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ON THE STRUCTURE OF SELF-SIMILAR FRACTALS

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1. Introduction

A subset E of the Euclidean n-space R^n is called *self-similar* if there are similitudes $S_1, \ldots, S_N$ of R^n such that

$$E = \bigcup_{i=1}^{N} S_i E$$

and the different parts $S_i E$ are "nearly" disjoint; more precisely, if s is the Hausdorff dimension of E then $\mathscr{H}^s(S_i E \cap S_j E) = 0$ for $i \neq j$. Here $\mathscr{H}^s$ is the s-dimensional Hausdorff measure. By a *similitude* we mean a map $S\colon R^n \to R^n$ such that for some r, $0 < r < 1$, $|Sx - Sy| = r|x-y|$ for all $x, y \in R^n$. The term fractal, appearing in the title, is a general name introduced by B. Mandelbrot for sets whose Hausdorff and topological dimensions differ from each other.

In [3] Mandelbrot has studied the connections of self-similar fractals to various physical phenomena. In [2] J. E. Hutchinson showed that to any finite family $\mathscr{S} = \{S_1, \ldots, S_N\}$ of similitudes of R^n corresponds a unique compact set $K \subset R^n$ such that $K = \cup_{i=1}^{N} S_i K$. This set will be denoted by $|\mathscr{S}|$. Also several properties of selfsimilar fractals were proved in [2]. One of them was that if m is a positive integer, $0 < m < n$, and $\mathscr{S}$ satisfies certain natural separation conditions (which are valid, e.g. if the sets $S_i K$ are disjoint), then the intersection of $K = |\mathscr{S}|$ with any m-dimensional C^1 submanifold of R^n has m-dimensional Hausdorff measure zero. In this paper we prove that under a slightly stronger separation condition, the Hausdorff dimension of such an intersection is always at most $m - \varepsilon$ where $\varepsilon > 0$ depends only on $\mathscr{S}$ and not on the submanifold in question. This result has some content only if the Hausdorff dimension of K, $\dim K$, is not less than m. We also study the case $s = \dim K \leqq m$, and we show that then there are only two possibilities; either K lies on an m-dimensional affine subspace of R^n or $\mathscr{H}^s(K \cap M) = 0$ for every m-dimensional C^1 submanifold M of R^n.

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2. Preliminaries

We follow the notation and terminology of Hutchinson [2]. In the whole paper

— m and n are integers, $0<m<n$,

— $\mathscr{S}=\{S_1, \ldots, S_N\}$ is a finite family of similitudes of R^n,

— $K=|\mathscr{S}|$,

— $s=\dim K$, the Hausdorff dimension of K,

— $C_p(N)=\{1, \ldots, N\}^p$ for $p=1, 2, \ldots$; we denote $\alpha\prec\beta$ if $\alpha\in C_p(N)$, $\beta\in C_q(N)$, $p\leqq q$, and β is an extension of α, that is, $\alpha=(i_1, \ldots, i_p)$, $\beta=(i_1, \ldots, i_p, i_{p+1}, \ldots, i_q)$,

— $S_\alpha=S_{i_1}\circ\ldots\circ S_{i_p}$ for $\alpha=(i_1, \ldots, i_p)\in C_p(N)$

— $A_\alpha=S_\alpha A$ for $A\subset R^n$,

— r_i is the Lipschitz constant of S_i for $i=1, \ldots, N$ and $r_\alpha=r_{i_1}\cdot\ldots\cdot r_{i_p}$ that of S_α for $\alpha=(i_1, \ldots, i_p)\in C_p(N)$. We assume $0<r_1\leqq\ldots\leqq r_N<1$.

Observe that if $B\subset R^n$ is bounded, then for $\alpha\in C_p(N)$

$$d(B_\alpha) = r_\alpha d(B) \leqq r_N^p d(B) \to 0 \quad \text{as} \quad p\to\infty,$$

where d denotes diameter.

We say that $\mathscr{S}$ satisfies the *open set condition* if there is an open bounded set $O\subset R^n$ such that $\cup_{i=1}^N O_i\subset O$ and $O_i\cap O_j=\emptyset$ for $i\neq j$.

If $\mathscr{S}$ satisfies the open set condition with O, then $\bar{O}\supset\bar{O}_{i_1}\supset\bar{O}_{i_1 i_2}\supset\ldots$, $K\subset\bar{O}$, $K_\alpha\subset\bar{O}_\alpha$ for all α, and

$$K = \bigcap_{p=1}^{\infty} \bigcup_{\alpha\in C_p(N)} \bar{O}_\alpha.$$

(See [2, 5.2 (3) (ii) and 3.1 (3) (viii)].) Moreover, s is uniquely determined by the condition $\sum_{i=1}^N r_i^s=1$, and $0<\mathscr{H}^s(K)<\infty$, [2, 5.1 (2) and 5.3 (1)]. Observe that neither the sets K_i nor $\bar{O}_i$ need to be disjoint. But if K_i's are disjoint, then $\mathscr{S}$ always satisfies the open set condition with O=an ε-neighborhood of K for sufficiently small $\varepsilon>0$ [2, 5.2 (2)].

The lower and upper s-dimensional densities of a set $A\subset R^n$ at a point $x\in R^n$ are defined by

$$\Theta_*^s(A, x) = \liminf_{r\downarrow 0} \frac{\mathscr{H}^s(A\cap B(x, r))}{\alpha(s)r^s},$$

$$\Theta^{*s}(A, x) = \limsup_{r\downarrow 0} \frac{\mathscr{H}^s(A\cap B(x, r))}{\alpha(s)r^s},$$

where $B(x, r)$ is the closed ball with centre x and radius r and $\alpha(s)$ is a positive normalization constant.

3. The case $m \leqq s$

In this section we prove

3.1. Theorem. *Suppose that $\mathcal{S}$ satisfies the open set condition with the open bounded sets O and U, $O \subset U$, and that the following separation condition holds:*

If l is a line in R^n such that $\bar{O}_i \cap l \neq \emptyset \neq \bar{O}_j \cap l$ for some $i \neq j$, then $(U \sim \bigcup_{i=1}^N \bar{O}_i) \cap l \neq \emptyset$.

Then there is $\varepsilon > 0$ such that for any m-dimensional C^1 submanifold M of R^n, $\dim (M \cap K) \leqq m - \varepsilon$.

Proof. We shall show that there is $\varepsilon > 0$ such that, given M, every point $x \in M$ has a neighborhood $W \subset R^n$ such that $\dim (M \cap K_\alpha) \leqq m - \varepsilon$ whenever $\bar{U}_\alpha \subset W$. To see that this implies $\dim (M \cap K) \leqq m - \varepsilon$, we choose compact subsets M_i of M such that $\bigcup_i M_i = M$. Each M_i can be covered with a finite number of such neighborhoods W, and for every i there is p such that each $\bar{U}_\alpha$, $\alpha \in C_p(N)$, which meets M_i is contained in some such neighborhood W. It follows that $\dim (M_i \cap K) \leqq m - \varepsilon$; hence also $\dim (M \cap K) \leqq m - \varepsilon$.

Since the closures $\bar{O}_i$ are compact, since the function

$$l \mapsto \sup \left\{ \lambda \colon B(a, \lambda) \cap l \subset \left(U \sim \bigcup_{i=1}^{N} \bar{O}_i \right) \cap l \text{ for some } a \in l \right\}$$

is lower semicontinuous in the space of all lines in R^n and since it is positive on the compact set of all lines l for which $\bar{O}_i \cap l \neq \emptyset \neq \bar{O}_j \cap l$ for some $i \neq j$, it has a positive minimum on this set (cf. [2, 5.4 (1)]). This means that there is $\lambda > 0$ such that $\bar{O}_i \cap l \neq \emptyset \neq \bar{O}_j \cap l$ for some $i \neq j$ implies $B(a, \lambda) \cap l \subset (U \sim \bigcup_{i=1}^N \bar{O}_i) \cap l$ for some $a \in l$. Applying the similitudes S_α, $\alpha \in C_p(N)$, we find that for any line l

(1) $$\bar{O}_{\alpha i} \cap l \neq \emptyset \neq \bar{O}_{\alpha j} \cap l \quad \text{for some} \quad i \neq j \quad \text{implies}$$

$$B(a, \lambda r_\alpha) \cap l \subset \left(U_\alpha \sim \bigcup_{i=1}^{N} \bar{O}_{\alpha i} \right) \cap l$$

for some $a \in l$.

Let $x \in M$ and let V be the tangent plane of M at x. Given u, $1 < u < 2$, there are a neighborhood W of x in R^n and a diffeomorphism f of $W \cap M$ into V such that the Lipschitz constants $\operatorname{Lip}(f)$ and $\operatorname{Lip}(f^{-1})$ are $\leqq u$. We denote

$$A' = f(A \cap W \cap M)$$

for any $A \subset R^n$. Choosing u sufficiently close to 1 we obtain from (1) for any line l in R^n

(2) $$\bar{O}'_{\alpha i} \cap l \neq \emptyset \neq \bar{O}'_{\alpha j} \cap l \quad \text{for some} \quad i \neq j \text{ implies}$$

$$B(a, \lambda r_\alpha) \cap l \subset \left(U'_\alpha \sim \bigcup_{i=1}^{N} \bar{O}'_{\alpha i} \right) \cap l$$

for some $a \in l$ whenever $\bar{U}_\alpha \subset W$.

Let $\varrho>0$ be such that U contains a ball radius ϱ, let

$$\varkappa=(8/\varrho r_1)^n \quad\text{and}\quad \eta=\min\{1/(4\varkappa),\ \lambda/(8\varkappa d(U))\},$$

and let t be the smallest positive number with the following property: If I is any interval, $A\subset I$ and $I\sim A$ contains an interval of length $\eta d(I)$, then A can be covered with two intervals $I_1, I_2\subset I$ such that $d(I_1)^t+d(I_2)^t\leqq d(I)^t$ and $\operatorname{dist}(I_1, I_2)\geqq\eta d(I)$. One of the intervals I_1, I_2 may be degenerate. Clearly $0<t<1$, and t depends only on η.

Let α_0 be such that $\overline{U}_{\alpha_0}\subset W$ and set

$$H=f(K_{\alpha_0}\cap M).$$

Let l be a line in R^n. We shall show that there exist closed intervals

$$I_{j_1\dots j_p}\subset l,\quad j_i=1,2,\ i=1,\dots,p,\quad p=1,2,\dots,\quad \text{such that for all } p$$

$$H\cap l\subset\bigcup_{j_1\dots j_p} I_{j_1\dots j_p}, \tag{3}$$

$$I_{j_1\dots j_p 1}\cup I_{j_1\dots j_p 2}\subset I_{j_1\dots j_p}, \tag{4}$$

$$d(I_{j_1\dots j_p 1})^t+d(I_{j_1\dots j_p 2})^t\leqq d(I_{j_1\dots j_p})^t, \tag{5}$$

$$\operatorname{dist}(I_{j_1\dots j_p 1}, I_{j_1\dots j_p 2})\geqq\eta d(I_{j_1\dots j_p}), \tag{6}$$

$$d(I_{j_1\dots j_p j})\leqq(1-\eta)d(I_{j_1\dots j_p}),\quad j=1,2. \tag{7}$$

We first choose two closed intervals I_1, I_2 so that $H\cap l\subset I_1\cup I_2$ and $d(I_j)\leqq d(U_{\alpha_0})$ for $j=1,2$. Suppose then that $q\geqq 1$ and that $I_{j_1\dots j_p}$ for $p=1,\dots,q$ have been chosen so that (3)—(7) hold. Fix $I=I_{j_1\dots j_q}$. We shall show that $I\sim H$ contains an interval of length $\eta d(I)$, whence the required intervals $I_{j_1\dots j_q j}$, $j=1,2$, can be found by the choice of t.

Let $J\subset I$ be the interval in the middle of I of length $d(I)/2$. Let A be the set of all those multi-indices α for which $\alpha_0\prec\alpha$, $\overline{O}_\alpha\cap J\neq\emptyset$,

$$r_1 d(I)/8\leqq d(U_\alpha)<d(I)/8, \tag{8}$$

and which are maximal (in the order $\prec$) with respect to these properties. Then $U'_\alpha\cap l\subset I$ for $\alpha\in A$, because $\overline{U}'_\alpha\cap J\neq\emptyset$ and $d(U'_\alpha)<d(I)/4$. For $\alpha\in A$ let α' be the (unique) minimal sequence such that $\alpha\prec\alpha'$ and $\overline{O}'_{\alpha' i}\cap l\neq\emptyset\neq\overline{O}'_{\alpha' j}\cap l$ for some $i\neq j$, if such a sequence exists. We then set $P_\alpha=\overline{O}'_{\alpha'}\cap l$. If such a sequence does not exist, then either there are $\alpha\prec\alpha_1\prec\alpha_2\prec\dots$ such that $\bigcap_{i=1}^\infty \overline{O}'_{\alpha_i}\cap l$ is a singleton $\{x_\alpha\}$ or there is α' such that $\alpha\prec\alpha'$, $\overline{O}'_{\alpha'}\cap l\neq\emptyset$ and $\overline{O}'_{\alpha' i}\cap l=\emptyset$ for all i. In the first case we set $P_\alpha=\{x_\alpha\}$, in the second $P_\alpha=\emptyset$, and we agree $d(\emptyset)=0$. Then

$$H\cap J\subset\bigcup_{\alpha\in A}P_\alpha. \tag{9}$$

To see this let $x\in H\cap J$. Then there is an infinite sequence $(i_1, i_2, \dots)$ such that $x\in\overline{O}'_{\alpha_p}\cap l$ where $\alpha_p=(i_1, i_2, \dots, i_p)$ for $p=1,2,\dots$ and $\alpha_0=\alpha_{p_0}$ for some p_0. Since for $j=1$ or 2, $d(I)\leqq d(I_j)\leqq d(U_{\alpha_0})$, and since $r_1 d(U_{\alpha_{p+1}})\leqq r_1 d(U_{\alpha_p})\leqq d(U_{\alpha_{p+1}})$, it follows from (8) that for some $p \geqq p_0$, $\alpha_p \in A$. Then one readily checks that $x \in P_{\alpha_p}$ which proves (9).

By the definition of ϱ, each U_α contains a ball of radius $\varrho d(U_\alpha)$. Since the elements of A are maximal, the sets U_α, $\alpha \in A$, are disjoint. Moreover, they are contained in a ball of radius $d(I)$. If k is the number of the elements of A, we see comparing volumes that $k(\varrho r_1 d(I)/8)^n \leqq d(I)^n$, that is, $k \leqq \varkappa$.

Let $\beta \in A$ be such that $d(P_\beta)$ is the largest of the diameters $d(P_\alpha)$, $\alpha \in A$. If $2\varkappa d(P_\beta) < d(J)$, then by (9) $I \sim H$ contains an interval of length $d(J)/(2\varkappa) \geqq \eta d(I)$. If $2\varkappa d(P_\beta) \geqq d(J) > 0$, then $P_{\beta'} = \bar{O}'_{\beta'} \cap l$ and $\bar{O}'_{\beta' i} \cap l \neq \emptyset \neq \bar{O}'_{\beta' j} \cap l$ for some $i \neq j$, and (2) implies that $(U'_{\beta'} \sim \bigcup_{i=1}^N \bar{O}'_{\beta' i}) \cap l$ contains an interval I_0 of length

$$\lambda r_{\beta'} = \lambda d(U_{\beta'})/d(U) \geqq \lambda d(P_\beta)/(2d(U)) \geqq \lambda d(J)/(4\varkappa d(U)) \geqq \eta d(I).$$

If $\beta' \in C_p(N)$ then $U'_{\beta'} \cap \bar{U}'_\gamma = \emptyset$ for $\gamma \in C_p(N)$, $\gamma \neq \beta'$; hence $H \cap U_{\beta'} \subset \bigcup_{i=1}^N \bar{O}'_{\beta' i}$, and it follows that I_0 is contained in $I \sim H$. This completes the induction.

From (5) we obtain for $p = 1, 2, \dots$

$$\sum_{j_1 \dots j_p} d(I_{j_1 \dots j_p})^t \leqq d(I_1)^t + d(I_2)^t,$$

hence (3) and (7) imply

$$\mathscr{H}^t(H \cap l) \leqq d(I_1)^t + d(I_2)^t < \infty,$$

and $\dim(H \cap l) \leqq t$.

We set $\varepsilon = 1 - t$. Then $\dim H \leqq m - \varepsilon$. For otherwise it would follow from [5, Theorem 6.6] that for some lines l, $\dim(H \cap l) = \dim H + 1 - m > t$. Since H is diffeomorphic to $K_{\alpha_0} \cap M$, we have $\dim(K_{\alpha_0} \cap M) \leqq m - \varepsilon$. This completes the proof.

3.2. Remarks. If the lines are replaced by m-planes in the assumptions of Theorem 3.1, then $\mathscr{H}^m(M \cap K) = 0$ by [2, 5.4 (1)]. I do not know whether $\dim(M \cap K) < m$ in this case (except if $m = 1$).

On the other hand Theorem 3.1 and also [2, 5.4 (1)] are false without some assumption in addition to the open set condition. For example, if O is an open equilateral triangle in R^2, $O_1, O_2, O_3 \subset O$ are homothetic to O in ratio 1/2 each having one vertex in common with O and two sides contained in boundary O, and if S_1, S_2, S_3 are the obvious similitudes (without rotation) with $S_i O = O_i$, then $\mathscr{S} = \{S_1, S_2, S_3\}$ satisfies the open set condition with O, $\dim K = \log 3/\log 2 > 1$, but K contains countably many line segments. In this case the other separation condition fails for the lines containing the boundary segments of O.

4. The case $s \leqq m$

Marstrand considered the concept weakly tangential in [4]. We give the following.

4.1. Definition. *Let $E \subset R^n$ and $0 \leqq t \leqq n$. We say that E is weakly (t, m) tangential at a point $a \in R^n$ if $\Theta^{*t}(E, a) > 0$ and there is an m-plane V such that $a \in V$ and for every $\delta > 0$*

$$\liminf_{r \downarrow 0} r^{-t} \mathscr{H}^t(E \cap B(a, r) \sim \{x: \operatorname{dist}(x, V) \leqq \delta r\}) = 0.$$

V is then called a weak (t, m) tangent plane of E at a.

In general there may be several or no weak tangent planes. In the following recall that $K = |\mathscr{S}|$, $s = \dim K$ and $0 < \mathscr{H}^s(K) < \infty$ due to the open set condition.

4.2. Theorem. *Suppose that $\mathscr{S}$ satisfies the open set condition. If K has a weak (s, m) tangent plane V at some point $a \in K$, then $K \subset V$.*

Proof. We first show that K lies on some m-plane. Suppose this is not true. Then there is a subset $\{a_1, \ldots, a_{m+2}\}$ of K which is not contained in any m-plane, and we can find ϱ, $0 < \varrho < 1$, such that if W is an m-plane, then $\operatorname{dist}(a_i, W) \geqq \varrho$ for some $i = 1, \ldots, m+2$. The lower densities $\Theta^s_*(K, a_i)$ are positive by [2, 5.3 (1)]. Hence there are r_0, η such that $0 < r_0 < \varrho/2$ and

$$\mathscr{H}^s(K \cap B(a_i, r_0)) \geqq \eta r_0^s \quad \text{for} \quad i = 1, \ldots, m+2.$$

Let $0 < r < r_1$ and let α be a minimal sequence such that $a \in K_\alpha \subset B(a, r/2)$. We may assume $d(K) = 1$. Then $r_\alpha \leqq r/2 \leqq r_\alpha / r_1$. By the choice of ϱ there is i such that $\operatorname{dist}(S_\alpha^{-1}(V), a_i) \geqq \varrho$. Then

$$B(S_\alpha a_i, r_\alpha r_0) \subset B(a, r) \sim \{x: d(x, V) \leqq \varrho r_\alpha / 2\}.$$

Hence with $\delta = \varrho r_1 / 4$, we have $\delta r \leqq \varrho r_\alpha / 2$ and

$$\begin{aligned} \mathscr{H}^s(K \cap B(a, r) \sim \{x: d(x, V) \leqq \delta r\}) &\geqq \mathscr{H}^s(K_\alpha \cap B(S_\alpha a_i, r_\alpha r_0)) \\ &= r_\alpha^s \mathscr{H}^s(K \cap B(a_i, r_0)) \geqq \eta r_\alpha^s r_0^s \geqq \eta (r_1 r_0 / 2)^s r^s. \end{aligned}$$

This contradicts the fact that V is a weak (s, m) tangent plane for K at a. Therefore $K \subset W$ for some m-plane W.

Suppose that $W \neq V$. Letting X be the orthogonal complement of $V \cap W$, we have

$$c = \operatorname{dist}(W \cap X \cap \{x: |x| = 1\}, V \cap X) > 0.$$

Then one checks $c \operatorname{dist}(x, V \cap W) \leqq \operatorname{dist}(x, V)$ for $x \in W$. Since $K \subset W$ and $\Theta^*_s(K, a) > 0$, it follows that $1 \leqq \dim(V \cap W) < m$ and that $V \cap W$ is a weak tangent plane for K at a.

Let k, $0 < k < m$, be the smallest integer such that K is weakly (s, k) tangential at a. Then by the above proof K lies on some k-plane $U \subset W$. Then $U \subset V$, because otherwise $U \cap V$ would again be a weak tangent plane for K at a with $1 \leqq \dim (U \cap V) < k$, which would contradict the choice of k. Thus $K \subset V$.

4.3. Corollary. *Either K lies on an m-plane or $\mathscr{H}^s(M \cap K) = 0$ for every m-dimensional C^1 submanifold M of R^n.*

Proof. Suppose $\mathscr{H}^s(M \cap K) > 0$ for some M. For $\mathscr{H}^s$ a.a. $x \in M \cap K$, $\Theta^{*s}(K \sim M, x) = 0$ by [1, 2.10.19 (4)], and at these points the tangent m-plane of M is a weak (s, m) tangent plane of K. Hence K lies on an m-plane.

4.4. Corollary. *If either s is non-integral or s is an integer and K lies on no s-plane, then*

$$\Theta^s_*(K, x) < \Theta^{*s}(K, x) \text{ for } \mathscr{H}^s \text{ a.a. } x \in K.$$

Proof. If s is non-integral this follows directly from Marstrand's result [4]. If s is an integer set $E = \{x \in K\colon \Theta^*_s(K, x) = \Theta^{*s}(K, x)\}$, then by [4] K is weakly (s, s) tangential for $\mathscr{H}^s$ a.a. $x \in E$. Hence by Theorem 4.2, $\mathscr{H}^s(E) = 0$.

4.5. Remark. If s is an integer and K lies on no s-plane then Theorem 4.2 implies that K is purely $(\mathscr{H}^s, s)$ unrectifiable in the sense of [1, 3.2.14]. It is not known whether the conclusion of 4.4 is true for all purely $(\mathscr{H}^s, s)$ unrectifiable sets $K \subset R^n$ with $\mathscr{H}^s(K) < \infty$, except if $s = 1$.

References

[1] Federer, H.: Geometric measure theory. - Die Grundlehren der mathematischen Wissenschaften 153, Springer-Verlag, Berlin—Heidelberg—New York. 1969.

[2] Hutchinson, J. E.: Fractals and self-similarity. - Indiana Univ. Math. J. 30, 1981, 713—747.

[3] Mandelbrot, B. B.: Fractals: form, chance, and dimension. - W. H. Freeman and Company, San Francisco, 1977.

[4] Marstrand, J. M.: The (φ, s) regular subsets of n space. - Trans. Amer. Math. Soc. 113, 1964, 369—392.

[5] Mattila, P.: Hausdorff dimension, orthogonal projections and intersections with planes. - Ann. Acad. Sci. Fenn. Ser. A I Math., 1975, 227—244.

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