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# ON FUCHSIAN GROUPS OF ACCESSIBLE TYPE

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## 1. Introduction

**1.1.** Let $\Gamma$ be a Fuchsian group acting on the open unit disk $\boldsymbol{D}$. The group $\Gamma$ is called of *accessible type* if there exists a measurable set $B\subset\partial\boldsymbol{D}$ of positive Lebesgue measure that contains no two $\Gamma$-equivalent points, and $\Gamma$ is called of *fully accessible type* if moreover

$$\partial\boldsymbol{D} \doteq \Gamma B \equiv \bigcup_{\gamma\in\Gamma} \gamma(B) \tag{1.1}$$

where $\doteq$ denotes equality up to a set of zero measure. Every group of accessible type is of convergence type and therefore has a Green's function.

We define

$$u(\zeta) = \sum_{\gamma\in\Gamma} |\gamma'(\zeta)| \quad \text{for} \quad \zeta\in\partial\boldsymbol{D}. \tag{1.2}$$

It was shown in [10] and [11] that the following conditions are equivalent:

(i) $\Gamma$ is of fully accessible type;

(ii) for almost all $\zeta\in\partial\boldsymbol{D}$, every horocycle at $\zeta$ contains only finitely many points $\gamma(0)$ with $\gamma\in\Gamma$;

(iii) $u(\zeta)<\infty$ for almost all $\zeta\in\partial\boldsymbol{D}$;

(iv) the Green's measure of $\Gamma$ is absolutely continuous with density $u$.

The Green's measure of $\Gamma$ is the reformulation [10] for the groups $\Gamma$ of the Green's measure (Brelot and Choquet [1]) of the Riemann surface $\boldsymbol{D}/\Gamma$.

D. Sullivan [16, Theorem III] has shown that (i) is also equivalent to the condition

(v) for almost all $\zeta\in\partial\boldsymbol{D}$, some horocycle at $\zeta$ contains no $\gamma(0)(\gamma\in\Gamma)$.

He considers the ergodic properties of the action of $\Gamma$ on $\partial\boldsymbol{D}$ and actually generalizes (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) to Kleinian groups of any dimension. In the present case we have

$$(\text{dissipative part of } \partial\boldsymbol{D}) \doteq \bigcup_{\gamma\in\Gamma} \gamma(\partial\boldsymbol{D}\cap\partial F)$$

where $F$ denotes a Ford fundamental domain of $\Gamma$. Thus "not accessible" means that $\partial\boldsymbol{D}$ is conservative while "fully accessible" means that $\partial\boldsymbol{D}$ is dissipative with respect to the action of $\Gamma$ on $\partial\boldsymbol{D}$.

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S. J. Patterson has proved [7, Theorem 3] that $\Gamma$ is of fully accessible type if the exponent of convergence satisfies $\delta(\Gamma)<1/2$, that is if $\sum_{\gamma\in\Gamma}|\gamma'(0)|^{\alpha}<\infty$ for some $\alpha<1/2$. On the other hand, he has constructed [8] an example of a group with $\delta(\Gamma)<1$ which is not even of accessible type.

**1.2.** As a motivation we briefly consider the analogue of the Martin boundary for the group $\Gamma$. The Martin boundary is discussed in the book of Constantinescu and Cornea [2] for hyperbolic Riemann surfaces.

Let $\Gamma$ be of fully accessible type. It follows from (1.2) and (iii) that, for almost all $\zeta\in\partial\boldsymbol{D}$, the "Poisson kernel"

$$p(z,\zeta)=\frac{1}{u(\zeta)}\sum_{\gamma\in\Gamma}\frac{1-|\gamma(z)|^2}{|\zeta-\gamma(z)|^2}=\sum_{\gamma\in\Gamma}\frac{1-|z|^2}{|\gamma(\zeta)-z|^2}\frac{|\gamma'(\zeta)|}{u(\zeta)} \tag{1.3}$$

is a positive harmonic function of $z\in\boldsymbol{D}$ with $p(0,\zeta)=1$ and $p(\gamma(z),\zeta)=p(z,\zeta)$ for $\gamma\in\Gamma$. It is minimal among functions with these properties. If $v$ is a bounded harmonic function in $\boldsymbol{D}$ with $v\circ\gamma=v$ for $\gamma\in\Gamma$ then, by the Poisson integral formula and by (1.1) and (1.3),

$$\begin{aligned} v(z)&=\frac{1}{2\pi}\int_{\partial\boldsymbol{D}}\frac{1-|z|^2}{|\zeta-z|^2}v(\zeta)\,|d\zeta|=\sum_{\gamma\in\Gamma}\frac{1}{2\pi}\int_{\gamma(B)}\frac{1-|z|^2}{|\zeta-z|^2}v(\zeta)\,|d\zeta|\\ &=\frac{1}{2\pi}\int_{B}p(z,\zeta)v(\zeta)u(\zeta)\,|d\zeta|, \end{aligned} \tag{4.1}$$

and conversely the last integral always represents a $\Gamma$-invariant bounded harmonic function. It follows [2, p. 138] that almost all points of the Martin boundary of $\boldsymbol{D}/\Gamma$ can be represented by points of $B$; the canonical measure becomes $u(\zeta)|d\zeta|$. In particular, if $A$ is a $\Gamma$-invariant measurable set on $\partial\boldsymbol{D}$ then, by (1.4),

$$\omega(z,A,\boldsymbol{D})=\frac{1}{2\pi}\int_{B\cap A}p(z,\zeta)\,|u(\zeta)|\,|d\zeta|\quad(z\in\boldsymbol{D}) \tag{1.5}$$

is the harmonic measure of $A$ with respect to $\boldsymbol{D}$.

I want to thank the referee for his helpful comments. The present form of Theorem 1 and its proof is due to him.

## 2. Characterizations in the disk

**2.1.** Let $\Gamma$ be a Fuchsian group with identity $\iota$ and limit set $L(\Gamma)$. The Ford fundamental domain is

$$F=\{z\in\boldsymbol{D}\colon\ |\gamma'(z)|<1\quad\text{for}\quad\gamma\in\Gamma,\ \gamma\neq\iota\}. \tag{2.1}$$

Let mes denote the Lebesgue measure on $\partial\boldsymbol{D}$.

The group $\Gamma$ is [10] of accessible type if and only if $\operatorname{mes}(\partial F\cap\partial\boldsymbol{D})>0$, and it is of fully accessible type if and only if

$$\partial\boldsymbol{D} \doteq \Gamma(\partial F\cap\partial\boldsymbol{D}) = \bigcup_{\gamma\in\Gamma}\gamma(\partial F\cap\partial\boldsymbol{D}); \tag{2.2}$$

this is a disjoint union except for countably many points. Since always

$$\partial\boldsymbol{D} = L(\Gamma)\cup\Gamma(\partial F\cap\partial\boldsymbol{D}) \tag{2.3}$$

we see that all groups with $\operatorname{mes} L(\Gamma)=0$ are of fully accessible type. This holds in particular for finitely generated groups of the second kind.

Theorem 1. *Let there exist measurable sets* $A_n\subset\partial\boldsymbol{D}$ *with*

$$\operatorname{mes} A_n \to 2\pi \quad \textit{as} \quad n\to\infty, \tag{2.4}$$

$$\gamma(A_n)\cap A_n = \emptyset \quad \textit{for} \quad \gamma\in\Gamma\setminus\Gamma_n \tag{2.5}$$

*for* $n=1,2,\ldots$ *where either*

(a) $\Gamma_n$ *is a finite set, or*
(b) $\Gamma_n$ *is a subgroup of* $\Gamma$ *of fully accessible type, or*
(c) $\Gamma$ *is infinitely generated and* $\Gamma_n$ *is a finitely generated subgroup.*

*Then* $\Gamma$ *is of fully accessible type.*

*Proof.* We prove first that there exist sets $C_n\subset A_n$ $(n=1,2,\ldots)$ such that

$$\Gamma A_n = \Gamma C_n,\quad \gamma(C_n)\cap C_n = \emptyset \quad \text{for} \quad \gamma\in\Gamma\setminus\{\iota\}. \tag{2.6}$$

In case (a) the set $A_n\cap\{\gamma(\zeta)\colon \gamma\in\Gamma\}$ is finite for each $\zeta\in A_n$ and therefore has a point with smallest argument $\in[0,2\pi]$. These points together form a measurable set $C_n$ satisfying (2.6).

Let now (b) hold. Then $\Gamma_n A_n$ satisfies (2.4) and also (2.5) because $\Gamma_n$ is a group. Hence we may assume that $A_n$ is $\Gamma_n$-invariant. Since $\Gamma_n$ is of fully accessible type there exist measurable sets $B_n\subset\partial\boldsymbol{D}$ containing no two $\Gamma_n$-equivalent points such that $\partial\boldsymbol{D}\doteq\Gamma_n B_n$. Hence

$$\Gamma A_n \doteq \Gamma(A_n\cap\Gamma_n B_n) = \Gamma(\Gamma_n A_n\cap\Gamma_n B_n) = \Gamma(A_n\cap B_n),$$

and this proves (2.6) with $C_n=A_n\cap B_n$ because of (2.5).

Let finally (c) be satisfied. Since $\Gamma$ is not finitely generated $F$ has infinite non-euclidean area [3, p. 210]. Hence the Ford fundamental domain of $\Gamma_n$ has also infinite area. Therefore $\Gamma_n$ is of the second kind and thus of fully accessible type so that case (b) applies. This proves (2.6) for all three cases.

We define now

$$E_n = C_n\setminus\bigcup_{k=1}^{n-1}\Gamma C_k$$

and $B=\bigcup_n E_n$. It follows from (2.6) that $\gamma(B)\cap B=\emptyset$ for $\gamma\in\Gamma\setminus\{\iota\}$. Furthermore, by (2.6),

$$\Gamma E_n\supset\Gamma A_n\setminus\bigcup_{k=1}^{n-1}\Gamma A_k,\quad\text{thus}\quad\Gamma B\supset\bigcup_{n=1}^{\infty}\Gamma A_n,$$

and we conclude from (2.4) that $\partial\boldsymbol{D}\doteq\Gamma B$.

We consider the symmetric Stolz angle of opening $2\alpha$ at $\zeta\in\partial\boldsymbol{D}$, namely

$$\Delta_\alpha(\zeta,\delta)=\{z\in\boldsymbol{D}\colon |\arg(1-\bar\zeta z)|<\alpha,\ |z-\zeta|<\delta\}\quad(0<\delta<1). \tag{2.7}$$

Corollary 1. *Let* $G_n\subset\boldsymbol{D}$ *be open sets and let*

$$A_n=\{\zeta\in\partial\boldsymbol{D}\colon \Delta_\alpha(\zeta,\delta)\subset G_n\ \text{ for some }\ \alpha=\alpha_n(\zeta)>0,\ \ \delta=\delta_n(\zeta)>0\}. \tag{2.8}$$

*If* $\operatorname{mes}A_n\to 2\pi$ *as* $n\to\infty$ *and if, for* $n=1,2,\dots,$

$$\gamma(G_n)\cap G_n=\emptyset\quad\textit{for}\quad\gamma\in\Gamma\setminus\Gamma_n \tag{2.9}$$

*where* $\Gamma_n$ *satisfies one of the conditions* (a)—(c) *of Theorem 1. Then* $\Gamma$ *is of fully accessible type.*

*Proof.* If $\gamma(A_n)\cap A_n\neq\emptyset$ it follows from (2.8) by a simple geometric argument that $\gamma(G_n)\cap G_n\neq\emptyset$. Hence (2.9) implies (2.5), and $\Gamma$ is of fully accessible type by Theorem 1.

**2.2.** We give now a characterization in terms of the Ford fundamental domain $F$. Let $l$ denote the length.

Theorem 2. *The group* $\Gamma$ *is of fully accessible type if and only if there are domains* $G_n\subset\boldsymbol{D}$ *bounded by rectifiable Jordan curves such that*

$$G_n\to\boldsymbol{D},\quad l(\boldsymbol{D}\cap\partial G_n)\to 0\quad\textit{as}\quad n\to\infty \tag{2.10}$$

*and, for some finite set* $\Gamma_n\subset\Gamma$,

$$\gamma(\bar F)\cap G_n=\emptyset\quad\textit{for}\quad\gamma\in\Gamma\setminus\Gamma_n. \tag{2.11}$$

We can express this also by saying that, for each $\varepsilon>0$, there exist disks $\boldsymbol{D}_k$ such that

$$\bigcup_{\gamma\in\Gamma\setminus\Gamma_n}\gamma(\bar F)\subset\bigcup_k D_k,\quad\sum_k\operatorname{diam}D_k<\varepsilon; \tag{2.12}$$

note that, for general Fuchsian groups, this relation only holds with $(\operatorname{diam}D_k)^2$ instead of $\operatorname{diam}D_k$.

*Proof.* (a) Let $\Gamma=\{\gamma_\nu\colon \nu=1,2,\dots\}$ be of fully accessible type and let

$$H_n=\bigcup_{\nu=1}^{n}\gamma_\nu(F)\cup(\text{intermediate sides})\quad(n=1,2,\dots). \tag{2.13}$$

Since the curve $\partial F$ is rectifiable it has a tangent almost everywhere. Hence, by (2.2), there are sets $A_n \subset \partial \boldsymbol{D}$ $(n=1, 2, \ldots)$ with

(2.14) $$\operatorname{mes} A_n = \sum_{\nu=1}^{n} \operatorname{mes}(\partial\gamma_\nu(F) \cap \partial \boldsymbol{D}) \to 2\pi \quad (n \to \infty)$$

such that $\Delta(\zeta, \delta_n(\zeta)) \subset H_n$ for every $\zeta \in A_n$ and some $\delta_n(\zeta) > 0$ where $\Delta \equiv \Delta_{\pi/4}$ as in (2.7). Since

$$A_n = \bigcup_{k=1}^{\infty} \{\zeta \in A_n \colon \Delta(\zeta, 1/k) \subset H_n\} \quad (n = 1, 2, \ldots)$$

we can find $k_n$ such that

(2.15) $$B_n \equiv \{\zeta \in A_n \colon \Delta(\zeta, 1/k_n) \subset H_n\}, \operatorname{mes} B_n \to 2\pi \quad (n \to \infty).$$

Let $D_n = \{|z| < 1 - 1/(5k_n)\}$. Then

(2.16) $$G_n = D_n \cup \bigcup_{\zeta \in B_n} \Delta(\zeta, 1/k_n)$$

is a starlike domain with $B_n \subset \partial G_n$ and, under the projection mapping

$$z \in \partial G_n \mapsto \frac{z}{|z|} \in \partial \boldsymbol{D},$$

lengths are decreased at most by a factor $1/\sqrt{2}$. Hence

$$l(\boldsymbol{D} \cap \partial G_n) \leqq \sqrt{2}(2\pi - \operatorname{mes} B_n) \to 0 \quad (n \to \infty)$$

because of (2.15). Since $\Gamma$ is discontinuous there exist $N_n \geqq n$ such that $D_n \subset H_{N_n}$. Then it follows from (2.15) and (2.16) that $G_n \subset H_{N_n}$. Hence we conclude from (2.13) that

$$\gamma_\nu(\bar{F}) \cap G_n \subset \gamma_\nu(\bar{F}) \cap H_{N_n} = \emptyset \quad \text{for} \quad \nu > N_n.$$

Thus (2.9) holds with $\Gamma_n = \{\gamma_\nu \colon \nu = 1, \ldots, N_n\}$.

(b) Conversely let there exist domains $G_n$ with (2.10) and (2.11). Since the rectifiable curve $\partial G_n$ has a tangent almost everywhere and since $l(\partial \boldsymbol{D} \cap \partial G_n) \to 2\pi$ $(n \to \infty)$ by (2.10), the sets $A_n$ defined by (2.8) satisfy $\operatorname{mes} A_n \to 2\pi$ as $n \to \infty$. If $\Gamma_n^* = \{\alpha \circ \beta^{-1} \colon \alpha, \beta \in \Gamma_n\}$ then $\gamma(G_n) \cap G_n = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma_n^*$. Since $\Gamma$ is finite it therefore follows from Corollary 1 that $\Gamma$ is of fully accessible type.

### 3. Characterization as a Riemann surface

Let $\cong$ denote conformal equivalence. The Riemann surface $R \cong \boldsymbol{D}/\Gamma$ is said to be of *finite topological type* if it has finite genus and finite connectivity. This holds if and only if the Fuchsian group $\Gamma$ is finitely generated; see for instance [3, p. 200]. If $R$ is of finite topological type and not a punctured compact surface then $\Gamma$ is also of the second kind and thus of fully accessible type; see Section 2. We will then denote by $\omega(p, E, R)$ the harmonic measure of the boundary set $E$ at the point $p \in R$.

Theorem 3. *Let $R \cong \boldsymbol{D}/\Gamma$ be a Riemann surface of infinite topological type and let $p \in R$. Then $\Gamma$ is of fully accessible type if and only if there exist surfaces $R_n$ of finite topological type with $p \in R_n \subset R$ such that*

$$\omega(p, \partial R \cap \partial R_n, R_n) \to 1 \quad as \quad n \to \infty. \tag{3.1}$$

Since $R_n$ is a compact bordered surface (possibly with finitely many punctures) the set $\partial R \cap \partial R_n$ lies on this border and we do not have to consider ideal boundaries.

*Proof.* (a) Let there first exist subsurfaces $R_n$ of finite topological type such that (3.1) holds. We may assume that $R \setminus R_n$ has no relatively compact components because "filling in the holes" increases the harmonic measure. We may furthermore assume that $R = \boldsymbol{D}/\Gamma$ and $\pi(0) = p$ where $\pi$ denotes the projection map of $\boldsymbol{D}$ onto $\boldsymbol{D}/\Gamma$.

Let $G_n$ be the component of $\pi^{-1}(R_n) \subset \boldsymbol{D}$ that contains 0 and let $\Gamma_n = \{\gamma \in \Gamma: \gamma(G_n) = G_n\}$. Then $G_n \cap \gamma(G_n) = \emptyset$ for $\gamma \in \Gamma \setminus \Gamma_n$. Furthermore $G_n$ is simply connected in $\boldsymbol{D}$ because $R \setminus R_n$ does not possess a relatively compact component. Let $f_n$ map $\boldsymbol{D}$ conformally onto $G_n$ such that $f_n(0) = 0$. Then

$$\Gamma_n^* = \{f_n^{-1} \circ \gamma \circ f_n: \gamma \in \Gamma_n\}$$

is a Fuchsian group in $\boldsymbol{D}$ with $R_n \cong G_n/\Gamma_n \cong \boldsymbol{D}/\Gamma_n^*$. Since $R_n$ is of finite topological type the group $\Gamma_n^*$ is finitely generated hence also $\Gamma_n$. Since $\Gamma$ is infinitely generated $\Gamma_n$ is of the second kind.

The projection map of $\boldsymbol{D}$ onto $R_n \cong \boldsymbol{D}/\Gamma_n^*$ is $\pi \circ f_n$. If we define $C_n = f_n^{-1} \circ \pi^{-1}(\partial R \cap \partial R_n)$ then

$$\omega(\pi \circ f_n(z), \partial R \cap \partial R_n, R_n) = \omega(z, C_n, \boldsymbol{D}) \quad (z \in \boldsymbol{D}).$$

Since $f_n(0) = 0$ and $\pi(0) = p$ it follows from (3.1) that

$$\operatorname{mes} C_n = 2\pi\, \omega(0, C_n, \boldsymbol{D}) \to 2\pi \quad \text{as} \quad n \to \infty.$$

The set $f_n(C_n)$ lies on $\partial \boldsymbol{D}$. Hence we conclude from a generalisation of Löwner's lemma [4, Lemma 1], [6, p. 57] that $\operatorname{mes} f_n(C_n) \geqq \operatorname{mes} C_n \to 2\pi$ as $n \to \infty$. By the McMillan twist point theorem [5], [9, p. 326] the function $f_n$ has a finite angular derivative at almost all points of $C_n$. It follows [9, p. 303] that (2.10) holds for some set $A_n$ with $\operatorname{mes} A_n = \operatorname{mes} f(C_n)$. Hence $\operatorname{mes} A_n \to 2\pi$ as $n \to \infty$, and $\Gamma$ is therefore of fully accessible type by the corollary.

We need the following known lemma to prove the converse.

Lemma. *Let $G_n$ be Jordan domains with $0 \in G_n$ and let $f_n$ map $\boldsymbol{D}$ conformally onto $G_n$ such that $f_n(0) = 0$. If*

$$l(\partial G_n) \to 2\pi, \quad |f_n'(0)| \to 1 \quad as \quad n \to \infty \tag{3.2}$$

*then, for all measurable sets* $C_n \subset \partial \boldsymbol{D}$,

$$l(f(C_n)) - \text{mes } C_n \to 0 \quad as \quad n \to \infty. \tag{3.3}$$

*Proof.* Let $a_n = \sqrt{f_n'(0)}$; note that $|a_n| \to 1$ by (3.2). We write

$$\sqrt{f_n'(z)} = a_n + g_n(z) \ (z \in \boldsymbol{D}),\ g_n(0) = 0. \tag{3.4}$$

Then it follows from Parseval's formula and from (3.2) that

$$\frac{1}{2\pi} \int_{\partial \boldsymbol{D}} |g_n(z)|^2 |dz| = \frac{1}{2\pi} \int_{\partial \boldsymbol{D}} |f_n'(z)|\,|dz| - |a_n|^2 = l(\partial G_n)/2\pi - |a_n|^2 \to 0 \tag{3.5}$$

as $n \to \infty$. Furthermore, by (3.4),

$$l(f(C_n)) - |a_n|^2 \text{ mes } C_n = \int_{C_n} (2 \operatorname{Re}[\bar{a}_n g_n(z)] + |g_n(z)|^2)\,|dz|.$$

Hence (3.3) follows from (3.5) by the Schwarz inequality.

*Proof of Theorem* 3. (b) Let now $\Gamma$ be of fully accessible type. Let $G_n$ be the Jordan domains defined by (2.8) in the proof of Theorem 1. Since $(G_n)$ is an exhaustion of $\boldsymbol{D}$ and since $l(\boldsymbol{D} \cap \partial G_n) \to 0$ as $n \to \infty$, we see that (3.2) holds. We can write $\partial \boldsymbol{D} \cap \partial G_n = f_n(C_n)$ with $C_n \subset \partial \boldsymbol{D}$. Since $l(f_n(C_n)) = l(\partial G_n) - l(\boldsymbol{D} \cap \partial G_n) \to 2\pi$ as $n \to \infty$ by (2.3), it follows from the lemma that

$$\omega(0, \partial \boldsymbol{D} \cap \partial G_n, G_n) = \omega(0, C_n, \boldsymbol{D}) = \frac{\text{mes } C_n}{2\pi} \to 1\,(n \to \infty). \tag{3.6}$$

Let $H_n$ be the open set defined by (2.5). Since $G_n \subset H_n^* \equiv H_{N_n}$ it follows from the principle of domain extension [6, p. 69] and from (3.6) that

$$\omega(0, \partial \boldsymbol{D} \cap \partial H_n^*, H_n^*) \geqq \omega(0, \partial \boldsymbol{D} \cap \partial G_n, H_n^*) \geqq \omega(0, \partial \boldsymbol{D} \cap \partial G_n, G_n) \to 1 \tag{3.7}$$

as $n \to \infty$. We define $R_n = \pi(H_n^*)$, the projection of $H_n^*$ into $R$. Since $\omega(p, \partial R \cap \partial R_n, R_n) = \omega(0, \partial \boldsymbol{D} \cap \partial H_n^*, H_n^*)$ the assertion (3.1) follows from (3.7).

We see from (2.5) that

$$H_n^* = \bigcup_{\nu=1}^{N_n} \gamma_\nu(F) \cup (\text{intermediate sides}).$$

Since $\pi(F)$ is simply connected we conclude that $R_n = \pi(H_n^*)$ is of finite topological type.

### 4. Some examples

In the following examples we assume that $\Gamma$ is a Fuchsian group without elliptic elements and that $\boldsymbol{D}/\Gamma$ is conformally equivalent to a domain $R\subset\hat{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$. Let $f$ denote a universal covering map of $\boldsymbol{D}$ onto $R$.

Example 1 (compare [10]). Let

$$R=\boldsymbol{D}\setminus E, \quad \operatorname{cap} E=0.$$

In his example mentioned before, Patterson [8, p. 289] chooses $E=\{\lambda(0)\colon \lambda\in\Lambda, \lambda\neq\iota\}$ where $\Lambda$ is a finitely generated Fuchsian group of the first kind. Let

$$T=\{e^{i\theta}\colon \Delta_\alpha(e^{i\theta},\delta)\subset R \text{ for some } \alpha>0, \delta>0\}; \tag{4.1}$$

see (2.1). We shall show that

$$\Gamma \text{ of fully accessible type} \Leftrightarrow \operatorname{mes} T=2\pi. \tag{4.2}$$

This motivates the term "fully accessible".

The Green's function of $R$ with respect to 0 is $\log 1/|w|$ because $\operatorname{cap} E=0$. Hence the singular Green's lines [1], [15] are the radial segments $(\varrho e^{i\theta}, e^{i\theta})$ where $\varrho e^{i\theta}\in E$. The starlike domain $R^*$ obtained by deleting these segments from $R$ is the Green's star domain. It is easy to see from (4.1) that

$$T\setminus\{e^{i\theta}\colon \varrho e^{i\theta}\in E\}=\{e^{i\theta}\colon \Delta_\alpha(e^{i\theta},\delta^*)\subset R^* \text{ for some } \alpha>0, \delta^*>0\}.$$

Since $\operatorname{cap} E=0$ it follows that the set $T^*$ of points $e^{i\theta}$ where $R^*$ is tangential to $\partial\boldsymbol{D}$ satisfies $\operatorname{mes} T^*=\operatorname{mes} T$. Hence (4.2) holds by [11, Corollary]; the set denoted by $g(G)$ in [11, p. 163] is our $R^*$.

Example 2. Let $R$ be a domain in $\hat{\boldsymbol{C}}$ with

$$\partial R=E_0\cup\bigcup_{k=1}^{\infty} J_k, \quad \operatorname{cap} E_0=0 \tag{4.3}$$

where $J_k$ are disjoint open Jordan arcs with $J_k\cap\overline{\partial R\setminus J_k}=\emptyset$. Since the function $f$ omits an arc it is of bounded characteristic. Hence the angular limit $f(\zeta)$ exists for almost all $\zeta\in\partial\boldsymbol{D}$. If $f(\zeta)\in J_k$ for some $k$ then $f$ is continuous at $\zeta$, by Carathéodory's theorem on conformal mappings, and it follows that $\zeta$ does not belong to the limit set $L(\Gamma)$. Since $\operatorname{cap} E_0=0$ the Privalov uniqueness theorem [14, p. 210] shows that

$$\operatorname{mes}\{\zeta\in\partial\boldsymbol{D}\colon f(\zeta)\in E_0\}=0.$$

Since $f(\zeta)\in\partial R$ we conclude from (4.3) that $\operatorname{mes} L(\Gamma)=0$. Hence $\Gamma$ is of fully accessible type.

Example 3. Let $E=C\setminus R$ lie on the rectifiable Jordan arc $C$ and let $l$ denote the arc length measure on $C$. We shall show that

$$(4.4) \qquad l(E)>0 \Rightarrow \quad \Gamma \text{ of accessible type.}$$

The curve $C$ has a tangent for almost all $a\in E$. Since $C\setminus C\subset G$ we conclude that there is an open triangle of vertex $a$ that lies in $G$. If $l(E)>0$ it follows [13, Lemma 1] that $f$ has a finite angular derivative on a set on $\partial \boldsymbol{D}$ of positive measure. Hence $\Gamma$ is of accessible type by [12, Remark on p. 293].

Example 4. We prove now the converse of (4.4) for the special case that $E=C\setminus R\subset \boldsymbol{R}$. Let cap $E>0$ so that $\Gamma$ is of convergence type [6, p. 213]. We claim that

$$(4.5) \qquad l(E)>0 \Leftrightarrow \quad \Gamma \text{ of accessible type.}$$

Let $f(0)=\infty$. Since $f(F)$ is a simply connected subdomain of $R$ that is symmetric with respect to $\boldsymbol{R}$ we see that $f(F)=\hat{\boldsymbol{C}}\setminus I$ where $I$ is the smallest closed interval containing $E$.

Let $B=\partial F\cap\partial \boldsymbol{D}$. Then $f(B)\subset E=\partial R$. Since $F$ is bounded by a rectifiable Jordan curve and since $f$ maps $F$ conformally onto $\hat{C}\setminus I$, it follows from $l(E)=0$ by the Riesz theorem [9, p. 320] that mes $B=0$. Hence $\Gamma$ is not of accessible type. This makes the term "accessible" appear somewhat incongruous.

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