ON SOME EXTENSIONS OF A THEOREM OF HARDY AND LITTLEWOOD

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1. Introduction. Let $G \subseteq \mathbb{C}$ be a domain with $\partial G \neq \emptyset$ and $\omega(t)$ a modulus of continuity, that is, a continuous and increasing function $\omega(t)$ ($t \geq 0$) with

$$
\begin{aligned}
(i) & \quad \omega(t) > 0 \quad \text{for} \quad t > 0, \\
(ii) & \quad \lim_{t \to 0^+} \omega(t) = 0 \quad \text{and} \\
(iii) & \quad \omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2).
\end{aligned}
$$

With $G$ and $\omega$ we associate the following class of functions:

$$
A_\omega(G) = \{ f : G \to \mathbb{C}; \ f \text{ regular in } G \text{ and continuous in } \overline{G} \text{ with} \}
$$

$$
|f(z) - f(w)| \leq \omega(\delta) \forall z, w \in \overline{G} \quad \text{with} \quad |z - w| \leq \delta.
$$

If $\omega(t) = O(t^\alpha)$ ($0 < \alpha \leq 1$) (that is, $f \in A_\omega$ is Lipschitz-continuous in $\overline{G}$), we will write $A_2(G)$. For $G = U = \{ z : |z| < 1 \}$ we have the following result due to Hardy and Littlewood [1]:

$$
f \in A_2(\overline{U}) \quad \text{if and only if for all } z \in U \text{ with } |z| = r
$$

$$
|f'(z)| \leq C \cdot (1 - r)^{\alpha - 1} = C \cdot (\text{dist}(z, \partial U))^\alpha - 1.
$$

In [7] it was shown that (2) has a natural extension to so-called uniform domains (see [5]). This result contains the previous generalizations [3] and [9] as special cases. We will show that (2) remains valid if we replace $U$ by a uniform domain $D$ and $\alpha$ by any modulus of continuity $\omega(t)$. Hence our theorems will contain the result of [7] as a special case.

The main idea in what follows is to sharpen the following known necessary condition (see Theorem 1 below):

For $f \in A_\omega(\overline{G})$, $z \in \overline{G}$ and $\delta = \text{dist}(z, \partial G)$ we have

$$
|f'(z)| \leq \frac{\omega(\delta)}{\delta}.
$$

Proof. We write $B = B_\delta(z) = \{ w \in G : |w - z| \leq \delta \}$. Since $f \in A_\omega(\overline{G})$ we have

$$
|f'(z)| \leq \left| \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta) - f(z)}{(\zeta - z)^2} \, d\zeta \right| \leq \frac{1}{2\pi i} \int_{\partial B} \omega(\delta)/\delta^2 d\zeta = \omega(\delta)/\delta.
$$

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It seems that (3) is too weak to show that this condition is also sufficient for $f$ to belong to $A_b(G)$ (even in the case $G=U$; see [3]).

2. Uniform domains and moduli of continuity. As in [6] a domain $G \subseteq \mathbb{C}$ is called an $(\alpha, \beta)$-John domain $0<\alpha \leq \beta < \infty$ if there is a point $z_0 \in G$ such that each point $z \in G$ can be joined with $z_0$ by means of a rectifiable path $\gamma$: $[0, d] \to G$ (arc length as parameter) with

$$
\begin{align*}
(i) & \quad \gamma(0) = z, \quad \gamma(d) = z_0, \\
(ii) & \quad d \equiv \beta \quad \text{and} \\
(iii) & \quad \text{dist (} \gamma(s), \partial G \text{) } \equiv \alpha \cdot s/d \quad (0 \leq s \leq d).
\end{align*}
$$

A domain $D \subseteq \mathbb{C}$ is called an $(\alpha, \beta)$-uniform domain $(0<\alpha \leq \beta < \infty)$ if for all $z_1, z_2 \in D$, $z_1 \neq z_2$, there is an $(\alpha|z_1-z_2|, \beta|z_1-z_2|)$-John domain $G$ in $D$ containing $z_1$ and $z_2$.

At first sight the definition for a uniform domain looks complicated, but it turns out that a simply connected domain $D \neq \mathbb{C}$ is uniform if and only if it is a quasiconformal disc (see [6]). Because of this the boundary $\partial D$ need not be lipschitzian and its Hausdorff dimension may be arbitrary near 2 (see [5]). There is also an interesting conformally invariant condition (see [5]).

In [4; Ch. 3] Lorentz shows that if $\omega$ is a modulus of continuity as in (1), then there is a concave modulus of continuity $\omega^*$ with

$$
\omega(t) \equiv \omega^*(t) \equiv 2 \cdot \omega(t)
$$

for all $t \geq 0$. Hence in the rest of this note all moduli of continuity $\omega$ will be assumed concave.

If $\omega(t)$ ($t \geq 0$) is concave, then $\omega(t)$ is continuous for $t \geq 0$, has a right hand derivative $D^+ \omega(t)$ at each $t \geq 0$ (with, possibly, $D^+ \omega(0) = \infty$) and a left hand derivative $D^- \omega(t)$ at each $t > 0$. For $0 \leq t_1 \leq t_2$, we have

$$
D^+ \omega(t_1) \equiv D^- \omega(t_2) \equiv D^+ \omega(t_2).
$$

Hence $\omega'(t)$ exists and is continuous except for at most countably many $t$, and we will also have (see [2; Theorem 18.14])

$$
\int_0^\delta D^+ \omega(t) \, dt = \int_0^\delta \omega'(t) \, dt \equiv \omega(\delta) - \omega(0) = \omega(\delta).
$$

From (iii) of (1) one can deduce the inequality

$$
\omega(\lambda \cdot t) \leq (\lambda + 1) \cdot \omega(t) \quad \forall \lambda > 0.
$$

Finally we have for $0 \leq t_1 \leq t_2$

$$
D^+ \omega(t_1) \equiv \frac{\omega(t_2) - \omega(t_1)}{t_2 - t_1}.
$$
Proof. Since \( \omega(t) \) is concave we have for \( 0 \leq \lambda \leq 1 \)

\[
\omega(t_1 + \lambda (t_2 - t_1)) \leq \omega(t_1) + \lambda \cdot (\omega(t_2) - \omega(t_1))
\]

and hence

\[
D^+ \omega(t_1) = \lim_{\lambda \to 0} \frac{\omega(t_1 + \lambda \cdot (t_2 - t_1)) - \omega(t_1)}{\lambda \cdot (t_2 - t_1)}
\]

\[
= \lim_{\lambda \to 0} \frac{\omega(t_1) + \lambda \cdot (\omega(t_2) - \omega(t_1)) - \omega(t_1)}{t_2 - t_1} = \frac{\omega(t_2) - \omega(t_1)}{t_2 - t_1}.
\]

3. The main results. First we will improve the estimate (3).

Theorem 1. Let \( G \subseteq \mathbb{C} \) be a domain with \( \partial G \neq \emptyset \), \( \omega(t) \) any modulus of continuity and \( f \in A_\omega(G) \). Then we have for all \( z \in G \) with \( d = \text{dist} (z, \partial G) \)

\[
|f'(z)| \leq C \cdot D^+ \omega(d_z),
\]

where \( C \) is a constant independent of \( z \).

Proof. Suppose there is a sequence \( \{z_n\}, n=1, 2, \ldots \) of points in \( G \) with \( d_{z_n} = \text{dist} (z_n, \partial G) \) and

\[
\lim_{n \to \infty} \frac{|f'(z_n)|}{D^+ \omega(d_{z_n})} = +\infty.
\]

Without loss of generality we may assume that \( d_{z_n} = d_{z_{n+1}} > 0 \) for \( n \geq 1 \). Let \( \Omega_n = |f'(z_n)|/D^+ \omega(d_{z_n}) \), choose \( n(0) \geq 1 \) such that \( \Omega_{n(0)} \geq 1 \), and let \( d_0 = d_{z_{n(0)}} > 0 \). Suppose that \( \delta_0, \delta_1, \ldots, \delta_k \) and \( n(0), \ldots, n(k) \) have already been defined. Let \( \delta_{k+1} \) be the unique number in \((0, \delta_k)\) with

\[
\omega(\delta_{k+1}) = \frac{1}{2} \cdot \omega(\delta_k)
\]

and look for \( n = n(k+1) \geq n(k) \) with

\[
0 < d_{z_{n(k+1)}} \leq \delta_{k+1} \quad \text{and}
\]

\[
Q_{n(k+1)} \geq 2^{k+1}.
\]

Because of \( \lim_{n \to \infty} d_{z_n} = 0 \) and \( \lim_{n \to \infty} Q_n = +\infty \) this is always possible. Now define \( \delta_{k+1} = d_{z_{n(k+1)}} \).

By (10), (11) and (12) we therefore have

\[
|f'(z_{n(k)})| \leq 2^k \cdot D^+ \omega(\delta_k) \quad \text{and}
\]

\[
0 < \omega(\delta_{k+1}) = \frac{1}{2} \cdot \omega(\delta_k), \quad k = 1, 2, \ldots.
\]

For the sake of simplicity let us write \( z_k \) instead of \( z_{n(k)} \). Now let \( k \geq 1 \), \( \delta = \delta_k/2 \),
and \( B = B_\delta(z_k) \). Because of \( \operatorname{dist}(z_k, \partial G) = \delta_k \geq \delta \) we have
\[
|f'(z_k)| = \left| \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta) - f(z_k)}{(\zeta - z_k)^2} d\zeta \right| \\
\leq \max_{\zeta \in \partial B} |f(\zeta) - f(z_k)| \cdot \frac{1}{2\pi} \int_{\partial B} d\zeta/\delta^2 = 2 \cdot |f(\zeta_0) - f(z_k)|/\delta_k
\]
with \( |\zeta_0 - z_k| = \delta_k/2 \). Hence with (13), (8) and (14) we have
\[
|f(\zeta_0) - f(z_k)| \leq \frac{1}{2} \cdot \delta_k \cdot |f'(z_k)| \leq 2^{k-1} \cdot \delta_k \cdot D^+ \omega(\delta_k) \\
\leq 2^{k-1} \cdot \delta_k \cdot \frac{\omega(\delta_k) - \omega(\delta_{k+1})}{\delta_k - \delta_{k+1}} \leq 2^{k-1} (\omega(\delta_k) - \omega(\delta_{k+1}))
\]
(15)
\[
\leq 2^{k-1} \left( \omega(\delta_k) - \frac{1}{2} \cdot \omega(\delta_k) \right) = 2^{k-2} \cdot \omega(\delta_k).
\]
On the other hand we have by (7)
\[
|f(\zeta_0) - f(z_k)| \leq \sup_{|z - w| \leq \delta} |f(z) - f(w)| \leq \omega(\delta) = \omega \left( \frac{1}{2} \cdot \delta_k \right) \leq \frac{3}{2} \cdot \omega(\delta_k),
\]
which is a contradiction to (15) if \( k \geq 4 \).

Remark. If \( \omega(t) = C \cdot t^\alpha (0 < \alpha \leq 1) \), then (3) and (9) give the same bound (up to a constant). But, if \( \lim_{t \to 0} \omega(t)/t^\alpha = +\infty \) for every \( \alpha > 0 \), then (9) is much stronger than (3).

Theorem 2. Suppose that \( D \subseteq C, D \neq \emptyset \), is an \((\alpha, \beta)\)-uniform domain and that \( f: D \to C \) is regular in \( D \). If, for any \( z \in D \) and \( d_z = \operatorname{dist}(z, \partial D) \)
\[
|f'(z)| \leq C \cdot D^+ \omega(d_z),
\]
then \( f \in A_\omega(\overline{D}) \).

Proof. The proof is similar to that of [7]. If \( z_1, z_2 \in D, z_1 \neq z_2 \), then there is an \((|z_1 - z_2|, \beta|z_1 - z_2|)\)-John domain \( G \subseteq D \) containing \( z_1 \) and \( z_2 \). Let \( z_0 \) be the point as in the definition of a John domain and let \( \gamma_k: [0, d_k] \to G \) be the corresponding paths joining \( z_k \) to \( z_0 \) \((k = 1, 2)\). Then
\[
I_k = \int_{\gamma_k} f'(\zeta) d\zeta \equiv \int_0^{d_k} f'(\gamma_k(s)) ds \leq C \cdot \int_0^{d_k} D^+ \omega(\operatorname{dist}(\gamma_k(s), \partial G)) ds.
\]
Hence by (iii) of (4), (5), (6) and (ii) of (4) we have
\[
I_k \equiv C \cdot \int_0^{d_k} D^+ \omega(\alpha|z_1 - z_2| \cdot s/d_k) ds = C \cdot \int_0^{d_k} \omega'(\alpha|z_1 - z_2| \cdot s/d_k) ds
\]
\[
= \frac{C \cdot d_k}{\alpha \cdot |z_1 - z_2|} \int_0^{\alpha|z_1 - z_2|} \omega'(t) dt \leq \frac{C \cdot d_k}{\alpha \cdot |z_1 - z_2|} \cdot \omega(\alpha|z_1 - z_2|) \equiv \frac{C(\alpha + 1)}{\alpha} \cdot \omega(|z_1 - z_2|).
Finally we obtain from this last estimate uniformly in $D$

$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} f'(\zeta) d\zeta \right| \leq I_1 + I_2 \leq \frac{2C(\alpha+1)}{\alpha} \cdot \omega(|z_1 - z_2|).$$

Remark. Theorem 4 of [7] shows that an $(\alpha, \beta)$-uniform domain is fat in the sense of [8] if its complement $C \setminus D$ only has a finite number of components. Hence by [7; Theorem 2.6] we also have

Theorem 3. Let $D$ be an $(\alpha, \beta)$-uniform domain so that $C \setminus D$ has a finite number of components. Let $f: D \to C$ be regular in $D$ and continuous in $\overline{D}$. Then

$$\sup_{\|z_1 - z_2\| \leq \delta} |f(z_1) - f(z_2)| \leq C_1 \cdot \omega(\delta)$$

if and only if (9) holds in $D$.

Note that even in the Lipschitz case of the unit disc Theorem 3 is not true any longer if “regular” is replaced by “harmonic”.

References


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