ON THE GROWTH OF THE SPHERICAL DERIVATIVE OF A MEROMORPHIC FUNCTION

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1. Introduction

Let \( f \) be meromorphic in the plane. We denote

\[
\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2},
\]

\[
\mu(r, f) = \sup \{ \varrho(f(z)): |z| = r \}
\]

and

\[
\lambda(r, f) = \inf \{ \varrho(f(z)): |z| = r \}.
\]

In this paper, we shall give some estimates on the growth of \( \mu(r, f) \) and \( \lambda(r, f) \).

2. On the growth of \( \lambda(r, f) \)

We shall employ the usual notation of the Nevanlinna theory. First we shall estimate the growth of \( \lambda(r, f) \) from below.

**Theorem 1.** Let \( f \) be a transcendental meromorphic function of finite lower order. Then

\[
\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} = -\infty.
\]

This result need not hold for functions of infinite lower order. If we take \( f(z) = \exp \{e^z\} \), then

\[
\log \lambda(r, f) \equiv \log q(f(r)) \equiv -(1 + o(1))e^r
\]

and \( T(r, f) = o(e^r) \) as \( r \to \infty \). These estimates imply that

\[
\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} = -\infty.
\]

If $f$ is a meromorphic function of lower order zero, then $f$ satisfies

$$\liminf_{r \to \infty} \frac{T(2r, f)}{T(r, f)} = 1. \tag{2.3}$$

For this class of functions we prove

Theorem 2. Let $f$ be a transcendental meromorphic function satisfying (2.3). Then

$$\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} \equiv -1. \tag{2.4}$$

This does not hold for all functions of positive order. If we take $f(z) = e^z$, then

$$\lambda(r, f) \equiv \varrho(f(r)) \equiv e^{-r},$$

and since

$$T(r, f) = (1 + o(1)) \frac{r}{\pi}$$

we deduce that

$$\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} \equiv -\pi < -1 \tag{2.5}$$

for this function $f$.

In the other direction we have

Theorem 3. Let $f$ be a transcendental meromorphic function of finite order. Then

$$\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} \equiv -\delta_{\infty}(f). \tag{2.6}$$

Theorems 2 and 3 together show that if $f$ is an entire transcendental function of finite order satisfying (2.3), then

$$\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} = -1. \tag{2.7}$$

The condition (2.6) does not hold for all meromorphic functions of infinite order.

Theorem 4. There exists an entire function of infinite order and of lower order zero such that

$$\limsup_{r \to \infty} \frac{\log \lambda(r, f)}{T(r, f)} = \infty. \tag{2.8}$$

For meromorphic functions without Nevanlinna deficient values the following theorem gives in some cases a sharper estimate than (2.6).
Theorem 5. Let $f$ be a transcendental meromorphic function. Then

\begin{equation}
\limsup_{r \to \infty} \frac{r \lambda(r, f)}{T(2r, f)} < \infty.
\end{equation}

On the other hand, we have

Theorem 6. Given any increasing and positive function $\varphi(r)$ such that $\varphi(r) \to \infty$ as $r \to \infty$, there exists a transcendental meromorphic function $f$ satisfying

\begin{equation}
T(r, f) = O(\varphi(r)(\log r)^2) \quad \text{as} \quad r \to \infty
\end{equation}

such that

\begin{equation}
\limsup_{r \to \infty} \frac{r \lambda(r, f)}{T(2r, f)} > 0
\end{equation}

and that

\begin{equation}
\limsup_{r \to \infty} \frac{r \lambda(r, f)}{T(r, f)} = \infty.
\end{equation}

The function $\varphi(r)$ in (2.10) cannot be replaced by a positive constant. We have

Theorem 7. If $f$ is a transcendental meromorphic function satisfying

\begin{equation}
T(r, f) = O((\log r)^2) \quad \text{as} \quad r \to \infty,
\end{equation}

then

\begin{equation}
\limsup_{r \to \infty} \frac{r \lambda(r, f)}{T(r, f)} = 0.
\end{equation}

3. On the growth of $\mu(r, f)$

The following theorem gives a lower bound for the growth of $\mu(r, f)$.

Theorem 8. Let $f$ be a transcendental meromorphic function. Then

\begin{equation}
\liminf_{r \to \infty} \frac{\log \mu(r, f)}{T(r, f)} \equiv -1.
\end{equation}

In the other direction, we have

Theorem 9. Let $f$ be a transcendental meromorphic function satisfying (2.3). Then

\begin{equation}
\liminf_{r \to \infty} \frac{\log \mu(r, f)}{T(r, f)} \equiv -\delta(\infty, f).
\end{equation}
Combining Theorems 8 and 9, we deduce that

(3.3) \[ \liminf_{r \to \infty} \frac{\log \mu(r, f)}{T(r, f)} = -1 \]

for transcendental entire functions satisfying (2.3).

The following theorem shows that (3.2) need not hold if the order of $f$ is positive.

**Theorem 10.** Given $d, 0 < d < 1$, there exists a meromorphic function $f$ of order $d$ with $\delta(\infty, f) > 0$ such that

(3.4) \[ \liminf_{r \to \infty} \frac{r\mu(r, f)}{T(r, f)} > 0. \]

On the other hand, the following theorem shows that a transcendental meromorphic function of lower order zero cannot satisfy (3.4).

**Theorem 11.** Let $f$ be a transcendental meromorphic function. Then

(3.5) \[ \liminf_{r \to \infty} \frac{r\mu(r, f)}{T(2r, f)} < \infty, \]

and if the lower order of $f$ is finite, then

(3.6) \[ \liminf_{r \to \infty} \frac{r\mu(r, f)}{T(r, f)} < \infty. \]

If, further, the lower order of $f$ is zero, then

(3.7) \[ \liminf_{r \to \infty} \frac{r\mu(r, f)}{T(r, f)} = 0. \]

If the order of $f$ is infinite, then (3.6) need not hold. We take $f(z) = \exp\{ie^z\}$.

Then $f'(z) = ie^z f(z)$, and since $|f(z)| = 1$ on the positive real axis, we get

(3.8) \[ \mu(r, f) \equiv \frac{1}{2} e^r. \]

Since

\[ T(r, f) = o(e^r) \quad \text{as} \quad r \to \infty, \]

we get from (3.8)

\[ \liminf_{r \to \infty} \frac{\mu(r, f)}{T(r + K, f)} = \infty \]

for any positive constant $K$. 

4. Proof of Theorem 1

We use Lemma 3 of Hayman [4] in the following form.

Lemma A. Let \( a_n, n=1, \ldots, p, \) lie in \( 0<|z|<\infty. \) For any \( B\geq 9, \) there exists a set \( E \) which is a countable union of discs \( |z-c_k|<d_k, \) such that

\[
\sum_{|z|/2<|a_n|<2|z|} \frac{d_k}{|c_k|} < 4000e^{-B}
\]

and that

\[
\sum_{r<|c_k|<2r} \frac{d_k}{|c_k|} < 16000e^{-B}
\]

when \( z \neq 0 \) and \( z \) lies outside \( E. \)

Let \( f \) be meromorphic in the plane and let \( a_n \) be the \( a \)-points of \( f. \) We write

\[
S(z,a) = \sum_{|z|/2<|a_n|<2|z|} \log \frac{|z|+|a_n|}{|z-a_n|}.
\]

Lemma 1. Let \( f \) be a non-constant meromorphic function. Then for any complex value \( a \) and any \( B\geq 12 \) there exists a set \( E \) which is a countable union of discs \( |z-c_k|<d_k \) such that

\[
\sum_{r<|c_k|<2r} \frac{d_k}{|c_k|} < 16000e^{-B}
\]

for all positive \( r \) and that

\[
S(z,a) \equiv n(4|z|, a, f)B
\]

when \( z \neq 0 \) and \( z \) lies outside \( E. \)

Proof. It follows from Lemma A that

\[
S(z,a) \equiv n(2^{k+1}, a, f)B \equiv n(4|z|, a, f)B
\]

if \( 2^{k-1}<|z|\leq2^k \) and \( z \) lies outside a set \( E_k \) satisfying (4.1). We select from each \( E_k \) those discs which have at least one common point with the annulus \( 2^{k-1}<|z|<2^k \) and denote the union of these discs by \( E. \) It follows from (4.5) that \( S(z,a) \) satisfies (4.4) outside \( E \) when \( z \neq 0. \) Since \( B\geq 12, \) it follows from (4.1) that all discs which are selected from \( E_k \) are contained in \( 2^{k-2}<|z|<2^{k+1}, \) and (4.3) follows from (4.1). Lemma 1 is proved.

Let \( f \) be as in Lemma 1. We choose \( B=20 \) and \( a=\infty \) in Lemma 1, and denote the corresponding exceptional set \( E \) by \( E. \)

Lemma 2. With the above notation we have

\[
\log |f(z)| \equiv 37T(8|z|, f)
\]

when \( |z|\geq 1 \) and \( z \) lies outside \( E, \) where \( E \) satisfies (4.3) with \( B=20. \)
\textbf{Proof.} Let \( b_k \) be the poles of \( f \) and \( |z| \equiv 1 \). Applying the Poisson—Jensen formula with \( R = 2|z| \), we get

\[
\log |f(z)| \leq 3m(R, f) + \sum_{|b_k| < R} \log \frac{R^2 - b_k^2}{R(z - b_k)}.
\]

If \( R/4 < |b_k| < R \), then

\[
\log \frac{R^2 - b_k^2}{R(z - b_k)} \leq \log \frac{|z| + |b_k|}{|z - b_k|} + \log 8,
\]

and for other terms in the sum of (4.7) we get the upper bound \( \log 8 < 3 \). These estimates together with (4.7) imply that

\[
\log |f(z)| \leq 3T(2|z|, f) + 3n(2|z|, \infty, f) + S(z, \infty),
\]

and we deduce from Lemma 1 that

\[
\log |f(z)| \leq 3T(2|z|, f) + 23n(4|z|, \infty, f)
\]

when \( |z| \equiv 1 \) and \( z \) lies outside \( E \). Since

\[
n(r, \infty, f) \log 2 \leq \int_{r}^{2r} n(t, \infty, f) t^{-1} dt = N(2r, \infty, f) - N(r, \infty, f) \leq T(2r, f)
\]

for \( r \equiv 1 \), (4.6) follows from (4.8). Lemma 2 is proved.

Now we prove Theorem 1. Let \( f \) be a transcendental meromorphic function of finite lower order. Then there exist \( K_1 > 0 \) and a sequence \( r_n, r_n \to \infty \) as \( n \to \infty \), such that

\[
T(32r_n, f) < K_1 T(r_n, f)
\]

for all \( n \). Since

\[
m(r, f') \leq m(r, f) + m(r, f'|f),
\]

we deduce from the lemma on the logarithmic derivative [Nevanlinna p. 245] that

\[
m(16r_n, f') \leq m(16r_n, f) + o(T(32r_n, f)) \quad \text{as} \quad n \to \infty,
\]

which together with the fact that \( N(r, f') \equiv 2N(r, f) \) implies that

\[
T(16r_n, f') \leq (2 + o(1))T(32r_n, f) \quad \text{as} \quad n \to \infty.
\]

Using Lemma 2, we deduce that there exists \( t_n, r_n \equiv t_n < 2r_n \), such that

\[
\log |f(z)| \leq O(T(8|z|, f)) = O(T(16r_n, f))
\]

and

\[
-\log |f'(z)| \leq O(T(8|z|, |f'|)) = O(T(16r_n, f'))
\]

for all \( z \) lying on \( |z| = t_n \), and we deduce from (4.9) and (4.10) that

\[
-\log \frac{f(t_n, f) \equiv O(T(32r_n, f)) = O(T(16r_n, f)) \quad \text{as} \quad n \to \infty,
\]

which proves Theorem 1.
5. Some properties of functions satisfying (2.3)

We denote by \( f^{(k)} \) the \( k \)-th derivative of \( f \).

Lemma 3. Let \( f \) be a transcendental meromorphic function satisfying (2.3), and let \( k \) be a positive integer. Then there exists a sequence \( r_p, r_p \to \infty \) as \( p \to \infty \), such that
\[
T(p^2 r_{p}, f^{(k)}) = (1 + o(1)) T(r_{p}, f) \quad \text{as} \quad p \to \infty
\]
and
\[
T(p^2 r_{p}, f^{(k)}) = T(r_{p}, f) + o(T(r_{p}, f))
\]
as \( p \to \infty \), and that for any complex value \( a \)
\[
n(p^2 r_{p}, a, f) = o(T(r_{p}, f)),
\]
\[
N(p^2 r_{p}, a, f) = N(r_{p}, a, f) + o(T(r_{p}, f)),
\]
\[
n(p^2 r_{p}, a, f^{(k)}) = o(T(r_{p}, f))
\]
and
\[
N(p^2 r_{p}, a, f^{(k)}) = N(r_{p}, a, f^{(k)}) + o(T(r_{p}, f))
\]
as \( p \to \infty \).

Proof. It follows from (2.3) that there exists a sequence \( t_n, t_n \to \infty \) as \( n \to \infty \), such that
\[
T(2t_{n}, f) = (1 + o(1)) T(t_{n}, f) \quad \text{as} \quad n \to \infty.
\]
We choose \( r_1 = 2 \). Let \( p \geq 2 \) be a positive integer and
\[
x = \frac{1}{p^4 e^{2p}}.
\]
Since \( T(r, f) \) is an increasing and convex function of \( \log r \), we get
\[
T(t_{n}, f) - T(xt_{n}, f) \leq \frac{\log \left(1/x\right)}{\log 2} (T(2t_{n}, f) - T(t_{n}, f)),
\]
and we deduce from (5.7) that
\[
T(t_{n}, f) = (1 + o(1)) T(xt_{n}, f) \quad \text{as} \quad n \to \infty.
\]
We choose \( n \) so large that \( xt_{n} > p^2 r_{p-1} \) and
\[
T(t_{n}, f) < 2T(xt_{n}, f)
\]
and we set \( r_p = xt_{n} \).

Let \( a \) be a complex value. We get from (5.8)
\[
n(e^p r_{p}, a, f) \leq \int_{e^p r_{p}}^{e^{2p} r_{p}} n(t, a, f) t^{-1} dt \leq (1 + o(1)) T(e^{2p} r_{p}, f) \leq (2 + o(1)) T(r_{p}, f),
\]
and
which proves (5.3). From (5.9) we get

\[(5.10) \quad N(p^2r_p, a, f) - N(r_p, a, f) \equiv 2n(p^2r_p, a, f) \log p \equiv \frac{1}{p} (4 + o(1)) T(r_p, f) \log p,\]

which proves (5.4). If we choose \( a \) such that \( A(a, f) = 0 \), we get from (5.4)

\[(5.11) \quad T(p^2r_p, f) - T(r_p, f) = N(p^2r_p, a, f) - N(r_p, a, f) + o(T(p^2r_p, f)) = o(T(p^2r_p, f)),\]

which proves (5.1).

It follows from the lemma on the logarithmic derivative that

\[T(t_n/2, f^{(k)}) \equiv (k + 1) N(t_n/2, \infty, f) + m(t_n/2, \infty, f) + m(t_n/2, \infty, f^{(k)}/f) \equiv (k + 1 + o(1)) T(t_n, f),\]

and we deduce from (5.8) that

\[(5.12) \quad T(t_n/2, f^{(k)}) \equiv (2k + 2 + o(1)) T(r_p, f)\]
as \( p \to \infty \). Just as in the proof of (5.9), we deduce from (5.12) that

\[(5.13) \quad n(e^p r_p, a, f^{(k)}) \equiv \frac{1}{p} (1 + o(1)) T(e^{2p} r_p, f^{(k)}) \equiv \frac{1}{p} (2k + 2 + o(1)) T(r_p, f),\]

which proves (5.5), and just as in the proof of (5.10) and (5.11), we deduce that (5.6) and (5.2) follow from (5.13). Lemma 3 is proved.

We choose \( B = 20 \) in Lemma 1, and deduce that there exist sets \( E_1, E_2, E_3 \) and \( E_4 \) each of them satisfying (4.3) such that

\[(5.14) \quad S(z, \infty, f) = S(z, \infty) \equiv 20n(4|z|, \infty, f)\]
outside \( E_1 \), and that corresponding estimates hold for \( S(z, 0, f) \), \( S(z, \infty, f^{(k)}) \) and \( S(z, 0, f^{(k)}) \) outside the union of \( E_2, E_3 \) and \( E_4 \). We write

\[E = \bigcup_{k=1}^{4} E_k.\]

Let \( z = re^{ip} \) lie in \( r_p \equiv |z| \equiv pr_p \). Let \( b_s \) be the poles of \( f \). Applying the Poisson-Jensen formula with \( R = p^2 r_p \), we get

\[(5.15) \quad \log |f(z)| \equiv \frac{p+1}{p-1} m(R, \infty, f) - \frac{p-1}{p+1} m(R, 0, f) + \sum_{|b_s| < R} \log \left| \frac{R^2 - b_s z}{R(z - b_s)} \right|.\]
If \(|z|/2 < |b| < 2|z|\), then
\[
\log \left| \frac{R^2 - b_z z}{R(z - b_z)} \right| \leq \log \frac{|z| + |b|}{|z - b_z|} + \log (2p^2),
\]
and for other terms in the sum of (5.15) we get the upper bound \(\log (4p^2)\). These estimates together with (5.15) imply that
\[
\log |f(z)| \leq \frac{p}{p-1} m(R, \infty, f) - \frac{p-1}{p+1} m(R, 0, f) + S(z, \infty) + n(R, \infty, f) \log (4p^2),
\]
and we deduce from (5.14) that
\[
\log |f(z)| \leq N(R, 0, f) - N(R, \infty, f) + 2 \left( \frac{p+1}{p-1} - 1 + o(1) \right) T(R, f) + n(R, \infty, f)(20 + \log (4p^2))
\]
if \(z\) lies outside \(E\). Therefore we get from (5.1), (5.4) and (5.9)
\[
\log |f(z)| \leq N(r_p, 0, f) - N(r_p, \infty, f) + o(T(r_p, f))
\]
for all \(z\) lying in \(r_p \leq |z| \leq pr_p\) outside \(E\). A similar estimate holds for \(1/f\), and using (5.13) and Lemma 3 we get a similar estimate for \(1/f^{(k)}\) and \(f^{(k)}\). We have proved the following result.

**Lemma 4.** Let \(f\) and \(k\) be as in Lemma 3. The sequence \(r_p\) in Lemma 3 can be chosen such that
\[
\log |f(z)| = N(r_p, 0, f) - N(r_p, \infty, f) + o(T(r_p, f))
\]
and
\[
\log |f^{(k)}(z)| = N(r_p, 0, f^{(k)}) - N(r_p, \infty, f^{(k)}) + o(T(r_p, f))
\]
as \(z \to \infty\) through the union of the annuli \(r_p \leq |z| \leq pr_p\) outside a set \(E\) which is the union of four sets satisfying (4.3) with \(B = 20\).

### 6. Proof of Theorem 2

Let \(f\) be as in Theorem 2. We may suppose that \(f(0) = 0\), because in other cases we can consider \(1/f\) or the function
\[
\frac{f(z) - f(0)}{1 + f(0)f(z)}.
\]
Let
\[
f(z) = c_k z^k + c_{k+1} z^{k+1} + \ldots
\]
be the Laurent expansion of \( f \) at the origin. Then \( k \equiv 1 \), and from (6.1) we deduce that there exists \( t, \ 0 < t < 1 \), such that if \( 0 < |a| < t \), then

\[
n((|a|/(2|c_k|))^{1/k}, a, f) = 0
\]

and

\[
n((2|a|/|c_k|)^{1/k}, a, f) = k.
\]

These estimates imply that if \( 0 < |a| < t \) and

\[
r > 9(1+1/|c_k|) = s_0,
\]

then

(6.2) \[
|\log|a|| - N(r, a) \leq |\log|a|| - k \log(r(|c_k|/(2|a|)^{1/k}) \leq \log \frac{2|a|}{r^k|c_k||a|} = 0.
\]

Let \( 0 < |a| < 9 \). It follows from the first main theorem of the Nevanlinna theory that

\[
m(r, a) \equiv T(r, f) - N(r, a) + |\log|a|| + \log^+|a| + \log 2,
\]

and we deduce from (6.2) that

(6.3) \[
m(r, a) \equiv T(r, f) + \log 2 + \log (1/t) + 2 \log 9 \equiv T(r, f) + 6 + \log (1/t)
\]

if \( 0 < |a| < 9 \) and \( r > s_0 \).

We write

\[
M(r, f) = \sup \{|f(z)|: |z| = r\}.
\]

We apply Lemma 4 with \( f^{(k)} = f' \), and choose a sequence \( r_p \) as in Lemmas 3 and 4. We may assume that the circle \( |z| = r_p \) lies outside the exceptional set \( E \) of Lemma 4, because in other cases we may choose \( r'_p \), \( r_p < r'_p < 2r_p \), such that the circle \( |z| = r'_p \) lies outside \( E \), and consider \( \lambda(r'_p, f) \) instead of \( \lambda(r_p, f) \).

Let \( z_p, f(z_p) \neq 0 \), lie on \( |z| = r_p \). We get for any \( z \) lying on \( |z| = r_p \)

(6.4) \[
|f(z) - f(z_p)| = \left| \int_{z_p}^{z} f'(w)dw \right| \leq \pi r_p M(r_p, f').
\]

It follows from (5.18) that

(6.5) \[
\log |f'(z)| = \log M(r_p, f') + o(T(r_p, f))
\]

on \( |z| = r_p \), and from (5.17) we get

(6.6) \[
\log |f(z)| = \log |f(z_p)| + o(T(r_p, f))
\]

for all \( z \) lying on \( |z| = r_p \).

We consider first those values of \( p \) for which \( |f(z_p)| < 4 \). From (6.4) we deduce that

\[
m(r_p, f(z_p), f) \equiv - \log M(r_p, f') - \log (\pi r_p),
\]
which implies together with (6.3) that
\begin{equation}
\log M(r_p, f') \equiv -T(r_p, f) + o(T(r_p, f)).
\end{equation}
Combining this with (6.5) and (6.6) we get
\[ \log \lambda(r_p, f) \equiv \log M(r_p, f') + o(T(r_p, f)) \equiv -T(r_p, f) + o(T(r_p, f)), \]
which proves Theorem 2 in the case where \(|f(z_p)|<4\) for an infinite number of values of \(p\).

Let us suppose that \(|f(z_p)|\equiv 4\). Applying the Jensen formula we get
\[ \log |f(0) - f(z_p)| \equiv \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_p) - f(r_p e^{i\alpha})| \, d\alpha + N(r_p, \infty, f), \]
and since \(f(0) = 0\), it follows from (6.4) that
\[ \log |f(z_p)| \equiv \log M(r_p, f') + N(r_p, \infty, f) + o(T(r_p, f)). \]
This implies together with (6.5) and (6.6) that
\[ \log \lambda(r_p, f) \equiv \log M(r_p, f') - 2 \log M(r_p, f) + o(T(r_p, f)) \]
\[ \equiv -N(r_p, \infty, f) - \log |f(z_p)| + o(T(r_p, f)), \]
and using (5.17) we deduce that
\[ \log \lambda(r_p, f) \equiv -T(r_p, f) + o(T(r_p, f)). \]
This completes the proof of Theorem 2.

\section{7. Proof of Theorem 3}

Let \(f\) be as in Theorem 3. Let \(z\) lie on the circle \(|z| = r\). Since
\[ \log^+ |f(z)| \equiv \log \left( \frac{|f'(z)|}{f(z)} \right) \equiv \log^+ \left( \frac{f'(z)}{f(z)} \right) - \log \lambda(r, f), \]
we get
\begin{equation}
\log^+ |f(z)| \equiv \log \left( \frac{f'(z)}{f(z)} \right) - \log \lambda(r_p, f).
\end{equation}
Since \(f\) has finite order, it follows from the lemma on the logarithmic derivative that
\[ m(r, f') = o(T(r, f)) \quad \text{as} \quad r \to \infty, \]
and we deduce from (7.1) that
\[ \log \lambda(r, f) \equiv -m(r, f') + o(T(r, f)) \equiv (-\delta(\infty, f) + o(1))T(r, f) \quad \text{as} \quad r \to \infty. \]
This proves Theorem 3.
8. Proof of Theorem 4

We set
\[ f(z) = \sum_{n=1}^{\infty} \left( \frac{z}{r_n} \right)^{s_n}, \]
where \( r_n = \log s_n, s_1 = 100, \) and for \( n > 1, \) \( s_n \) is a positive integer such that
\[ \log \log s_n > s_n - 1. \]
Then
\[ f'(z) = \sum_{n=1}^{\infty} \left( \frac{s_n}{r_n} \right) \left( \frac{z}{r_n} \right)^{s_n - 1}. \]
It follows from (8.1) that
\[ \log M(r_n, f) \leq s_n - 1 \log r_n \leq (\log r_n)^2, \]
which implies that the lower order of \( f \) is zero. For \( |z| = r_n \) it follows from (8.1) that
\[ \log |f'(z)| \leq \log \left( \frac{s_n}{r_n} - \sum_{k=1}^{n-1} s_k r_n \right) \leq (1 + o(1)) \log s_n = (1 + o(1)) r_n, \]
and we deduce from (8.2) that
\[ \frac{\log \lambda(r_n, f)}{\log M(r_n, f)} \leq \frac{(1 + o(1)) r_n}{(\log r_n)^2} \to \infty \quad \text{as} \quad n \to \infty. \]
This proves Theorem 4.

9. Proof of Theorem 5

Let \( f \) be a transcendental meromorphic function. We write
\[ n(r) = n(r, 0, f) + n(r, 1, f) + n(r, \infty, f). \]
We have
\[ n(3r/2) \log (4/3) \leq \int_{3r/2}^{2r} n(t) t^{-1} dt \leq (3 + o(1)) T(2r, f) \leq 4T(2r, f) \]
for \( r \geq r_0. \) This implies that for all large values of \( r \) we may choose \( z_r \) lying on \( |z| = r \)
such that \( f \) does not take any of the values 0, 1 and \( \infty \) in the disc
\[ |z - z_r| < \frac{r}{16T(2r, f)}. \]
It follows from Schottky's theorem that there exists an absolute constant \( K_1 \) such that if \( |f(z_r)| = 1, \) then
\[ |f(z)| < K_1 \]
in

$$D_r = \left\{ z : |z - z_r| < \frac{r}{32T(2, f)} \right\},$$

and if $|f(z_r)| > 1$, then $1/f$ satisfies (9.3) in $D_r$.

We write $g(z) = f(z)$ if $|f(z_r)| \equiv 1$ and $g(z) = 1/f(z)$ if $|f(z_r)| > 1$. Integrating along the boundary of $D_r$ we get from (9.3)

$$|g'(z_r)| = \left| \frac{1}{2\pi} \int g(w) \frac{dw}{(z_r - w)^2} \right| = \frac{32T(2r, f)K_1}{r},$$

which implies that

$$\frac{|f'(z_r)|}{1 + |f(z_r)|^2} = \frac{|g'(z_r)|}{1 + |g(z_r)|^2} = \frac{32K_1T(2r, f)}{r},$$

and we deduce that

$$\frac{r \lambda(r, f)}{T(2r, f)} \leq 32K_1$$

for all large values of $r$. This proves Theorem 5.

10. Proof of Theorem 6

It does not mean any restriction to assume that the function $\varphi(r)$ given in Theorem 6 satisfies the condition

$$\varphi(r) = o(\log \log r) \quad \text{as} \quad r \to \infty.$$  

We choose $s_1 = 8$, $r_1 = 100$, and for $n \geq 2$, $s_n$ and $r_n$ are chosen such that $s_n$ is a positive integer,

$$\log \log r_n > r_{n-1},$$  

$$s_n > 8ns_{n-1} \log r_n.$$  

and

$$s_n < \varphi(\sqrt[r_n]{r_n}) \log r_n.$$  

We set

$$f_n(z) = \frac{(-1)^n(z/r_n)^{s_n}}{1 + (z/r_n)^{s_n}},$$

and

$$f(z) = \sum_{n=1}^{\infty} f_n(z).$$

Then

$$f'(z) = \frac{(-1)^n(s_n/r_n)(z/r_n)^{s_n-1}}{(1 + (z/r_n)^{s_n})^2}$$
and

\[ f'(z) = \sum_{n=1}^{\infty} f'_n(z). \]

We choose \( t_n \) such that

\[ (t_n/r_n)^s_n = 2. \]  

(10.5)

Suppose that \( t_n \equiv |z| \leq \sqrt{r_{n+1}} \). For \( k < n \) we get from (10.2) and (10.3)

\[ |f_k(z) - (-1)^k| = \frac{1}{1 + (z/r_k)^{s_k}} \leq 2(r_k/r_n)^{s_k} < e^{-k}. \]  

For \( k \geq n \) we get from (10.2) and (10.3)

\[ |f_k(z)| \leq 2(\sqrt{r_{n+1}}/r_k)^{s_k} < e^{-k}. \]  

(10.7)

It follows from (10.5) that \( |f_n(z)| \leq 2 \), which implies together with (10.6) and (10.7) that

\[ |f(z)| \leq 4 \]

for all \( z \) lying in \( t_n \equiv |z| \equiv \sqrt{r_{n+1}} \).

Let \( t_n \equiv r \equiv \sqrt{r_{n+1}} \). We get from (10.3)

\[ N(r, f) \equiv s_n \log (r/r_n) + \log r \sum_{k=1}^{n-1} s_k \equiv s_n \log (r/r_n) + 2s_{n-1} \log r. \]  

(10.9)

This implies together with (10.8) and (10.4) that

\[ T(r, f) \equiv 2s_n \log r + 4 \equiv 2\varphi(\sqrt{r_n})(\log r)^2 + \log 4 \]

for \( t_n \equiv r \equiv \sqrt{r_{n+1}} \). Since \( T(r, f) \) is an increasing function of \( r \), we get for \( \sqrt{r_n} < r < t_n \)

\[ T(r, f) \equiv T(2r_n, f) \equiv 2\varphi(r)(\log (2r^2))^2 + \log 4, \]

which together with (10.10) proves (2.10).

From (10.9) and (10.5) we deduce that

\[ N(t_n, f) \equiv \log 2 + 2s_{n-1} \log t_n \]

and

\[ N(2t_n, f) \equiv s_n \log 3 + 3s_{n-1} \log r_n. \]

These estimates combined with (10.3) and (10.8) yield

\[ T(t_n, f) \equiv 3s_{n-1} \log r_n \equiv s_{n}/n \]

(10.11)

and

\[ T(2t_n, f) \equiv 4s_n. \]  

(10.12)
Suppose that \(|z| = t_n\). We get from (10.5)

\[
|f'(z)| = \frac{s_n}{9r_n} - 2 \sum_{k < n} (s_k/r_k)(r_k/r_n)^{s_k+1} - 2 \sum_{k > n} (s_k/r_k)((2r_n)/r_k)^{s_k-1}
\]

and we deduce from (10.3) and (10.2) that

\[
|f'(z)| = \frac{s_n}{9r_n} - \frac{2}{r_n} \sum_{k < n} s_k - \frac{2}{r_n} \sum_{k > n} s_k/r_k,
\]

This implies together with (10.8) that

\[
t_n^\lambda(t_n, f) \equiv \frac{s_n}{18 \cdot 17}.
\]

This together with (10.11) and (10.12) proves (2.12) and (2.11). Theorem 6 is proved.

11. Proof of Theorem 7

Let \(f\) be as in Theorem 7. Let \(n(r)\) be defined by (9.1). We have

\[
n(r) \log r \equiv \int_r^2 n(t) t^{-1} dt \equiv (3 + o(1))T(r^2, f),
\]

and we deduce from (2.13) that there exists \(K > 0\) such that

\[
n(r) \leq K \log r
\]

for all large values of \(r\). This implies that for any large \(r\), there exists \(z_r, \ |z_r| = r\),

such that \(f\) does not take any of the values 0, 1 and \(\infty\) in

\[
|z - z_r| < \frac{r}{K \log r},
\]

and just as in the proof of (9.4), we deduce from (11.1) that

\[
\lambda(r, f) = O\left(\log\frac{r}{r}\right) \quad \text{as} \quad r \to \infty.
\]

Since \(f\) is transcendental, we get

\[
r\lambda(r, f) = O(\log r) = o(T(r, f)) \quad \text{as} \quad r \to \infty,
\]

which proves Theorem 7.
12. Proof of Theorem 8

Contrary to the assertion of Theorem 8, let us suppose that there exists a transcendental meromorphic function $f$ such that

\[
\liminf_{r \to \infty} \frac{\log \mu(r, f)}{T(r, f)} \leq -1 - 9d
\]

for some $d > 0$. Just as in the proof of Theorem 2, we may assume that $f(0) = 0$, and deduce (6.3) so that there exist $t > 0$ and $s_0 > 0$ such that

\[
m(r, a) \equiv T(r, f) + 6 + \log \left(1 + \frac{1}{t}\right)
\]

if $0 < |a| < 1$ and $r > s_0$.

It follows from (12.1) that there exists an increasing sequence $r_n, r_n \to \infty$ as $n \to \infty$, such that

\[
\log g(f(z)) \leq -(1 + 8d)T(r_n, f)
\]
on $|z| = r_n$ for all $n$. Since $m(r, \infty, f) \equiv T(r, f)$ and

\[
m(r, 0, f) \equiv (1 + o(1))T(r, f)
\]as $r \to \infty$,
we may choose a point $z_n$ lying on $|z| = r_n$ such that

\[
|\log |f(z_n)|| \equiv (1 + d)T(r_n, f)
\]
if $n$ is large enough.

Let us suppose that

\[
|\log |f(w_1)|| > (1 + 2d)T(r_n, f)
\]
for some $w_1$ lying on $|z| = r_n$. Then we may choose an arc $J_n$ contained in $|z| = r_n$ such that $z_n \in J_n$,

\[
|\log |f(w)|| \equiv (1 + 2d)T(r_n, f)
\]
for all $w \in J_n$, and that

\[
|\log |f(w_2)|| = (1 + 2d)T(r_n, f)
\](k = 2, 3) for the end points $w_2$ and $w_3$ of the arc $J_n$. For $z \in J_n$ we get from (12.3) and (12.6)

\[
|\log |f(z)/f(z)|| = \log (g (f(z)) + \log (|f(z)| + 1/|f(z)|)) \leq -(6d + o(1))T(r_n, f),
\]
and integrating along $J_n$ we deduce that

\[
(12.8) \quad |\log |f(z)/f(z)|| \leq \left| \int_{z_n}^{z} \frac{f'(w)}{f(w)} \, dw \right| \leq \exp\{- (6d + o(1))T(r_n, f)\} = o(1)
\]
for all $z \in J_n$. This implies together with (12.4) that

\[
|\log |f(w_2)|| \equiv (1 + d + o(1))T(r_n, f).
\]
This is a contradiction with (12.7) if \( n \) is large, and we conclude that

\[
(12.9) \quad |\log |f(z)|| \leq 1 + 2d + o(1))T(r_n, f)
\]

for all \( z \) lying on \( |z| = r_n \), and that (12.8) holds for all \( z \) lying on \( |z| = r_n \).

Let us suppose that there exist large values of \( n \) such that

\[
(12.10) \quad |f(z_n)| \leq 4.
\]

Then it follows from (12.8) that \( |f(z)| \leq 8 \) on \( |z| = r_n \), and from (12.3) we get

\[
(12.11) \quad \log |f'(z)| \leq - (1 + 8d) T(r_n, f) + \log 65
\]

on \( |z| = r_n \). Integrating along the circle \( |z| = r_n \), we deduce from (12.11) that

\[
\log |f(z) - f(z_n)| \leq - (1 + 8d) T(r_n, f) + \log 65 + 2 \log r_n
\]

on \( |z| = r_n \), which implies that

\[
m(r, f(z_n), f) \geq (1 + 8d + o(1))T(r_n, f).
\]

This contradicts (12.2) and we deduce that (12.10) is not possible if \( n \) is large. Therefore we have

\[
(12.12) \quad |f(z_n)| \geq 4
\]

for all large \( n \).

It follows from (12.8) that

\[
|f(z)| = (1 + o(1))|f(z_n)|
\]

for all \( z \) lying on \( |z| = r_n \), and we get from (12.12)

\[
(12.13) \quad \log |f(z)| = (1 + o(1))m(r_n, \infty, f)
\]

for all \( z \) lying on \( |z| = r_n \). This implies together with (12.3) that

\[
\log |f'(z)| \leq - (1 + 8d + o(1))T(r_n, f) + 2m(r_n, \infty, f)
\]

on \( |z| = r_n \), and integrating along the circle \( |z| = r_n \), we get

\[
(12.14) \quad \log |f(z) - f(z_n)| \leq - (1 + 8d + o(1))T(r_n, f) + 2m(r_n, \infty, f)
\]

on \( |z| = r_n \). Applying the Jensen formula to the function \( f(z) - f(z_n) \), we get from (12.14), since \( f(0) = 0 \),

\[
\log |f(z_n)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(r_n e^{ix}) - f(z_n)| dx + N(r_n, \infty, f)
\]

\[
\leq - (1 + 8d + o(1))T(r_n, f) + N(r_n, \infty, f) + 2m(r_n, \infty, f),
\]

which together with (12.13) implies that

\[
(1 + 8d + o(1))T(r_n, f) \leq T(r_n, f) \text{ as } n \to \infty.
\]

This is a contradiction, and we deduce that (3.1) holds for all transcendental meromorphic functions. Theorem 8 is proved.
13. Proof of Theorem 9

Let \( f \) be as in Theorem 9. Since

\[
m(r, f') \leq m(r, f) + o(T(r, f))
\]

outside a set of finite linear measure, it follows from Lemma 4 that we may choose a sequence \( r_n, r_n \to \infty \) as \( n \to \infty \), such that

\[
\log |f(z)| = m(r_n, \infty, f) - m(r_n, 0, f) + o(T(r_n, f))
\]

and

\[
\log |f'(z)| = m(r_n, \infty, f') - m(r_n, 0, f') + o(T(r_n, f))
\]

(13.1)

\[
\leq m(r_n, \infty, f) + o(T(r_n, f))
\]

on \( |z| = r_n \).

If \( m(r_n, 0, f) > 0 \), we deduce from (13.1) that

\[
m(r_n, \infty, f) = o(T(r_n, f)),
\]

and if \( m(r_n, \infty, f) > 0 \), then

\[
m(r_n, 0, f) = o(T(r_n, f)).
\]

These estimates imply together with (13.1) that

\[
\log (1 + |f(z)|^2) = 2m(r_n, f) + o(T(r_n, f))
\]

(13.3)

on \( |z| = r_n \) for all \( n \). From (13.2) and (13.3) we get

\[
\log \mu(r_n, f) \equiv -m(r_n, \infty, f) + o(T(r_n, f)) \equiv -\left(\delta(\infty, f) + o(1)\right)T(r_n, f)
\]

as \( n \to \infty \), which proves Theorem 9.

14. Proof of Theorem 10

Let \( d \) satisfy \( 0 < d < 1 \). We set

\[
f(z) = \prod_{n=1}^{\infty} \frac{r_n + z}{r_n - z},
\]

where \( r_n = n^{1/d} \). We have

\[
\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{2r_n}{r_n^2 - z^2},
\]

and since \( |f(ir)| = 1 \) for any real \( r \), we get

\[
\mu(r, f) \equiv \frac{|f'(ir)|}{1 + |f(ir)|^2} \equiv \sum_{n=1}^{\infty} \frac{r_n}{r_n^2 + r^2}
\]

(14.1)

for any \( r > 0 \).
From the choice of \( r_n \) we deduce that

\[
n(r, 0, f) = (1 + o(1))r^d \quad \text{as} \quad r \to \infty,
\]

which implies that

\[
(14.2) \quad n(2r, 0, f) - n(r, 0, f) = (2^d - 1 + o(1))r^d
\]
as \( r \to \infty \). It follows from (14.1) and (14.2) that

\[
(14.3) \quad \mu(r, f) \equiv \sum_{r \leq r_n \leq 2r} \frac{r_n}{r_n^2 + r^2} \equiv \left( n(2r, 0, f) - n(r, 0, f) \right) \frac{r}{5r^2}
\]
\[
\equiv \left( 2^d - 1 + o(1) \right) \frac{r^d}{5r} \quad \text{as} \quad r \to \infty.
\]

If \( |z| = r \) and \( |\arg z| < \pi/6 \), then

\[
\log |f(z)| \equiv \sum_{r \leq r_n \leq 2r} \log \left| \frac{r_n + z}{r_n - z} \right|
\]
\[
\equiv (n(2r, 0, f) - n(r, 0, f)) \log (5/4) \equiv (2^d - 1 + o(1))r^d \log (5/4).
\]

This implies that

\[
(14.4) \quad m(r, \infty, f) \equiv \frac{1}{6} \log (5/4)(2^d - 1 + o(1))r^d.
\]

On the other hand, since

\[
n(r, 0, f) = n(r, \infty, f) = O(r^d) \quad \text{as} \quad r \to \infty,
\]
we have (see e.g. Nevanlinna [7, p. 223])

\[
(14.5) \quad T(r, f) = O(r^d) \quad \text{as} \quad r \to \infty.
\]

From (14.4) and (14.5) we deduce that \( \delta(\infty, f) = 0 \) and that the order of \( f \) is \( d \). From (14.3) and (14.5) it follows that \( f \) satisfies (3.4). Theorem 10 is proved.

15. Proof of Theorem 11

Let \( f \) be a transcendental meromorphic function. Let \( n(r) \) be defined by (9.1). Just as in the proof of Theorem 5, we deduce that there exists \( r_0 \) such that

\[
(15.1) \quad n(3r/2) \equiv \frac{4T(2r, f)}{\log (4/3)}
\]
for \( r \geq r_0 \).

If the lower order of \( f \) is infinite, we choose \( r_n = e^n \) for any positive integer \( n \). Let us suppose that

\[
(15.2) \quad \lim \inf_{r \to \infty} \frac{T(2r, f)}{T(r, f)} = \infty.
\]
Then we have for any $K>1$,
\[(15.3) \quad T(2r, f) > KT(r, f)\]
for $r \equiv r_K$. This implies that
\[T(2^n r_K, f) > K^n T(r_K, f)\]
for all $n$, and we deduce that if $2^{n-1} r_K \leq t \leq 2^n r_K$, then
\[
\frac{\log T(t, f)}{\log t} \geq \frac{(n-1) \log K + \log T(r_K, f)}{n \log 2 + \log r_K}.
\]
This implies that
\[(15.4) \quad \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r} > \log K,
\]
and we deduce that if (15.2) holds, then the lower order of $f$ is infinite. Therefore, if the lower order of $f$ is finite and positive, we may choose a sequence $r_n$, $r_n \to \infty$ as $n \to \infty$, such that
\[(15.5) \quad \frac{T(2r_n, f)}{T(r_n, f)} < A
\]
for all $n$, $A$ being a constant.

If $f$ satisfies (15.3) for some $K > 1$, we deduce from (15.4) that the lower order of $f$ is positive. Therefore, if the lower order of $f$ is zero, then $f$ satisfies (2.3), and it follows from Lemma 3 that there exists a sequence $r_n$, $r_n \to \infty$ as $n \to \infty$, such that
\[(15.6) \quad n(3r_n/2) = o(T(r_n, f)) \quad \text{as} \quad n \to \infty.
\]
In all cases, we write, if $n(r_n) \equiv 1$,
\[d_n = \frac{r_n}{4n(3r_n/2)}.
\]
Then there exists $t_n, r_n \equiv t_n \equiv 3r_n/2$, such that $f$ does not take any of the values 0, 1 and $\infty$ in the annulus
\[B_n = \{z: t_n - d_n < |z| < t_n + d_n\}.
\]
Since the disc $|z-w| < d_n$ is contained in $B_n$ for all $w$ satisfying $|w| = t_n$, we get, just as in the proof of (9.4),
\[e(f(w)) \equiv \frac{2K_1}{d_n} = \frac{8K_1 n(3r_n/2)}{r_n}\]
for all $w$ lying on $|z| = t_n$. This implies that
\[(15.7) \quad t_n \mu(t_n, f) \equiv 16K_1 n(3r_n/2).
\]
Combining (15.1) and (15.7), we get
\[(15.8) \quad t_n \mu(t_n, f) = O(T(2r_n, f)) \quad \text{as} \quad n \to \infty.
\]
This proves (3.5), since \( r_n \equiv t_n \equiv 2r_n \) and \( T(r,f) \) is an increasing function of \( r \). Similarly, combining (15.8) and (15.5), we get (3.6) for functions of finite positive lower order. If the lower order of \( f \) is zero, then (3.7) follows from (15.7) and (15.6). Theorem 11 is proved.

References


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