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F-HARMONIC MEASURE IN SPACE

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1. Introduction

The use of the harmonic measure is well established in the theory of harmonic and analytic functions. In this paper we present a similar concept which is based on a non-linear Euler equation of the variational integral

$$\int F(x, \nabla u)\, dm$$

where $F(x, h) \approx |h|^n$. The form of F is essential for our applications in conformal geometry. The purpose of the paper is to show that this concept, called the F-harmonic measure, is useful even in the non-linear case in space although it has several drawbacks, e.g. it does not define a measure.

The paper is a continuation of [GLM] by the same authors and the same notation and terminology will be used. After constructing the F-harmonic measure in Chapter 2 we show that several classical results of the harmonic measure have analogous statements for the F-harmonic measure. Among these are Carleman's and Phragmén—Lindelöf's principles. Sets of F-harmonic measure zero are considered in Chapter 4 and a simple sufficient metric condition for this is introduced. The connection of F-harmonic measures and quasiregular mappings is studied in the last chapter. We prove the invariance of F-harmonic measures under quasiconformal mappings. Note that in this respect the usual harmonic measure is not an invariant, see e.g. [BA], [HP], or even a quasi-invariant. We also present the principle of the F-harmonic measure for quasiregular mappings. These principles include the classical invariance properties of the harmonic measure under conformal and analytic functions, respectively.

2. Definitions for F-harmonic measure

2.1. Let G be a domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$, $n \geqq 2$. Except in § 3.9, G is assumed to be bounded. We only consider domains which are regular in the following topological sense.

2.2. Definition. The domain $G \subset \boldsymbol{R}^n$ is called *regular*, if no component of its boundary ∂G reduces to a single point.

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2.3. Remark. The topological regularity condition on ∂G is used to assure the solvability of Dirichlet's problem in the class $\{v\in C(\bar{G})\cap W_n^1(G)\colon v|\partial G=\varphi|\partial G\}$, φ a given function in the same class. For more general conditions see [M] and [Maz], cf. also [GLM, 3.25 (a)].

2.4. *Variational integral I_F.* In the variational integral

$$I_F = I_F(u, D) = \int_D F(x, \nabla u(x))\,dm(x), \quad D\subset G, \tag{2.5}$$

the kernel $F\colon G\times\mathbf{R}^n\to\mathbf{R}$ is assumed to satisfy the following conditions:

(a) Given any $\varepsilon>0$, there is a compact set $K_\varepsilon\subset G$ such that $m(K_\varepsilon)>m(G)-\varepsilon$ and the restriction $F|K_\varepsilon\times\mathbf{R}^n$ is continuous.

(b) For a.e. $x\in G$ the mapping $h\mapsto F(x, h)$ is strictly convex.

(c) There are constants $0<\alpha\leqq\beta<\infty$ such that for a.e. $x\in G$

$$\alpha|h|^n \leqq F(x, h) \leqq \beta|h|^n \tag{2.6}$$

when $h\in\mathbf{R}^n$.

(d) For a.e. $x\in G$ the function $h\mapsto F(x, h)$ is differentiable.

(e) For a.e. $x\in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h) \tag{2.7}$$

when $h\in\mathbf{R}^n$ and $-\infty<\lambda<\infty$.

2.8. Remarks. For a thorough analysis of our assumptions, we refer the reader to [GLM]. In (a) a natural condition of measurability is expressed. The strict convexity (b), i.e. the validity of the inequality

$$F(x, \lambda h_1+(1-\lambda)h_2) < \lambda F(x, h_1)+(1-\lambda)F(x, h_2),$$

$0<\lambda<1$, $h_1\neq h_2$, is needed to establish the uniqueness of extremals with given boundary values. This uniqueness property is essential for the comparison principle [GLM, 4.18]. The use of (2.7) is exhibited in the fact that λu is an extremal, whenever u is an extremal.

2.9. Remark. If (2.6) and (2.7) are valid with the exponent n replaced by any exponent p in the range $(1, \infty)$, most of our theory still holds. For $1<p<n$ the proofs in the regularity theory become more involved.

2.10. *Boundary sequences and generating sequences.* The construction of the F-harmonic measure, F satisfying (a), (b), and (c), proceeds via certain auxiliary sequences.

Let $C\subset\partial G$ be any closed set. We say that the sequence (φ_i) of functions in $C(\bar{G})\cap W_n^1(G)$ is a *(C, G)-boundary sequence*, if

1°) $1\geqq\varphi_1\geqq\varphi_2\geqq\ldots\geqq 0$,

2°) $\varphi_i|C=1$, $i=1, 2, 3, \ldots,$ and

3°) to each compact set $K \subset \bar{G}$ with $K \cap C = \emptyset$ there corresponds an index, say i_K, such that $\varphi_i | K = 0$, when $i > i_K$.

Obviously, there exist (C, G)-boundary sequences in great profusion.

Corresponding to each (C, G)-boundary function φ_i there is a unique F-extremal $u_i \in C(\bar{G}) \cap W_n^1(G)$ with boundary values $u_i | \partial G = \varphi_i | \partial G$. For this existence result we refer to [GLM; 3.24]. We call the sequence (u_i) (C, G)-*generating* or, more precisely, $(C, G;\, F)$-*generating*.

By the comparison principle [GLM, 4.18] the monotonicity expressed in 1°) is reflected on the sequence (u_i), i.e.

(2.11) $$1 \geqq u_1 \geqq u_2 \geqq \ldots \geqq 0.$$

Thus the limit

(2.12) $$1 \geqq u(x) = \lim u_i(x) \geqq 0$$

exists for every fixed $x \in G$. Actually, the convergence (2.12) is uniform on compact subsets of G. This follows from Harnack's principle [GLM, 4.22]. The same principle also proves the following theorem.

2.13. Theorem. *The limit function u of the $(C, G;\, F)$-generating sequence (u_i) is a free F-extremal*, i.e., *if D is any open set with compact closure in G, then*

$$I_F(u, D) \leqq I_F(v, D)$$

for all $v \in C(\bar{D}) \cap W_n^1(D)$ with boundary values $v | \partial D = u | \partial D$.

Next we show that u is well defined.

2.14. Theorem. *The limit function u is independent of the particular choice of the (C, G)-boundary sequence (φ_i), i.e. all $(C, G;\, F)$-generating sequences have the same limit function.*

Proof. Let (φ_i) and (ψ_i) be (C, G)-boundary sequences. Suppose that the corresponding $(C, G;\, F)$-generating sequences (u_i) and (v_i) converge to u and v, respectively.

Take $\varepsilon > 0$. To every index i there corresponds an index j_i such that

$$\varphi_i + \varepsilon > \psi_{j_i} \quad \text{in } \bar{G}.$$

Then also

$$u_i + \varepsilon \geqq v_{j_i} \quad \text{in } \bar{G}$$

be the comparison principle (obviously, $u_i + \varepsilon$ is the unique F-extremal with boundary values $\varphi_i + \varepsilon$). Hence $u + \varepsilon \geqq v$ in G. Since $\varepsilon > 0$ was arbitrary, $u \geqq v$ in G. By symmetry $v \geqq u$. This proves the desired uniqueness $u = v$.

2.15. *Definitions for F-harmonic measure.* If $C \subset \partial G$ is any closed set and if F satisfies (a), (b), and (c), then the previous construction defines the unique free F-extremal $u \in C(G) \cap W_n^1(G)$ via $(C, G;\, F)$-generating sequences.

2.16. Definition. The limit function u of the $(C, G; F)$-generating sequences is called the *F-harmonic measure of C with respect to G*. This is denoted by

$$u = \omega(C, G;\ F).$$

2.17. Remark. It is easily seen that the F-harmonic measure $\omega(C, G; F)$ formally can be defined via the generating process without any regularity assumptions at all on ∂G. Then the boundary values of the $(C, G; F)$-generating functions must be considered in the W_n^1-sense. We could also allow C to be a more general set. However, such generalizations are not essential in this paper.

In certain applications an equivalent definition for $\omega(C, G; F)$ seems to be natural. To this end, fix a closed set $C \subset \partial G$ and consider the class $\mathscr{F}(C, G)$ of all F-extremals $v \in C(\bar{G}) \cap W_n^1(G)$ such that $v \geqq 0$ and $v|C \geqq 1$. In particular, this class contains all $(C, G; F)$-generating sequences.

The following theorem can be regarded as a definition for the F-harmonic measures.

2.18. Theorem. *Let* $u = \omega(C, G; F)$. *Then*

$$u(x) = \inf \{v(x):\ v \in \mathscr{F}(C, G)\}$$

for each $x \in G$.

Proof. Obviously, $u(x) \geqq \inf_v v(x)$ for every $x \in G$.

Fix $x \in G$. There are functions $v_i \in \mathscr{F}(C, G)$ such that

$$\lim_{i \to \infty} v_i(x) = \inf \{v(x):\ v \in \mathscr{F}(C, G)\}.$$

Let (φ_i) be a (C, G)-boundary sequence. Then also $\min\{\varphi_1, v_1\}$, $\min\{\varphi_2, v_1, v_2\}$, $\min\{\varphi_3, v_1, v_2, v_3\}, \ldots$ is a (C, G)-boundary sequence. The $(C, G; F)$-generating functions u_i corresponding to this latter sequence certainly satisfy the inequality $u_i \leqq v_i$ in G, since $u_i|\partial G \leqq v_i|\partial G$. Thus $u(x) = \lim u_i(x) \leqq \lim v_i(x)$, whence $u(x) \leqq \inf_v v(x)$. This concludes our proof, since $x \in G$ was arbitrary.

2.19. Remark. We can allow the class $\mathscr{F}(C, G)$ to contain all super-F-extremals, cf. [GLM, Chapter 4], $v \in C(\bar{G}) \cap W_n^1(G)$ such that $v \geqq 0$ and $v|C \geqq 1$.

2.20. Remark. For each $y \in \partial G \setminus C$, $\lim_{i \to \infty} u_i(y) = 0$ and since u_i decreases to the F-harmonic measure u of C,

$$\lim_{x \to y} u(x) = 0$$

for $y \in \partial G \setminus C$. In 4.3 we present quantitative estimates for this.

3. Classical principles

Some classical principles are extended to the F-harmonic measure. Most of these extensions are straightforward but some of them, e.g. Phragmén—Lindelöf's principle, have a wide scope of applications, especially if Chapter 4 is used.

3.1. *Basic principles.* The first theorem is an immediate consequence of Theorem 2.18. We assume that F satisfies (a)—(c), unless otherwise stated, in a bounded regular domain G.

3.2. Theorem. *Let $C_1 \subset C_2$ be closed sets in ∂G. Then $\omega(C_1, G; F) \leqq \omega(C_2, G; F)$.*

3.3. Theorem. *Suppose that the sets*

$$C_1 \supset C_2 \supset \dots, \quad C = \cap C_i,$$

are closed in ∂G. Then

$$\omega(C_i, G; F) \to \omega(C, G; F)$$

uniformly on compact subsets of G.

Proof. Write $u_i = \omega(C_i, G; F)$, $i = 1, 2, \dots$, and $u = \omega(C, G; F)$. By Theorem 3.2

$$u_1 \geqq u_2 \geqq \dots \geqq u.$$

Fix a compact set $K \subset G$. Given $\varepsilon > 0$ choose a (C, G)-boundary function φ such that

$$\sup_{x \in K} |u(x) - u_\varphi(x)| < \varepsilon/2.$$

Here u_φ is the F-extremal with boundary values φ.

There is an integer i_0 such that the F-extremal u_ψ, $\psi = \varphi + \varepsilon/2$, is in $\mathscr{F}(C_{i_0}, G)$ and thus, by Theorem 2.18,

$$u_{i_0} \leqq u_\psi = u_\varphi + \varepsilon/2.$$

Hence for $i \geqq i_0$

$$u_i \leqq u_{i_0} \leqq u_\varphi + \varepsilon/2 < u + \varepsilon/2 + \varepsilon/2$$

in K and since $u_i \geqq u$ in K, we have proved the uniform convergence on compact subsets of G.

The classical principle of Carleman holds for F-harmonic measures.

3.4. Theorem. *Suppose that G_1 and G_2 are regular domains. If $C \subset \partial G_1 \cap \partial G_2$ is closed and $G_1 \subset G_2$, then*

$$\omega(C, G_1; F) \leqq \omega(C, G_2; F)$$

in G_1.

Proof. If $v \in \mathscr{F}(C, G_2)$, then $v | \bar{G}_1 \in \mathscr{F}(C, G_1)$. Hence Theorem 2.18 yields the desired result.

3.5. *Comparison principles.* An upper semi-continuous function $v\colon G\to \boldsymbol{R}\cup\{-\infty\}$ is called a sub-F-extremal if for all domains $D\subset\subset G$ and all F-extremals $h\in C(\bar{D})$ the condition $h\geqq v$ in ∂D implies $h\geqq v$ in D. The local comparison principle [GLM, 4.18] has a trivial improvement. If v is a sub-F-extremal in a bounded but not necessarily regular domain G, w an F-extremal or more generally, a super-F-extremal, see [GLM, Chapter 5], in G and eiter v or w is bounded with

$$\underline{\lim}\, w(x) \geqq \overline{\lim}\, v(x)$$

as x in G approaches any point in ∂G, then $w\geqq v$ in G. If this principle is applied to a (C, G)-generating sequence, the following corollary is obtained.

3.6. Corollary. *Let C be a closed set on the boundary of a bounded regular domain G. Suppose that v is a sub-F-extremal in G such that*

$$\overline{\lim_{x\to y}}\, v(x) \begin{array}{ll} \leqq 0 & \text{for } y\in\partial G\setminus C, \\ \leqq 1 & \text{for } y\in C. \end{array}$$

Then $v\leqq\omega(C, G; F)$ in G.

If a little more is assumed on F, Corollary 3.6 can be improved.

3.7. Theorem. *Suppose that F satisfies the assumptions* (a)—(c) *and* (e) *of Chapter* 2 *in G and that $C\subset\partial G$ is closed. Let v be a sub-F-extremal in G such that*

$$\overline{\lim_{x\to y}}\, v(x) \begin{array}{ll} \leqq M & \text{for } y\in C, \\ \leqq m & \text{for } y\in\partial G\setminus C, \end{array}$$

where $M\geqq m$. Then

$$v \leqq (M-m)\omega(C, G; F)+m \tag{3.8}$$

in G.

Proof. We may assume $M>m$. Then $w=(v-m)/(M-m)$ is a sub-F-extremal. By Corollary 3.6, $w\leqq\omega(C, G; F)$ in G, i.e. (3.8) holds.

Because of the strict convexity assumption (b) each F-extremal is also a sub-F-extremal, cf. [GLM, 4.19]. Hence Theorem 3.7 holds for F-extremals as well.

3.9. *Phragmén—Lindelöf's principle.* Suppose that $G\subset\boldsymbol{R}^n$ is an unbounded domain with non-empty boundary and regular in the sense of Definition 2.2. Let $\omega(x; r)$ denote the value of $\omega(\bar{G}\cap S^{n-1}(r), G\cap B^n(r); F)$ taken at the point $x\in G$, $|x|<r$. Note that the open set $G\cap B^n(r)$ is not necessarily connected, but this plays no role in the definition of $\omega(x; r)$. The open set $G\cap B^n(r)$ is always regular.

With an obvious modification of (a) for an unbounded domain we assume that F satisfies (a)—(c) and (e) in G. Phragmén—Lindelöf's principle now takes the following form.

3.10. Theorem. *Suppose that u is a sub-F-extremal in G with $\overline{\lim}_{x\to y} u(x)\leqq 0$ for all $y\in\partial G$. Then, either $u\leqq 0$ in G or*

$$M(r)=\sup_{|x|=r,\, x\in G} u(x)$$

grows so fast that

$$\underline{\lim}_{r\to\infty}\,[M(r)\omega(x;\, r)]>0 \tag{3.11}$$

for every $x\in G$.

Proof. Suppose that $u(x_0)>0$ at some point $x_0\in G$. Since

$$M(r)=\sup_{|x|\leqq r,\, x\in G} u(x),$$

Theorem 3.7 yields for $r>|x_0|$

$$u(x_0)\leqq M(r)\omega(x_0,\, r).$$

Thus (3.11) follows for $x=x_0$.

If $x, y\in G$, then for all sufficiently large radii r Harnack's inequality gives a constant C independent of r such that

$$\omega(x,\, r)\leqq C\,\omega(y,\, r).$$

Hence (3.11) is valid at every point $x\in G$, if it holds for some point $y\in G$. This completes the proof.

3.12. Remark. (a) Simple examples show that (3.11) is best possible. (b) V. Mikljukov [Mik] has also studied theorems of Phragmén—Lindelöf type for regular subsolutions of elliptic equations.

4. Sets of F-harmonic measure zero

4.1. Definitions. If $G\subset \boldsymbol{R}^n$ is a regular domain and $C\subset\partial G$ a closed set, then $\omega(C, G;\, F_1)$ may be zero for a kernel F_1 satisfying (a)—(e), but $\omega(C, G;\, F_2)\neq 0$ for another kernel F_2 satisfying the same assumptions, see Remark 5.4 (b). Hence the following definitions will turn useful.

4.2. Definition. Let G be a regular domain and $C\subset\partial G$ a closed set. We say that the F-harmonic measure of C with respect to G is zero if $\omega(C, G;\, F)(x)=0$ for all $x\in G$. The set C is said to be of total F-harmonic measure zero if the F-harmonic measure of C with respect to G is zero for all kernels F in G satisfying (a)—(c) and (e). Note that in the last definition we do not restrict the range of the values $0<\alpha\leqq\beta<\infty$.

The purpose of this chapter is to study sets of total F-harmonic measure zero. We assume that G is a regular domain in $\boldsymbol{R}^n$ and that F satisfies (a)—(c) and (e) in G.

4.3. *Boundary estimate.* In the sequel we need an estimate for an F-extremal near the boundary of G. Similar, and even stronger, estimates have been derived in many papers, see e.g. [Maz, Theorem, p. 51]. However, our proof is elementary.

4.4. Lemma. *Suppose that $u\in C(\bar{G})$ is a non-negative F-extremal in G, $x_0\in\partial G$, and $u(x)=0$ for $x\in\partial G\cap B^n(x_0, R)$. If $S^{n-1}(x_0, t)$ meets ∂G for all $t\in(0, R)$, then*

$$\sup_{B^n(x_0,t)\cap G} u \leqq c \sup_G u \left(\log \frac{R}{t}\right)^{-n}.$$

The constant c depends only on n and β/α.

Proof. Let $0<t<R$. Set $r=\sqrt{tR}$. We may assume $x_0=0$. For $s>0$ write $D(s)=B^n(x_0, s)\cap G$ and $D=D(R)$. Let $\varphi\in C_0^\infty(B^n(R))$ such that $0\leqq\varphi\leqq 1$ and $\varphi(x)=1$ for $x\in B^n(r)$. Set $v=u-\varphi^n u$. Then $v\in C(\bar{D})$ and $v=u$ in ∂D. Now $\nabla v=(1-\varphi^n)\nabla u-n\varphi^{n-1}u\nabla\varphi=(1-\varphi^n)\nabla u+\varphi^n\left(-\frac{n}{\varphi}u\nabla\varphi\right)$, whenever $\varphi(x)>0$ and the F-extremality of u gives together with (b) and (c)

$$I_F(u, D) \leqq I_F(v, D) \leqq \int_D (1-\varphi^n) F(x, \nabla u)\, dm + \beta n^n \int_D |u|^n |\nabla\varphi|^n dm.$$

Now $\varphi|B^n(r)=1$, hence

$$I_F(u, D(r)) \leqq \beta n^n (\sup_G u)^n \int_D |\nabla\varphi|^n dm.$$

Letting φ vary over all admissible functions for the condenser $(B^n(R), \bar{B}^n(r))$, see [GLM, 2.3], we get

$$I_F(u, D(r)) \leqq \beta n^n (\sup_G u)^n \omega_{n-1} \left(\ln \frac{R}{r}\right)^{1-n}. \tag{4.5}$$

This estimate is similar to the standard estimate, see [GLM, 4.2].

For each $s\in[t, r]$ choose a spherical cap $C(x_s, \varphi_s)$ on $S^{n-1}(s)\cap G$ centered at a point x_s where

$$u(x_s) = \max_{x\in S^{n-1}(s)\cap G} u(x)$$

and $\bar{C}(x_s, \varphi_s)$ meets ∂G. This is possible by the assumption. Now u is monotone, hence $u(x_s)\geqq u(x_{s'})$ for $s\geqq s'$. Thus [GLM, 2.7] yields

$$(\sup_{D(t)} u)^n = u(x_t)^n \leqq \left(\log \frac{r}{t}\right)^{-1} \int_t^r \frac{u(x_s)^n}{s}\, ds$$

$$\leqq A_n \left(\log \frac{r}{t}\right)^{-1} \int_{D(r)} |\nabla u|^n dm \leqq A_n \left(\frac{\beta}{\alpha}\right) n^n (\sup_G u)^n \omega_{n-1} \left(\log \frac{r}{t}\right)^{-1} \left(\log \frac{R}{r}\right)^{1-n}$$

where (4.5) and (c) have also been used. This gives the result since $r=\sqrt{Rt}$.

4.6. Remark. The argument used in [GLM, 4.7] yields a Hölder-estimate

$$\sup_{B^n(x_0,t)\cap G} u \leqq c' \sup_G u\, t^\varepsilon \tag{4.7}$$

where $\varepsilon=\varepsilon(n, \beta/\alpha)$. However, we need not the strong version (4.7). Note also that the proof of Lemma 4.4 implies that

$$u\in W_n^1(G\cap B^n(x_0, t)) \quad \text{for} \quad t\in(0, R).$$

Lemma 4.4. and the definition of a $(C, G; F)$-generating sequence give

4.7. Corollary. *Suppose that a point $x_0\in\partial G$ satisfies $S^{n-1}(x_0, t)\cap\partial G\neq\emptyset$ for all $t\in(0, R)$. Let C be a closed set in ∂G with $B^n(x_0, R)\cap C=\emptyset$ and let u be the F-harmonic measure of C. Then there is $\varkappa=\varkappa(n, \beta/\alpha)>0$ such that $u(x)<1/2$ for $x\in B^n(x_0, \varkappa R)\cap G$.*

4.8. *Sets of F-harmonic measure zero.* We begin with a simple result where the condition (e) plays an essential role.

4.9. Theorem. *A closed set $C\subset\partial G$ is of F-harmonic measure zero if and only if*

$$\overline{\lim_{x\to\partial G}}\, \omega(C, G; F)(x) < 1. \tag{4.10}$$

Proof. The condition (4.10) is clearly necessary. To prove that it is also sufficient set $u=\omega(C, G; F)$. Write $\lambda=\sup_G u$. By (4.10), $\lambda<1$ since u is monotone. Now F satisfies (e), hence if $\lambda>0$, u/λ is an F-extremal and by Lemma 3.6, $u/\lambda\leqq u$ in G. This shows that $u=0$. The theorem follows.

To the other direction it is easy to give a simple sufficient condition.

4.10. Theorem. *Suppose that a closed set C in ∂G has a non-empty interior with respect to ∂G. Then the F-harmonic measure of C is not zero.*

Proof. Consider the (C, G)-generating sequence u_i. By (e), $v_i=1-u_i$ is an F-extremal in G. Let $x_0\in\mathrm{int}_{\partial G}\, C$. Then there is $R>0$ such that the assumptions of Lemma 4.4 are satisfied for v_i. Consequently,

$$v_i(x)\leqq 1/2$$

for all $x\in B^n(x_0, t)\cap G$ for some $t\in(0, R)$. Thus $u_i\geqq 1/2$ in the same set. This shows that the F-harmonic measure u of C is not zero.

The next theorem states that the F-harmonic measure is a metric outer measure on the zero level.

4.11. Theorem. *Suppose that C_1 and C_2 are closed sets of F-harmonic measure zero in ∂G. If $C_1\cap C_2=\emptyset$, then $C_1\cup C_2$ is of F-harmonic measure zero.*

Proof. Let u be the F-harmonic measure of $C_1\cup C_2$. By Theorem 4.10, $C_1\cup C_2\neq\partial G$. For $0<t<1$ consider the set $A_t=\{x\in G\colon u(x)>t\}$. Since u is monotone and C_1 and C_2 are disjoint, Remark 2.20 gives a component A of A_t for some t such that either $\bar{A}\subset G\cup C_1$ or $\bar{A}\subset G\cup C_2$. Assume, for example, that $\bar{A}\subset G\cup C_1$. Set $\varphi(x)=u(x)-t$ for $x\in A$ and $\varphi(x)=0$ for $x\in G\setminus A$. The function φ is a regular sub-F-extremal, cf. [GLM, 5.4], and not identically zero. On the other hand

$$\overline{\lim_{x\to y}}\,\varphi(x)\begin{array}{ll} = 0, & y\in\partial G\setminus C_1, \\ \leqq 1, & y\in C_1. \end{array}$$

By the comparison principle, Corollary 3.6, $\varphi\leqq u_1$ where u_1 is the F-harmonic measure of C_1. Thus $\varphi\leqq 0$ and $u\leqq t$. By Theorem 4.9 the F-harmonic measure of $C_1\cup C_2$ is zero.

4.12. *Sets of total F-harmonic measure zero.* We give two sufficient conditions for a closed set $C\subset\partial G$ to have a total F-harmonic measure zero. In general, it seems difficult to exhibit simple necessary and sufficient conditions for this.

4.13. Theorem. *Suppose that $C\subset\partial G$ is of zero n-capacity. Then the total F-harmonic measure of C is zero.*

4.14. Remark. We recall that a compact set $C\subset\boldsymbol{R}^n$ is of zero n-capacity if $\operatorname{cap}_n(D, C)=0$, see [GLM, 2.3], for all open sets $D\supset C$.

Proof. Since C is of zero n-capacity, there exists a (C, G)-boundary sequence φ_i such that $\nabla\varphi_i\to 0$ in $L^n(G)$. The corresponding (C, G)-generating sequence u_i has by the extremality and (c) the property

$$\alpha\int_G|\nabla u_i|^n\,dm\leqq I_F(u_i, G)\leqq I_F(\varphi_i, G)\leqq\beta\int_G|\nabla\varphi_i|^n\,dm.$$

Hence $\nabla u_i\to 0$ in $L^n(G)$ and the F-harmonic measure u of C is a constant function. Since G is regular, C is a proper subset of ∂G and thus by Remark 2.20, $u\equiv 0$. The theorem follows.

The condition of Theorem 4.13 is far from necessary. In order to establish a relatively simple geometric condition we consider two concepts: δ-thin sets and boundary uniform domains.

4.15. Definition. Given two sets $C\subset B$ in $\boldsymbol{R}^n$ and $0<\delta<1$ we say that C is *δ-thin* in B if there is $r_0>0$ such that for all $y\in B$ and each $r\in(0, r_0)$ there exists a point $z\in B\cap B^n(y, r)$ with $B^n(z, \delta r)\cap C=0$.

4.16. Remark. If B is a smooth k-dimensional submanifold of $\boldsymbol{R}^n$ and $C\subset\boldsymbol{B}$ is δ-thin in B, then the Hausdorff-dimension $\dim_H$ of C satisfies

$$\dim_H C\leqq C(k, \delta)<k,$$

see e.g. [S, Theorem 3.2].

4.17. Definition. A domain G in $\boldsymbol{R}^n$ is called *$(\varkappa, M)$-uniform*, $\varkappa>1$, $0<M<\infty$, *at a boundary point* $y\in\partial G$ if there exists a neighbourhood U of y such that for all $x\in U\cap G$ and all $z\in\partial G\cap B^n(x, \varkappa R)$, $R=\operatorname{dist}(x, \partial G)$, there is a rectifiable path (arc length as parameter) $\gamma\colon [0, d]\to G\cup\{z\}$ with (i) $\gamma(0)=z$, $\gamma(d)=x$ (ii) $d\leqq MR$, and (iii) $\operatorname{dist}(\gamma(t), \partial G)\geqq t/M$ for $t\in[0, d]$. A domain G is called *boundary uniform* if there are $\varkappa$ and M such that G is $(\varkappa, M)$-uniform at each boundary point.

4.18. Remark. The concept of a uniform domain was introduced in [MS]. For equivalent definitions of uniformity see [M] and [GO]. It is not difficult to see that a uniform domain in the sense of [MS] is boundary uniform. The converse is not true. For example, the plane domain $B^2\setminus\{(x_1, x_2)\colon x_1\geqq 0,\ x_2=0\}$ is boundary uniform but not uniform in the sense of [MS].

Note also a connection between uniform domains in the sense of [MS] and δ-thin sets. A closed set $C\subset S^{n-1}$ is δ-thin in S^{n-1} if and only if $R^n\setminus C$ is a uniform domain. A relatively easy proof is left to the reader.

4.19. Theorem. *Suppose that G is a bounded boundary uniform domain whose complement is connected. If $C\subset\partial G$ is a closed δ-thin set in ∂G, then the total F-harmonic measure of C is zero.*

Proof. Note first that G is a regular domain. Fix a kernel F in G. Let $y\in C$. By Theorem 4.9 and Remark 2.20 it suffices to show (4.20) $\overline{\lim}_{x\to y}u(x)\leqq 1-t'$ where u is the F-harmonic measure of C and $t'>0$ is independent of y.

Since G is $(\varkappa, M)$-uniform at y, there is a neighbourhood U of y as in Definition 4.17. Let $x\in U\cap G$ and set $R=\operatorname{dist}(x, \partial G)$. Choose $y_1\in\partial G$ with $|y_1-x|=R$. Since C is δ-thin in ∂G, we may assume that $\varkappa R<r_0$ where r_0 is given by 4.15. This implies that there is a point $z\in\partial G\cap B^n(y_1, \varkappa R)$ with $B^n(z, \delta\varkappa R)\cap C=\emptyset$. Now G is $(\varkappa, M)$-uniform at the boundary point y, hence there is a rectifiable path $\gamma\colon [0, d]\to G\cup\{z\}$ with (i) $\gamma(0)=z$, $\gamma(d)=x$, (ii) $d\leqq MR$, and (iii) $\operatorname{dist}(\gamma(t), \partial G)\geqq t/M$.

Since $R^n\setminus G$ is connected, Corollary 4.7 yields

$$u(w)\leqq 1/2 \tag{4.21}$$

for $w\in G\cap\bar B^n(z, cr)$, $r=\delta\varkappa R$, $c=c(n, \beta/\alpha)\in(0, 1)$. Next we choose numbers $t_1, t_2, \dots$ and radii $r_1, r_2, \dots$ inductively as follows:

$$t_1=\sup\{t\in[0, d]\colon \gamma(t)\in B^n(z, cr)\},$$

$r_1=t_1/2M$ and

$$t_i=\sup\{t\in[t_{i-1}, d]\colon \gamma(t)\in B^n(\gamma(t_{i-1}),\ r_{i-1})\},$$

$r_i=t_i/2M$, $i=2, 3, \dots$. Let k be the first integer i such that $t_k=d$. If $i<k$, then

$$t_i\geqq t_{i-1}+r_{i-1}=t_{i-1}(1+1/2M)\geqq\dots\geqq t_1(1+1/2M)^{i-1}\geqq cr(1+1/2M)^{i-1}.$$

Thus

$$k\leqq\frac{\log(MR/cr)}{\log(1+1/2M)}+2=\frac{\log(M/c\delta\varkappa)}{\log(1+1/2M)}+2=c'. \tag{4.22}$$

On the other hand $B^n(\gamma(t_i), 2r_i)\subset G$, $i=1, 2, \dots, k$ and we apply Harnack's inequality, see [GLM, 4.15], to the F-extremal $1-u$ in the ball $B^n(\gamma(t_i), r_i)$. Since

$\gamma(t_{i+1})\in\bar{B}^n(\gamma(t_i), r_i)$, this gives

$$1-u(\gamma(t_{i+1}))\geqq e^{-\lambda}(1-u(\gamma(t_i)))$$

for $i=1, \dots, k-1$. Here $\lambda>0$ depends only on n and β/α. By iteration

$$1-u(x) = 1-u(\gamma(t_k))\geqq e^{-\lambda(k-1)}(1-u(\gamma(t_1))).$$

Now (4.21) yields together with (4.22)

$$u(x)\leqq 1-\frac{1}{2}e^{-\lambda(k-1)} = 1-t'$$

where $t'=t'(n, \beta/\alpha, \delta, M, \varkappa)>0$. This proves (4.20) and the theorem follows.

Since B^n is a boundary uniform domain and its complement is connected, Theorem 4.19 has the following corollary.

4.23. Corollary. *Suppose that $C\subset S^{n-1}$ is a closed δ-thin set in S^{n-1}. Then the total F-harmonic measure of C is zero in B^n.*

4.24 Remarks. (a) By Remark 4.16, $\dim_H C<n-1$. It is not difficult to construct δ-thin sets C in S^{n-1} whose Hausdorff-dimension is arbitrary near $n-1$.

(b) By Remark 4.18, Corollary 4.23 has an alternative formulation: If $C\subset S^{n-1}$ is a closed set such that $R^n\setminus C$ is a uniform domain in the sense of [MS], then the total F-harmonic measure of C is zero.

(c) Let $C=\{x\in S^{n-1}: x_n=0\}$. Then it is easy to see that C is (1/2)-thin in S^{n-1}. Consequently its total F-harmonic measure is zero in B^n and also, by Theorem 4.19, in the domain

$$G = B^n\setminus\{x\in\bar{B}^n: x_1 = x_2 = \dots = x_{n-1} = 0\} = B^n\setminus C_1$$

if $n\geqq 3$. Observe that the set C_1 is not of F-harmonic measure zero in G by Theorem 4.10. This kind of situation cannot happen in R^2.

(d) Theorem 4.19 does not imply Theorem 4.13 even in B^n, since there are countable closed sets C in S^{n-1} which are not δ-thin in S^{n-1} for any $\delta>0$.

5. *F*-harmonic measure and quasiregular mappings

In this chapter we extend the classical principle of harmonic measure to quasiconformal and quasiregular mappings. We apply this principle to the growth of a quasiregular mapping.

Suppose that $f\colon G\to G'$ is a quasiregular mapping, G and G' domains in R^n, see [GLM, 6.1]. If a kernel F satisfies (a)—(e) in G', then $f^{\#}F$ is defined in G by

$$f^{\#}F(x, h) = \begin{cases} F(f(x), J(x, f)^{1/n}f'(x)^{-1*}h), & \text{if} \quad J(x, f)\neq 0, \\ |h|^n, \quad \text{if} \quad J(x, f) = 0 \quad \text{or} \quad f'(x) \quad \text{does not exist.} \end{cases}$$

The kernel $f^{\#}F$ satisfies the same assumptions as F, see [GLM, 6.4].

We first consider the quasiregular case.

5.1. Theorem. *Suppose that G and G' are regular bounded domains, F satisfies* (a)—(d) *in G' and $f\colon G\to G'$ is a quasiregular mapping. If C, resp C', is closed in ∂G, resp. in $\partial G'$, and C' contains the cluster set $C(f,C)$ of f at C, then*

$$\omega(C', G';\ F)(f(x)) \geqq \omega(C, G;\ f^{\#}F)(x) \quad \textit{for all } x\in G. \tag{5.2}$$

Proof. Let v_i be a $(C', G';\ F)$-generating sequence. Then $u_i=v_i\circ f$ is an $f^{\#}F$-extremal in G and u_i satisfies

$$\lim_{x\to y} u_i(x) = 1, \quad y\in C \tag{5.3}$$

$$\varliminf_{x\to y} u_i(x) \geqq 0, \quad y\in\partial G\setminus C,$$

hence $u_i\geqq\omega(C, G;\ f^{\#}F)$ by Theorem 2.18. This yields the desired inequality (5.2).

In the quasiconformal case the inequality can be interpreted as follows.

5.4. Theorem. *Let G, G' and F be as in Theorem* 5.1. *Suppose that $f\colon G\to G'$ is a quasiconformal mapping. If C is a closed set in ∂G, then*

$$\omega(C(f,C), G';\ F)(f(x)) \geqq \omega(C, G;\ f^{\#}F)(x) \tag{5.5}$$

for all $x\in G$ where $C(f,C)$ is the cluster set of f at C. If the cluster set of f^{-1} at $C(f,C)$ is C, then (5.5) *holds as equality.*

Proof. Since f is a homeomorphism, $C(f,C)$ is a closed set in $\partial G'$ and (5.5) follows from (5.2). Suppose that the cluster set of f^{-1} at $C(f,C)$ is C. Consider a $(C, G;\ f^{\#}F)$-generating sequence u_i and let $v_i=u_i\circ f^{-1}$. Then (5.3) holds for v_i and the inequality in (5.5) follows as in the proof of Theorem 5.1. This completes the proof.

5.6. Remarks. (a) Theorems 5.1 and 5.4 give the classical principles of harmonic measure [N, pp. 37—38], if f is chosen conformal or analytic, respectively.

(b) Using quasiconformal mappings it is now easy to give an example of a set $C\subset\partial B^2$ such that the F-harmonic measure of C is positive for $F(x,h)=|h|^2$ but the F_1-harmonic measure of C is zero. To construct such a set choose a closed set $C\subset\partial B^2$ of linear measure zero and a quasiconformal mapping $f\colon \mathbf{R}^2\to\mathbf{R}^2$ such that $fB^2=B^2$ and f maps C onto a set of positive linear measure, see [BA]. Now $\omega(fC, B^2;\ F)>0$ but for $F_1=f^{-1\#}F$ and for all $y\in B^2$

$$\omega(fC, B^2;\ F_1)(y) = \omega(C, B^2;\ f^{\#}F_1)(f^{-1}(y)) = \omega(C, B^2;\ F)(f^{-1}(y)) = 0$$

since the ordinary harmonic measure of C is zero. It is an open problem if this situation is typical only in the plane.

5.7. *Growth of a quasiregular mapping.* The Phragmén—Lindelöf principle, Theorem 3.9, can be used to estimate a quasiregular mapping. However, here we give a simple estimate in a ball B^n based directly on Theorem 3.7.

5.8. Theorem. *Suppose that $f\colon B^n \to \mathbf{R}^n$ is a quasiregular mapping and $C\subset\partial B^n$ is a closed set. Suppose that there are $0<m\leqq M<\infty$ such that*

$$\overline{\lim_{x\to C}}\,|f(x)|\leqq M$$

and

$$\overline{\lim_{x\to\partial B^n\setminus C}}\,|f(x)|\leqq m,$$

then

$$|f(x)|\leqq m(M/m)^{u(x)}$$

for all $x\in B^n$. Here $u=\omega(C, B^n; f^{\#}F)$ and $F(x,h)=|h|^n$.

Proof. The function $v=\log r$ is an F-extremal in $\mathbf{R}^n\setminus\{0\}$ and a sub-F-extremal in $\mathbf{R}^n$. By [GLM, 7.10] the function $w=v\circ f$ is a sub-$f^{\#}F$-extremal in B^n. Now

$$\overline{\lim_{x\to y}}\, w(x)\leqq\log M\quad\text{for}\quad y\in C,$$

$$\overline{\lim_{x\to y}}\, w(x)\leqq\log m\quad\text{for}\quad y\in\partial G\setminus C.$$

Hence by Theorem 3.7

$$w\leqq(\log M-\log m)u+\log m.$$

This gives the required estimate.

5.9. Remarks. (a) If the set C in Theorem 5.8 is of F-harmonic measure zero, then Theorem 5.5 yields $|f(x)|\leqq m$ for all $x\in B^n$. In view of Theorems 4.13 and 4.19 this is a considerable extension of [MR, Corollary 3.9].

(b) Theorem 5.8 also gives the two constant theorem of quasiregular mappings, since the result can be written in the form

$$\log|f(x)|\leqq\big(1-u(x)\big)\log m+u(x)\log M. \tag{5.10}$$

The formula (5.10) has been applied by S. Rickman [R] in a special case to study Lindelöf's theorem for quasiregular mappings in $\mathbf{R}^3$.

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