$$\begin{aligned}|\gamma_Q(x)-f_Q(x)| &= |\varphi_{i-1}(\alpha_Q(x))-f(\alpha_Q(x))|/s_Q\\ &\leqq 2\delta_{i-1}s_R/s_Q \leqq 2^{\varkappa_2+1}\delta_{i-1}\end{aligned}$$

by 2.8, which implies that $d(\gamma_Q, h_Q|P_Q)<\delta^*(\delta_i)$. Hence there is a PL embedding $g_Q^*\colon P'_Q\to R^q$ with $g_Q^*|P_Q=\gamma_Q$ and $d(g_Q^*, h_Q|P'_Q)<\delta_i$. We can choose the maps g_Q^* such that the maps $g_Q^*|\alpha_Q^{-1}Q^*$ belong to a finite set G_i, depending only on n, q, η, and ε, of PL embeddings $g\colon I^n(9/8)\cap T\to R^q$, $T\in\mathcal{T}_n$.

We set $\varphi_i|F_{i-1}=\varphi_{i-1}$ and $\varphi_i|Q^*=\beta_Q^{-1}g_Q^*\alpha_Q^{-1}|Q^*$ for $Q\in\mathcal{K}_i$. Then $\varphi_i\colon F_i\to R^q$ is a well-defined PL map and (1) is satisfied with $A_i=A_{i-1}\cup G_i$. If $Q\in\mathcal{K}_i$, then $d(f|Q^*, \varphi_i|Q^*)=s_Qd(f_Q|\alpha_Q^{-1}Q^*, g_Q^*|\alpha_Q^{-1}Q^*)\leqq 2\delta_is_Q$, which implies (2). We prove that φ is injective. Observe first that φ_i is the embedding $\beta_Q^{-1}g_Q^*\alpha_Q^{-1}$ on $\alpha_QP'_Q$ for every $Q\in\mathcal{K}_i$. Thus it suffices to show that $\Delta=|\varphi_i(x)-\varphi_i(y)|>0$ if $Q\in\mathcal{K}_i$, $R\in\mathcal{K}_i^*$, $x\in Q^*$, $y\in R^*$, and either (a) $Q\cap R=\emptyset$ or (b) $Q\cap R\neq\emptyset$ and $y\notin\alpha_QI^n(2)$. We proceed as in (2.15). If either (a) holds or (b) holds with $s_Q\geqq s_R$, we get $\Delta\geqq s_Q(1/c_2-4\delta_i)>0$. If (b) holds with $s_Q<s_R$, we use the fact $\varrho_Q\geqq\varrho_R/c_1^2$, implied by 2.6(2), and get $\Delta\geqq s_R(1/c_1^2c_2-4\delta_i)>0$. □

2.16. Theorem. *Let* (n, q) *be admissible, let* $\eta\colon R_+^1\to R_+^1$ *be a homeomorphism, and let* $\varepsilon>0$. *Then there exist a homeomorphism* $\eta^*\colon R_+^1\to R_+^1$ *and a finite set* D *of* PL *embeddings* $g\colon T\to R^q$, $T\in\mathcal{T}_n$, *with the following property: Let* $X\subset R^n$, *let* $f\colon X\to R^q$ *be an* η*-quasisymmetric embedding, let* Y *be an open subset of* X *which is the union of a subfamily of* $\mathcal{L}_n$ *locally finite in* Y *and which is a manifold, and let* $\mathcal{K}$ *be a cubical decomposition of* Y *with respect to* X. *Then there exists an* η^**-quasisymmetric embedding* $f^*\colon X\to R^q$ *such that*

(1) $f^*|X\setminus Y=f|X\setminus Y$,
(2) $f^*|Y$ *is* PL,
(3) $d(f^*|Q, f|Q)\leqq\varepsilon\varrho_Q$ *for every* $Q\in\mathcal{K}$,
(4) $\beta_Qf^*\alpha_Q|T_Q\in D$ *for every* $Q\in\mathcal{K}$.

Proof. We define $f^*=\varphi_{M(n)}\cup(f|X\setminus Y)$. Then (1), (2), and (3) are satisfied. One can find D and prove (4) in the same way as one obtained the relation $\gamma_Q\in C_i$ in the proof of 2.14. Two auxiliary results will be proved next. The first one, (2.17), implies that f^* is an embedding.

Let $M=M(n)$. Define $a_0=4\delta_Mc_1^2c_2\in(0, 1)$ and $a=(1-a_0)^{-1}$. We show that

$$|f(x)-f(y)|/a \leqq |f^*(x)-f^*(y)| \leqq a|f(x)-f(y)| \tag{2.17}$$

if $x\in Q\in\mathscr{K}$ and $y\in X\setminus\alpha_Q J^n(2)$. Let $\Delta=\big||f^*(x)-f^*(y)|-|f(x)-f(y)|\big|$. Suppose first that $y\in R\in\mathscr{K}$. Then $\Delta\leqq 2\delta_M(s_Q+s_R)$. If $Q\cap R\neq\emptyset$, this implies, by 2.6, $\Delta\leqq 2\delta_M(1+c_1^2)\varrho_Q\leqq 2\delta_M(1+c_1^2)c_2|f(x)-f(y)|$, whereas if $Q\cap R=\emptyset$, then $\Delta\leqq 4\delta_M c_2|f(x)-f(y)|$. Suppose now that $y\in X\setminus Y$. Then $\Delta\leqq 2\delta_M\varrho_Q\leqq 2\delta_M c_2|f(x)-f(y)|$. Thus $\Delta\leqq a_0|f(x)-f(y)|$ in all cases. Hence (2.17) holds.

Define $b_0=4\delta_M c_1^2\eta(2)\eta(2\sqrt{n})\in(0, 1)$ and $b=(1-b_0)^{-1}$. We show that

$$|f(x)-f(y)|/b \leqq |f^*(x)-f^*(y)| \leqq b\,|f(x)-f(y)| \tag{2.18}$$

if $Q\in\mathscr{K}$, $x, y\in\alpha_Q I^n(2)\cap X$, and $|x-y|\geqq\lambda_Q$. Let Δ be as above. Then $\Delta\leqq 4\delta_M c_1^2\varrho_Q$. We may assume that $|x-z_Q|\geqq\lambda_Q/2$. Then $\varrho_Q\leqq\eta(2)|f(x)-f(z_Q)|$. Since $|x-z_Q|\leqq 2\sqrt{n}\,\lambda_Q\leqq 2\sqrt{n}\,|x-y|$, we have $|f(x)-f(z_Q)|\leqq\eta(2\sqrt{n})|f(x)-f(y)|$. Thus $\Delta\leqq b_0|f(x)-f(y)|$, whence (2.18).

Now we prove that f^* is quasisymmetric. Since every $g\in D$ is quasisymmetric and since α_Q and β_Q are similarities, (4) implies that $f^*|\alpha_Q I^n(2)\cap X$ is η_0-quasisymmetric for every $Q\in\mathscr{K}$ for some η_0 depending only on n, q, η, and ε. Set $\eta_1(t)=\sup\{\eta_0(s)\eta(s')|s, s'\in I, ss'\leqq t\}$ for $t\in I$. Then $\eta_1\colon I\to R^1_+$ is bounded and non-decreasing. If $\varepsilon_1>0$, there is $\delta\in(0, 1)$ such that $\eta_0(\delta)\leqq\varepsilon_1/\eta(1)$ and $\eta(\delta)\leqq\varepsilon_1/\eta_0(1)$, whence $\eta_1(\delta^2)\leqq\varepsilon_1$. Thus $\eta_1(t)\to 0$ as $t\to 0$. It follows that there is a homeomorphism $\eta_2\colon R^1_+\to R^1_+$ with $\eta_1\leqq\eta_2|I$.

Let $u, v, x\in X$, $v\neq x$, $t=|u-x|/|v-x|$, and $t'=|f^*(u)-f^*(x)|/|f^*(v)-f^*(x)|$. We need an estimate $t'\leqq\eta^*(t)$. We divide the consideration into four separate cases.

Case 1. Let $x\in Q\in\mathscr{K}$ and $u, v\in\alpha_Q I^n(2)$. Then $t'\leqq\eta_0(t)$.

Case 2. Let $x\in Q\in\mathscr{K}$, $u\in\alpha_Q I^n(2)$, and $v\notin\alpha_Q I^n(2)$.

Subcase 2a: $|u-x|\geqq\lambda_Q$. Then, by (2.18) and (2.17),

$$\begin{aligned}|f^*(u)-f^*(x)| &\leqq b|f(u)-f(x)| \leqq b\eta(t)|f(v)-f(x)|\\ &\leqq ab\eta(t)|f^*(v)-f^*(x)|.\end{aligned}$$

Subcase 2b: $|u-x|<\lambda_Q$. Choose $y\in Q$ with $|x-y|=\lambda_Q$. Then $|u-x|<|y-x|<|v-x|$. Hence by Subcase 2a,

$$\begin{aligned}t' &= \frac{|f^*(u)-f^*(x)|}{|f^*(y)-f^*(x)|}\cdot\frac{|f^*(y)-f^*(x)|}{|f^*(v)-f^*(x)|}\\ &\leqq \eta_0\left(\frac{|u-x|}{|y-x|}\right)ab\eta\left(\frac{|y-x|}{|v-x|}\right)\leqq ab\eta_2(t).\end{aligned}$$

Case 3. Let $x\in Q\in\mathscr{K}$, $u\notin\alpha_Q I^n(2)$, and $v\in\alpha_Q I^n(2)$.

Subcase 3a: $|v-x|\geqq\lambda_Q$. Now

$$\begin{aligned}|f^*(u)-f^*(x)| &\leqq a|f(u)-f(x)| \leqq a\eta(t)|f(v)-f(x)|\\ &\leqq ab\eta(t)|f^*(v)-f^*(x)|.\end{aligned}$$

Subcase 3b: $|v-x|<\lambda_Q$. Choose $y\in Q$ with $|x-y|=\lambda_Q$. Then by Subcase 3a and since $|v-x|<|y-x|<|u-x|$,

$$t'=\frac{|f^*(u)-f^*(x)|}{|f^*(y)-f^*(x)|}\cdot\frac{|f^*(y)-f^*(x)|}{|f^*(v)-f^*(x)|}$$

$$\leqq ab\eta\left(\frac{|u-x|}{|y-x|}\right)\eta_0\left(\frac{|y-x|}{|v-x|}\right)\leqq ab\eta(t)\eta_0(t).$$

Case 4. Let either $x\in Q\in\mathcal{K}$ and $u, v\notin\alpha_Q I^n(2)$ or $x\in X\setminus Y$. Then by (2.17),

$$|f^*(u)-f^*(x)|\leqq a|f(u)-f(x)|\leqq a\eta(t)|f(v)-f(x)|$$
$$\leqq a^2\eta(t)|f^*(v)-f^*(x)|.$$

Thus there exists a homeomorphism $\eta^*\colon R^1_+\to R^1_+$ which depends only on n, q, η, and ε such that $t'\leqq\eta^*(t)$, i.e., such that f^* is η^*-quasisymmetric. □

2.19. Remark. In Theorem 2.16 we assumed that Y is a manifold only in order to be able to use the PL approximation results 1.7 and 1.9. However, for $q\geqq n=1$ this assumption is redundant, and for $n\geqq 2$, $q\geqq n+3$ we could omit it by 1.10.3 (cf. also 1.10.1). This remark also applies to 2.20 and 3.2.

2.20. Corollary. *Let n, q, and η be as in* 2.16. *Then there exists a homeomorphism $\eta^*\colon R^1_+\to R^1_+$ with the following property: Let X, f, and Y be as in* 2.16 *and let $\varepsilon\in C_+(Y)$. Then there exists an η^*-quasisymmetric embedding $f^*\colon X\to R^q$ such that*

(1) $f^*|X\setminus Y=f|X\setminus Y$,

(2) $f^*|Y$ *is* PL,

(3) $|f^*(x)-f(x)|<\varepsilon(x)$ *for every* $x\in Y$.

Proof. This follows from 2.16 (choose $\varepsilon=1$) and 2.3. □

2.21. Corollary. *Let (n, q) be admissible and let $\eta\colon R^1_+\to R^1_+$ be a homeomorphism. Then there exists a homeomorphism $\eta^*\colon R^1_+\to R^1_+$ with the following property: Let $U\subset R^n$ be open, let $f\colon U\to R^q$ be an η-quasisymmetric embedding, and let $\varepsilon\in C_+(U)$. Then there exists an η^*-quasisymmetric* PL *embedding $g\colon U\to R^q$ such that $|g(x)-f(x)|<\varepsilon(x)$ for every $x\in U$, $\bar g|\partial U=\bar f|\partial U$, and, if $1\leqq n=q\leqq 3$, $gU=fU$. Here $\bar f$ and $\bar g$ are the closed embeddings $\bar U\to R^q$ extending f and g, respectively (cf.* 1.2*).*

Proof. Let η^* be the homeomorphism of 2.20. It is easy to see that U is the union of a subfamily of $\mathcal{L}_n$ which is locally finite in U. Thus 2.20 gives an η^*-quasisymmetric PL embedding $g\colon U\to R^q$ with $|g(x)-f(x)|<\varepsilon(x)$ and $|g(x)-f(x)|<d(f(x), \bar f\partial U)$ for every $x\in U$. Then $\bar g|\partial U=\bar f|\partial U$. Suppose that $n=q$. If V is a component of U, we have $\partial gV=\bar g\partial V=\bar f\partial V=\partial fV$ and $gV\subset fV$, whence $gV=fV$. Thus $gU=fU$. □

2.22. Theorem. *Let* (n, q) *be admissible and let* $H \geqq 1$. *Then there exists* $H^* \geqq 1$ *with the following property: Let* $X \subset R^n$, *let* $f\colon X \to R^q$ *be a weakly H-quasisymmetric embedding, let* $U \subset X$ *be open in* R^n, *and let* $\varepsilon \in C_+(U)$. *Then there exists a weakly* H^**-quasisymmetric embedding* $f^*\colon X \to R^q$ *such that*

(1) $f^*|X \setminus U = f|X \setminus U$,

(2) $f^*|U$ *is* PL,

(3) $|f^*(x) - f(x)| < \varepsilon(x)$ *for every* $x \in U$,

(4) $\bar{f}^*|\partial U = \bar{f}|\partial U$, *where* $\bar{f}$ *and* $\bar{f}^*$ *are the closed embeddings* $\bar{U} \to R^q$ *extending* $f|U$ *and* $f^*|U$, *respectively (cf.* 1.2*),*

(5) $f^*U = fU$ *if* $1 \leqq n = q \leqq 3$.

Proof. It suffices to prove the modification of Theorem 2.16 in which η is replaced by H, η^* is replaced by H^*, the embedding f is only assumed to be weakly H-quasisymmetric, Y is assumed to be open in R^n, and the embedding f^* is only required to be weakly H^*-quasisymmetric. To this end, we modify the proof of 2.16 (in 2.5—2.18). We first observe that in this proof, with two exceptions, one actually used, instead of the assumption that the embedding f is η-quasisymmetric, only the assumption that $f|Q'$ is η-quasisymmetric for every $Q \in \mathcal{K}$, where $Q' = \alpha_Q I^n(3) \cap X$. The exceptions were (a) the proof of 2.6(3) and (b) the estimation of t' in Cases 1—4. Now let f and Y be as in the present modification of 2.16, and let $\mathcal{K}$ be a cubical decomposition of Y with respect to X. Then $Q' = \alpha_Q I^n(3) \subset Y$ for every $Q \in \mathcal{K}$. It follows from [38, Theorem 2.16] that $f|Q'$ is η-quasisymmetric for every $Q \in \mathcal{K}$ with η depending only on n, q, and H. Hence it suffices to re-examine (a) and (b).

For (a) let $Q \in \mathcal{K}$, $x \in Q^*$, and $y \in X \setminus \alpha_Q J^n(15/8)$. Choose $u \in \partial \alpha_Q J^n(15/8) \subset X$ with $|x-u| \leqq |x-y|$. Then $|f(x)-f(u)| \leqq H|f(x)-f(y)|$. On the other hand, the proof of 2.6(3) shows that $|f(x)-f(u)| \geqq \varrho_Q/c$, where $c = \eta(8/15)\eta((3/2)\sqrt{n}+1)$. Therefore $\varrho_Q \leqq Hc|f(x)-f(y)|$. Hence 2.6(3) holds.

For (b) suppose that $t \leqq 1$. Then, proceeding as earlier, we get $t' \leqq \eta_0(1)$ in Case 1, $t' \leqq abH$ in Subcases 2a and 3a, $t' \leqq \eta_0(1)abH$ in Subcase 2b, and $t' \leqq a^2H$ in Case 4; Subcase 3b does not occur. This implies that the embedding f^* is weakly H^*-quasisymmetric with H^* depending only on n, q, H, and ε. □

3. PL approximation of bilipschitz embeddings in Euclidean spaces

3.1. In this section we apply 2.16 to proving bilipschitz analogues of 2.20 and 2.21. Both [41, Theorem 2.4] and [18, Theorem 3.1] are special cases of Corollary 3.3 for $1 \leqq n = q \leqq 3$.

3.2. Theorem. *Let* (n, q) *be admissible and let* $L \geqq 1$. *Then there exists* $L^* \geqq 1$ *with the following property: Let* $X \subset R^n$, *let* $f\colon X \to R^q$ *be an L-bilipschitz* *embedding, let Y be an open subset of X which is the union of a subfamily of $\mathcal{L}_n$ locally finite in Y and which is a manifold, and let $\varepsilon \in C_+(Y)$. Then there exists an L^*-bilipschitz embedding f^*: $X \to R^q$ such that*

(1) $f^*|X \setminus Y = f|X \setminus Y$,
(2) $f^*|Y$ *is* PL,
(3) $|f^*(x) - f(x)| < \varepsilon(x)$ *for every* $x \in Y$.

Proof. Every L-bilipschitz embedding is η_L-quasisymmetric, $\eta_L(t) = L^2 t$. Let D be the finite set of PL embeddings g: $T \to R^q$, $T \in \mathcal{T}_n$, given by 2.16 with substitutions $(n, q) \mapsto (n, q)$, $\eta \mapsto \eta_L$, $\varepsilon \mapsto 1/3L^2$. Let now X, f, Y, and ε be as in 3.2. By 2.3 there is a cubical decomposition $\mathcal{K}$ of Y with respect to X such that $\lambda_Q \leqq \varepsilon(x)$ if $x \in Q \in \mathcal{K}$. Then 2.16 gives an embedding f^*: $X \to R^q$ such that (1) and (2) hold and such that $d(f^*|Q, f|Q) \leqq \varrho_Q/3L^2 \leqq \lambda_Q/3L$ and $\beta_Q f^* \alpha_Q | T_Q \in D$ for every $Q \in \mathcal{K}$. It follows that also (3) holds.

There is $L_0 \geqq 1$ depending only on n, q, and L such that each $g \in D$ is L_0-bilipschitz. We show that f^* is L^*-bilipschitz with $L^* = \max(2LL_0, 3L)$. Let $x, y \in X$, $x \neq y$, and set $\Delta = |f^*(x) - f^*(y)|$, $\delta = |x - y|$. If $x, y \in X \setminus Y$, then $1/L \leqq \Delta/\delta \leqq L$. If $x, y \in \alpha_Q I^n(2)$ for some $Q \in \mathcal{K}$, then $\Delta/\delta \leqq s_Q L_0/\lambda_Q \leqq LL_0$ and $\Delta/\delta \geqq s_Q/L_0 \lambda_Q \geqq 1/2LL_0$. If $x \in Q \in \mathcal{K}$, $y \in R \in \mathcal{K}$, $x \notin \alpha_R I^n(2)$, and $y \notin \alpha_Q I^n(2)$, then $\lambda_Q, \lambda_R \leqq \delta$, whence $\Delta \leqq |f(x) - f(y)| + (\lambda_Q + \lambda_R)/3L \leqq (L + 2/3L)\delta$ and

$$\Delta \geqq |f(x) - f(y)| - (\lambda_Q + \lambda_R)/3L \geqq \delta/3L.$$

Finally, let either $x \in Y$, $y \notin Y$ or $x \notin Y$, $y \in Y$. We may assume that $x \in Q \in \mathcal{K}$ and $y \notin Y$. Then $\lambda_Q \leqq \delta$ and thus $2/3L \leqq \Delta/\delta \leqq L + 1/3L$. Hence f^* is L^*-bilipschitz. □

3.3. Corollary. *Let n, q, L, and L^* be as in 3.2. Let $U \subset R^n$ be open, let f: $U \to R^q$ be an L-bilipschitz embedding, and let $\varepsilon \in C_+(U)$. Then there exists an L^*-bilipschitz* PL *embedding g: $U \to R^q$ such that $|g(x) - f(x)| < \varepsilon(x)$ for every $x \in U$, $\bar{g}|\partial U = \bar{f}|\partial U$, and, if $1 \leqq n = q \leqq 3$, $gU = fU$. Here $\bar{f}$ and $\bar{g}$ are the closed embeddings $\bar{U} \to R^q$ extending f and g, respectively.* □

4. CAT approximation of embeddings in manifolds

4.1. Recall that CAT denotes either LQS or LIP. An *immersion* of a topological space into another is a continuous map which is locally an embedding. The following lemma for CAT=LIP and $1 \leqq n = q \leqq 3$ is very similar to [37, Theorem 1].

4.2. Lemma. *Let (n, q) be admissible, let (A, B) be either $(2J^n, J^n)$ or $(2J^n_+, J^n_+)$, let f: $A \to R^q$ be an embedding which is a* CAT *embedding on a neighborhood in A of a closed subset C of A, and let $\varepsilon \in C_+(B)$. Then there exists an embedding f^*: $A \to R^q$ such that*

(1) $f^*|A\setminus B=f|A\setminus B$,
(2) $f^*|B$ *is* PL,
(3) $f^*|U$ *is a* CAT *embedding for some open neighborhood* U *of* C *in* A,
(4) $|f^*(x)-f(x)|<\varepsilon(x)$ *for every* $x\in B$.

Consequently, $f^*|B\cup U$ *is a* CAT *embedding.*

Proof. We may assume that $C\subset(A\setminus B)\cap\bar{B}$. Then there exists a compact polyhedron X which is a neighborhood of C in A and such that $f|X$ is a CAT embedding, $Y=X\cap B$ is the union of a subfamily of $\mathcal{L}_n$ locally finite in Y, and Y is a PL manifold. Then also $Y_1=B\cap\overline{B\setminus Y}$ and $Y\cap Y_1$ are PL manifolds. Let $A_1=A\setminus I^n(3/2)$; then $\varepsilon_1=d(fB,fA_1)/2>0$. Define $\varepsilon_0\in C_+(B)$ by

$$\varepsilon_0(x) = \min\left(\varepsilon(x), \varepsilon_1, d(f(x), f(A\setminus B))\right).$$

By 1.9 (or also by 1.10.2 if $n=q=2$ or 3) there exists $\delta\in C_+(Y)$ such that if $g\in U_d(f|Y,\delta)$ is a PL embedding, there is a PL embedding $g^*\in U_d(f|B,\varepsilon_0)$ extending g. Here $d(x,y)=|x-y|$.

Now $f|X$ is quasisymmetric if CAT=LQS or bilipschitz if CAT=LIP. Hence 2.20 and 3.2 imply that there is a CAT embedding $f_1\colon X\to R^q$ such that $f_1(x)=f(x)$ if $x\in X\setminus Y$, $f_1|Y$ is PL, and $|f_1(x)-f(x)|<\delta(x)$ if $x\in Y$.

Let g^* be the extension of $g=f_1|Y$ given above. Define $f^*=(f|A\setminus B)\cup g^*$. Then f^* is a continuous injection, and f^* satisfies (1), ..., (4) since $f^*|X=f_1$. Since $d(f^*B,f^*A_1)\geqq\varepsilon_1$, we conclude that f^* is an embedding. □

4.3. Remarks. 1. Suppose that $1\leqq n=q\leqq 3$ and $(A,B)=(2J^n,J^n)$ in 4.2. Then obviously $f^*B=fB$. Moreover, Alexander's trick [34, Proposition 3.22(i)] gives an isotopy H rel $A\setminus B$ from f to f^* (i.e., an embedding $H\colon A\times I\to R^q\times I$ of the form $H(x,t)=(H_t(x),t)$ such that $H_0=f$, $H_1=f^*$, and $H_t(x)=f(x)$ if $(x,t)\in(A\setminus B)\times I$); then $H_tB=fB$ for every $t\in I$.

2. It is easy to see that if $n=q=1$ and $(A,B)=(2J^1_+,J^1_+)$ in 4.2, one can choose f^* in such a way that $f^*B=fB$.

4.4. Theorem. *Let* CAT=LQS *or* LIP, *let* M *and* N *be* CAT *manifolds with* $\partial N=\emptyset$ *such that* M *is* n*-dimensional and* N q*-dimensional with either* $n\leqq 1$, $q\geqq n$ *or* $n\geqq 2$, $q\geqq n+3$ *or* $2\leqq n\leqq q\leqq 3$, *let* $U\subset M$ *be an open neighborhood of a closed set* $C\subset M$, *let* $f\colon M\to N$ *be an immersion such that* $f|U$ *is a* CAT *immersion, let* $\varepsilon\colon M\setminus C\to(0,\infty)$ *be continuous, and let* d *be a metric defining the topology of* N. *Then there exists a* CAT *immersion* $g\colon M\to N$ *such that*

(1) $g|C=f|C$,
(2) $d(g(x),f(x))<\varepsilon(x)$ *for every* $x\in M\setminus C$,
(3) $gM=fM$ *if* $n=q\leqq 3$ *and* $\partial M=\emptyset$,
(4) g *is a* CAT *embedding if* f *is an embedding.*

Proof. In the case $n=0$ we can choose $g=f$ since M is discrete. For the rest of the proof we assume that $n\geqq 1$, i.e., that (n, q) is admissible. Let $\mathscr{A}$ and $\mathscr{B}$ be the CAT structures of M and N, respectively. Replacing ε by a smaller function we may assume that for every $x\in M\setminus C$ there is a chart $(Z, \psi)\in\mathscr{B}$ such that $\{y\in N|d(y,f(x))<\varepsilon(x)\}\subset Z$ and $\psi Z=R^q$.

Let $k=(n+1)M(n)$, where $M(n)=2^n$ is the number given in 2.9. We construct a locally finite family $\mathscr{V}$ of open relatively compact sets in M and functions $r_0, r_1, \ldots, r_k\in C_+(M\setminus C)$ with the following properties:

(a) $f|\bar{V}$ is an embedding for every $V\in\mathscr{V}$.

(b) For each $V\in\mathscr{V}$ there is a chart $(V, \varphi_V)\in\mathscr{A}$ such that $\varphi_V V$ is either $2J^n$ or $2J^n_+$. We set $V^*=\varphi_V^{-1}J^n$ or $V^*=\varphi_V^{-1}J^n_+$, respectively; observe that $\bar{V}^*\subset V$. Let $\mathscr{V}^*=\{V^*|V\in\mathscr{V}\}$.

(c) $M\setminus U\subset\cup\mathscr{V}^*\subset\cup\mathscr{V}\subset M\setminus C$.

(d) $\mathscr{V}=\mathscr{V}_1\cup\ldots\cup\mathscr{V}_k$, where each $\mathscr{V}_i$ consists of disjoint sets.

(e) $r_0+\ldots+r_{k-1}\leqq r_k=\varepsilon$.

(f) $d(fV)\leqq\inf r_0 V$ for every $V\in\mathscr{V}$.

(g) If $V\in\mathscr{V}$ and $i<k$, then $3\sup r_i V\leqq\inf r_{i+1}V$.

To find $\mathscr{V}$ and the functions r_i, we choose first a closed neighborhood $C_1\subset U$ of C in M. It is easy to find an open cover $\mathscr{U}$ of $M\setminus C$ consisting of chart neighborhoods relatively compact in $M\setminus C$ and functions $r_0, \ldots, r_k\in C_+(M\setminus C)$ such that (e) holds, (a), (f), and (g) hold if $\mathscr{V}$ is replaced by $\mathscr{U}$, and such that $V\cap C_1=\emptyset$ whenever $V\in\mathscr{U}$ and $V\not\subset U$. By [31, Lemma 2.7] there is an open refinement $\mathscr{W}$ of $\mathscr{U}$ locally finite in $M\setminus C$ such that $\mathscr{W}=\mathscr{W}_0\cup\ldots\cup\mathscr{W}_n$ with the members of each $\mathscr{W}_i$ being disjoint. Then for each $i\leqq n$ there is a CAT homeomorphism φ_i of $\cup\mathscr{W}_i$ onto an open subset W_i of R^n_+. By 2.3 choose a cubical decomposition $\mathscr{K}_i$ of W_i with respect to W_i, and let $\mathscr{K}_{i,1}, \ldots, \mathscr{K}_{i,M(n)}$ be its partition as in 2.9. For $Q\in\mathscr{K}_i$ define $Q^*\subset W_i$ by $Q^*=\alpha_Q I^n(9/8)$ if $Q\cap R^{n-1}=\emptyset$ and by $Q^*=\alpha_Q I^n(9/8)\cap R^n_+$ if $Q\cap R^{n-1}\neq\emptyset$. Then $Q_1^*\cap Q_2^*=\emptyset$ if $Q_1, Q_2\in\mathscr{K}_i$ and $Q_1\cap Q_2=\emptyset$. For $Q\in\mathscr{K}_i$ choose a PL embedding $\sigma\colon Q^*\to R^n$ such that $\sigma Q^*=2I^n$ and $\sigma Q=I^n(1/2)$ if $Q\cap R^{n-1}=\emptyset$ and such that $\sigma Q^*=2I^n\cap R^n_+$, $\sigma(Q^*\cap R^{n-1})=2I^{n-1}$, and $\sigma Q=I^n(1/2)\cap R^n_+$ if $Q\cap R^{n-1}\neq\emptyset$. Let $V_Q=\varphi_i^{-1}\sigma^{-1}(2J^n)$ or $V_Q=\varphi_i^{-1}\sigma^{-1}(2J^n_+)$, respectively, and let $\varphi_{V_Q}=\sigma\varphi_i|V_Q$. Then $(V_Q, \varphi_{V_Q})\in\mathscr{A}$ satisfies (b). Let $\{F_i|0\leqq i\leqq n\}$ be a closed cover of $M\setminus C$ with $F_i\subset\cup\mathscr{W}_i$. Define

$$\mathscr{V}_{iM(n)+j}=\{V_Q|Q\in\mathscr{K}_{i,j},\ Q\cap\varphi_i F_i\neq\emptyset,\ V_Q\cap C_1=\emptyset\}.$$

Then $\mathscr{V}=\mathscr{V}_1\cup\ldots\cup\mathscr{V}_k$ is the required family (observe that each $V\in\mathscr{V}$ is contained in some $W\in\mathscr{W}$), presented in such a way that (d) is satisfied.

By (c) there is a closed cover $\{C_V|V\in\mathscr{V}\}\cup\{C_U\}$ of M such that $C_V\subset V^*$ for each $V\in\mathscr{V}$ and such that $C\subset C_U\subset U$. Let $\mathscr{V}_0=\{U\}$, and define a closed set $C^i=\cup\{C_V|V\in\mathscr{V}_0\cup\ldots\cup\mathscr{V}_i\}$ for $0\leqq i\leqq k$.

We prove Theorem 4.4 by showing that there is a sequence $f=f_0, f_1, \ldots, f_k$ of continuous maps $M\to N$ such that

(i) f_i is a CAT immersion on a neighborhood of C^i if $0 \leqq i \leqq k$,
(ii) $f_i|\bar{V}$ is an embedding if $V \in \mathscr{V}$ and $0 \leqq i \leqq k$,
(iii) f_i is an embedding, $0 \leqq i \leqq k$, if f is an embedding,
(iv) $d(f_i V) \leqq \inf r_i V$ if $V \in \mathscr{V}$ and $0 \leqq i \leqq k$,
(v) $d(f_i(x), f_{i-1}(x)) < r_{i-1}(x)$ if $x \in M \setminus C$ and $1 \leqq i \leqq k$,
(vi) $f_i(x) = f_{i-1}(x)$ if $x \in M \setminus \cup \{\bar{V}^* | V \in \mathscr{V}_i\}$ and $1 \leqq i \leqq k$.

We have already set $f_0 = f$; this satisfies (i), ..., (iv). Assume that $1 \leqq m \leqq k$ and that we have defined f_i for $0 \leqq i \leqq m-1$ satisfying (i), ..., (vi). Let $V \in \mathscr{V}_m$ and $\varepsilon_V > 0$; we specify ε_V later. If $x, y \in V$, by (iv), (v), and (e), we have $d(f_{m-1}(y), f(x)) < r_0(x) + \ldots + r_{m-1}(x) \leqq \varepsilon(x)$. Hence there is a chart $(Z, \psi) \in \mathscr{B}$ such that $f_{m-1} V \subset Z$ and $\psi Z = R^q$. Set $A = \varphi_V V$, $B = \varphi_V V^*$, $h = \psi f_{m-1} \varphi_V^{-1}$: $A \to R^q$, and $X = \varphi_V(V \cap C^{m-1})$. Then h is an embedding by (ii), X is closed in A, and h is a CAT embedding on a neighborhood of X in A by (i). Let $\delta > 0$. Then 4.2 gives an embedding h^*: $A \to R^q$ such that $h^* = h$ on $A \setminus B$, $h^*|B \cup E$ is a CAT embedding for some open neighborhood E of X in A, and $d(h^*, h) < \delta$. It follows that $f_V^* = \psi^{-1} h^* \varphi_V$: $V \to N$ is an embedding which is a CAT embedding on a neighborhood of $V \cap C^m = (V \cap C^{m-1}) \cup C_V$ and coincides with f_{m-1} on $V \setminus V^*$. We choose δ so small that $d(f_V^*(x), f_{m-1}(x)) < \varepsilon_V$ if $x \in V$. We now require that $\varepsilon_V \leqq \min r_{m-1} \bar{V}^*$. Since $\{W \in \mathscr{V} | \bar{W} \cap V \neq \emptyset\}$ is finite, we can also require, by (ii), that

$$\varepsilon_V \leqq \tfrac{1}{3} d(f_{m-1}(\bar{W} \cap \bar{V}^*), f_{m-1}(\bar{W} \setminus V)) \quad \text{if} \quad W \in \mathscr{V} \quad \text{and} \quad \bar{W} \cap V \neq \emptyset.$$

By (iii) we can require the stronger condition

$$\varepsilon_V \leqq \tfrac{1}{3} d(f_{m-1} \bar{V}^*, f_{m-1}(M \setminus V))$$

if f is an embedding.

We choose a map f_V^*: $V \to N$ as above for each $V \in \mathscr{V}_m$. Then we can define a continuous map f_m: $M \to N$ by

$$f_m(x) = \begin{cases} f_V^*(x) & \text{if} \quad x \in V \in \mathscr{V}_m, \\ f_{m-1}(x) & \text{otherwise.} \end{cases}$$

Clearly f_m satisfies (vi), (i), (v), and, by (g), also (iv). To prove (ii) and (iii), assume first that f is an embedding. If $V \in \mathscr{V}_m$, $x \in \bar{V}^*$, and $y \in (M \setminus V) \setminus \cup \{\bar{W}^* | W \in \mathscr{V}_m\}$, then

$$\begin{aligned} d(f_m(x), f_m(y)) &\geqq d(f_{m-1}(x), f_{m-1}(y)) - \varepsilon_V \\ &\geqq \tfrac{2}{3} d(f_{m-1}(x), f_{m-1}(y)). \end{aligned}$$

If $V, W \in \mathscr{V}_m$, $V \neq W$, $x \in \bar{V}^*$, and $y \in \bar{W}^*$, then

$$\begin{aligned} d(f_m(x), f_m(y)) &\geqq d(f_{m-1}(x), f_{m-1}(y)) - \varepsilon_V - \varepsilon_W \\ &\geqq \tfrac{1}{3} d(f_{m-1}(x), f_{m-1}(y)). \end{aligned}$$

It follows that f_m is an embedding. In exactly the same way one sees that $f_m|\overline{W}$ is an embedding for every $W\in\mathscr{V}$ if f is only assumed to be an immersion. We have now constructed the sequence $f_0, \dots, f_k$.

Consider $g=f_k$. By (i), g is a CAT immersion of $C^k=M$ into N. Further, (vi) implies (1), (v) and (e) imply (2), and (iii) implies (4). Finally, suppose that $1\leqq n=q\leqq 3$ and $\partial M=\emptyset$. Then in the above construction $f_V^* V=f_{m-1}V$ for each $V\in\mathscr{V}_m$, whence $f_m M=f_{m-1}M$. Thus $gM=fM$. □

4.5. Remark. Theorem 4.4 holds for $n=q=1$ even if the supposition $\partial N=\emptyset$ is omitted, and we can choose g in such a way that $gM=fM$ even if $\partial M\neq\emptyset$. By 4.3.2 this follows from the proof of 4.4, the only modification being that one also allows the possibility $\psi Z=R^1_+$ in addition to $\psi Z=R^1$.

4.6. Corollary. *Every* CAT *n-manifold,* $n\geqq 0$, *is* CAT *homeomorphic to a closed subset of* R^{2n+1}.

Proof. This follows from 1.15 and 4.4. □

4.7. Remark. We give a stronger embedding result in 4.11. Corollary 4.6 implies that for every CAT manifold $(M, \mathscr{A})$ there is a metric d on M such that id: $(M, \mathscr{A})\to(M, d)$ is a CAT homeomorphism, in which case (M, d) is a metric CAT manifold. This shows that the two definitions of CAT manifolds are essentially equivalent. (To get full equivalence, we should identify two metric CAT manifolds $M_i=(M, d_i)$, $i=1, 2$, whenever id: $M_1\to M_2$ is a CAT homeomorphism.) It follows that the category of metric spaces and CAT immersions is a natural category in the sense of [42, 1.9].

4.8. Corollary. *Every component of a* CAT *1-manifold is* CAT *homeomorphic to exactly one of the following* CAT *1-manifolds:* $(0, 1)$, $[0, 1)$, $[0, 1]$, ∂I^2. □

4.9. Corollary. *Let* M *and* N *be homeomorphic* CAT *n-manifolds,* $n\leqq 3$, *and suppose that* $\partial M=\emptyset=\partial N$ *if* $n=2$ *or* 3. *Then* M *and* N *are* CAT *homeomorphic.* □

4.10. Corollary. *If* $n=2$ *or* 3, *every* CAT *n-manifold without boundary is* CAT *homeomorphic to a closed* C^∞*-differentiable submanifold of* R^{2n}.

Proof. This follows from the proof of 1.17 and 4.9. □

4.11. Corollary. *Every* CAT *n-manifold,* $n\geqq 1$, *can be closedly* CAT *embedded into* R^{2n}.

Proof. The case $n=1$ follows from 4.8 and the case $n=2$ or 3 from 1.4 and 4.10. If $n\geqq 4$ (or, in fact, if $n\neq 2$), then 4.11 follows from 1.17 and 4.4. □

4.12. Corollary. *If* M *is a* LIP *n-manifold,* $n\geqq 1$, *there is a closed locally* LIP *flat* LIP *embedding* $f\colon M\to R^{3n}$.

Proof. This follows from 4.11 and [26, Corollary 4.8]. □

4.13. Corollary. *Let* $n \geqq 0$, $q \geqq 2n+1$, *and let* M *be a* LIP *n-manifold,* N *a* LIP *q-manifold,* $f\colon M \to N$ *continuous,* $C \subset M$ *closed,* U *a closed neighborhood of* C, $f|U$ *a closed* LIP *embedding,* $fC \subset \operatorname{int} N$, *and* $\mathcal{U}$ *an open cover of* N. *Then* $N(f, \mathcal{U})$ *contains a* LIP *embedding* $g\colon M \to \operatorname{int} N$ *with* $g|C = f|C$.

This also holds if LIP *is replaced by* LQS *and* $C = \emptyset$.

Proof. We may assume that $fM \subset \operatorname{int} N$ by [25, Lemma 2.3 and 1.2]. By 1.13, $f|U$ is a Z^n-embedding. Hence we may assume by 1.12 that f is an embedding. The corollary now follows from 4.4. □

4.14. Remarks. 1. Consider the case CAT=LIP. The special case of 4.4 in which $n = q \leqq 3$ and f is a homeomorphism is [37, Theorem 2]. There the result is stated for manifolds with boundary, but the proof is only valid for manifolds without boundary. However, the first-named author could reduce (using 4.14.6) the case with possibly $\partial M \neq \emptyset$ to the case $\partial M = \emptyset$. Another proof will be mentioned in 4.14.3. Corollary 4.6 solves affirmatively [26, Problem 9.1(1)] and with 4.11 improves [26, Theorems 4.2 and 4.5], which only give a closed LIP embedding of a LIP n-manifold into $R^{n(n+1)}$ $(n \geqq 1)$. Similarly 4.12 improves [26, Theorem 4.9]. In [41, Theorem 3.8], 4.9 is proved for 2-manifolds with boundary. In fact, [41, Theorem 3.7] and PL approximation results imply a (stronger) special case of 4.4 for these manifolds.

2. Theorem 4.4 for CAT=LQS, $n = q \leqq 3$, and f a homeomorphism was conjectured in [37, § 4, p. 138]. Corollary 4.6 for CAT=LQS solves a problem in [44, 4.3]. The result that every compact metric LQS manifold can be quasisymmetrically embedded into a Euclidean space also follows from [1, Remarque 2, p. 732, and Proposition 2 (h), (i)] by [38, Theorem 2.10] (cf. also [38, Remark 3.20]). Kuusalo [21] and Cannon [9] have considered orientable quasiconformal 2-manifolds. They did not assume that these are metrizable, but they supposed there to exist a locally quasiconformal atlas the dilatations of whose coordinate changes have certain boundedness properties. They proved that these manifolds have a locally quasiconformally equivalent conformal structure and, hence, are metrizable. In particular, a corollary of [21, Satz 3] is that, in our terminology, every orientable connected LQS 2-manifold without boundary is LQS homeomorphic to a Riemann surface; cf. 4.10. The metrizability result is generalized for $n \geqq 2$ in [22].

3. Tukia and Väisälä [39, Theorems 4.4 and 4.8] have recently proved, using Sullivan's methods [36], that if M, N are CAT manifolds with $\dim M \neq 4 \neq \dim \partial M$, every homeomorphism of M onto N can be relatively approximated by CAT homeomorphisms. Sullivan proved this in [36, Corollary 3] for LIP manifolds without boundary. In [14, Theorem 3.11] one proves that every CAT manifold satisfying the above dimension condition and homeomorphic either to R^n, S^n, or I^n is CAT homeomorphic to it. Earlier this was proved for CAT=LIP in [26, Section 8].

4. We can omit the hypothesis $\partial N=\emptyset$ (but still have $gM \subset \operatorname{int} N$) in the weaker form of 4.4 where $C=\emptyset$ and (3) is omitted, provided that the map $M \to fM$ defined by f is proper (which is satisfied if f is an embedding). To see this, observe first that since fM is locally compact, we may assume that fM is closed in N, in which case by [19, Lemma, p. 47] there is $\delta \in C_+(N)$ with $\delta(f(x)) \leqq \varepsilon(x)/2$ if $x \in M$. Then, using a collar of ∂N in N, we can construct an embedding $h\colon N \to N$ in $U_d(\mathrm{id}_N, \delta)$ with $hN \subset \operatorname{int} N$ ([25, Lemma 2.3]). Hence 4.4 gives a CAT immersion $g\colon M \to N$ such that $gM \subset \operatorname{int} N$, 4.4(4) holds, and $g \in U_d(hf, \varepsilon/2)$, whence $g \in U_d(f, \varepsilon)$.

5. Let (n, q) be admissible, M a CAT n-manifold, N a PL q-manifold, $f\colon M \to N$ an embedding, $\varepsilon \in C_+(M)$, and d a metric for N such that $\mathrm{id}\colon N \to (N, d)$ is a LIP homeomorphism. Then $U_d(f, \varepsilon)$ contains a CAT embedding $g\colon M \to N$ such that there is a closed subset Y of gM whose n-dimensional Hausdorff measure is zero if CAT$=$LQS or whose Hausdorff dimension is $\leqq n-1$ if CAT$=$LIP and for which $gM \setminus Y$ is a PL submanifold of N. In fact, since we may assume that $\partial N=\emptyset$ by 4.14.4, the proof of 4.4 gives the required g if one chooses $U=\emptyset$ and each (Z, ψ) such that ψ is PL. To see this for CAT$=$LQS, one can use the fact ([43, Theorem 4.1]) that every quasisymmetric image in R^n of an open set in R^p, $p<n$, has zero Lebesgue measure.

6. Suppose that $1 \leqq n=q \leqq 3$, $\partial M=\emptyset$, and that f is an embedding in 4.4. Then the construction of g allows one also to construct, by 4.3.1, an isotopy H rel C from f to g such that $H_t M=fM$ if $t \in I$ and $d(H_t(x), f(x)) \leqq \varepsilon(x)$ if $(x, t) \in (M \setminus C) \times I$.

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