QUASICONFORMAL MAPPINGS WITH FREE BOUNDARY COMPONENTS

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1. Introduction

Let \( \Gamma \) denote a closed subset of the boundary of \( D = \{ w \mid |w| < 1 \} \) and \( T \) a compact subset of \( D \), such that \( D - T \) is a domain. We consider a quasiconformal mapping \( F, z = F(w) \), which maps \( D - T \) into \( D \), such that \( \delta D \) is mapped onto itself. The class of all such mappings which agree with \( F \) on \( \Gamma \) is denoted by \( Q_F \). We do not indicate the dependence on \( T \) and \( \Gamma \), since these sets are fixed throughout this paper (except in \( \S 6 \)). If \( F \) has minimal maximal dilatation in the class \( Q_F \), we call \( F \) absolutely extremal. We use the notation “absolutely extremal” to indicate that the image domains of competing mappings \( G \in Q_F \) are not fixed. So \( F \) is absolutely extremal, if

\[
K[F] = K_0 := \inf_{G \in Q_F} K[G],
\]

where \( K[G] \) denotes the maximal dilatation of \( G \).

By normality we conclude that \( Q_F \) contains at least one absolutely extremal mapping. If there is only one such mapping, it is called uniquely absolutely extremal.

To derive necessary and sufficient conditions for a mapping to be absolutely extremal, we use the method of E. Reich [5] in connection with a similar problem. We consider the inverse mapping \( f = F^{-1}, w = f(z) \), which is defined in \( F(D - T) \), and its complex dilatation

\[
\kappa(z) = \frac{f_z(z)}{f_z(z)} = \frac{F_w(w)}{F_w(w)}.
\]

The following Banach space of holomorphic functions plays a basic role in this problem too:

\[
\mathcal{B} = \mathcal{B}_{F, \Gamma} := \{ \varphi \mid \varphi \text{ holomorphic in } D, \varphi dz^2 \text{ real on } \delta D - F(\Gamma),
\]

\[
\| \varphi \| = \int_D \int |\varphi(z)| dx dy < \infty.
\]

First we derive a general necessary condition which leads to the possibility of the existence of a so-called “substantial” boundary point [4]. After this we derive a second necessary condition in the case when there is no such boundary point: Then
there is a quadratic differential \( \varphi \in \mathcal{B} \) such that \( f \) is a Teichmüller mapping with complex dilatation \( \kappa = k_0 \sqrt{\varphi}/|\varphi| (k_0 = (K_0-1)/(K_0+1)) \) and \( F(D-T) \) is a domain which is the unit disk slit along some subarcs of vertical trajectories and connected subsets of the vertical critical graph of \( \varphi \) (this is defined to be the union of all vertical critical trajectories and zeroes of \( \varphi \)). These subarcs then correspond to the set \( T \). We restrict ourself to the case where at most denumerably many components of \( D-F(D-T) \) are points. With this slight restriction the necessary conditions turn out to be sufficient for \( F \) to be absolutely extremal. Moreover \( F \) then is uniquely absolutely extremal.

2. The general necessary condition for absolute extremality

If \( F \) is as above, then the complex dilatation \( \kappa \) of \( f=F^{-1} \) is a measurable function in \( F(D-T) \). We extend \( \kappa \) by setting

\[
\kappa(z) = 0 \quad \text{for} \quad z \in D - F(D-T)
\]

to get a measurable function in \( D \). We prove the

**Theorem 1.** If \( F \) is absolutely extremal in \( Q_F \), then

\[
(1) \quad \sup_{\varphi \in \mathcal{B}, \|\varphi\| = 1} \left| \int \int_D \kappa \varphi \, dx \, dy \right| = \|\kappa\|_\infty.
\]

**Proof.** We apply a technique employed by Krushkal [3] and elaborated by Reich [5]:

Let \( k_0 = \|\kappa\|_\infty \). If \( k_0 = 0 \) nothing has to be shown. We assume \( k_0 > 0 \). If \( (1) \) does not hold, then

\[
\sup_{\varphi \in \mathcal{B}, \|\varphi\| = 1} \left| \int \int_D \kappa \varphi \, dx \, dy \right| = a < k_0.
\]

By the Hahn—Banach and Riesz representation theorems there exist a complex valued measurable function \( \alpha(z) \) with

\[
\int_D \kappa \varphi \, dx \, dy = \int_D \alpha \varphi \, dx \, dy, \quad \text{for every} \quad \varphi \in \mathcal{B},
\]

and \( \|\alpha\|_\infty = a \).

We form \( v(z) = \alpha(z) - \kappa(z), \ z \in D \). For \( 0 < t < 1/\|v\|_\infty \), let \( g \) denote the quasi-conformal selfmapping of \( D \) with complex dilatation \( tv \) and with \( g(1) = 1, \ g(i) = i, \ g(-1) = -1 \). Then we put \( h = f \circ g^{-1} \) and have

\[
(2) \quad \frac{h_\zeta}{h_\zeta} = \mu_h(\zeta) = \frac{\kappa(z) - tv(z)}{1 - tv(z) \kappa(z)} \frac{g_z(z)}{g_z(z)}, \quad \zeta = g(z).
\]

By the Fundamental variational lemma ([5], p. 107) there exists a \( (1+c)/(1-c) \) quasiconformal mapping \( g^* \) of \( D \) onto itself whose boundary values agree with those
of \( g \) at \( F(\Gamma) \) such that

\[
c \leq \frac{t^2\|v\|_\infty^2}{1-t\|v\|_\infty+t^2\|v\|_\infty^2}.
\]

We put \( \tilde{f} = h \circ g^* = f \circ g^{-1} \circ g^* \). Then \( \tilde{f} \) is defined in \( g^{*-1} \circ F(D-T) \) and maps this domain onto \( D-T \) with boundary values of \( f \) on \( F(\Gamma) \). Therefore \( \tilde{f}^{-1} \) belongs to \( Q_F \). We want to show that

\[
K[\tilde{f}] < K[F] \quad \text{for} \quad t > 0, \quad \text{sufficiently small.}
\]

This would contradict the absolute extremality of \( F \).

Let

\[
V_1 = \left\{ z \in F(D-T) \mid |\kappa(z)| \leq \frac{k_0+a}{2} \right\},
\]

\[
V_2 = \left\{ z \in F(D-T) \mid \frac{k_0+a}{2} < |\kappa(z)| \leq k_0 \right\}.
\]

Since \( k_0 > 0 \), it is immediately clear from (2) that there exist \( \delta_1 > 0, t_1 > 0 \), such that

\[
|\mu_h(\zeta)| \leq k_0 - \delta_1 t, \quad \text{if} \quad 0 \leq t \leq t_1, \quad \zeta \in g(V_1).
\]

Expanding (2) we obtain for \( \zeta \in g(V_2) \) and \( \zeta = g(z) \),

\[
|\mu_h(\zeta)| = |\kappa(z)| - t \frac{1-|\kappa(z)|^2}{|\kappa(z)|} \Re (v(z)\bar{\kappa}(z)) + O(t^2).
\]

Here \( O(t^2) \) is uniform with respect to \( z \) in \( V_2 \). We have \( v\bar{\kappa} = |\kappa|^2 - z \bar{\kappa} \) and therefore

\[
\Re v\bar{\kappa} \equiv |\kappa|^2 - |\kappa| |\kappa| \equiv |\kappa| (|\kappa| - a) \equiv |\kappa| (k_0 - a)/2, \quad \text{hence}
\]

\[
\frac{1-|\kappa|^2}{|\kappa|} \Re v\bar{\kappa} \equiv (1-|\kappa|^2) \frac{k_0-a}{2} \equiv (1-k_0^2) \frac{k_0-a}{2} > 0.
\]

Therefore there exist \( \delta_2 > 0, t_2 > 0 \) such that

\[
|\mu_h(\zeta)| \leq k_0 - \delta_2 t, \quad 0 \leq t \leq t_2, \quad \zeta \in g(V_2).
\]

We consider the effect of \( g^* \), using \( c = O(t^2) \), and obtain

\[
K[\tilde{f}] < K[f] = K_0
\]

for \( t > 0, \quad \text{sufficiently small.} \)
3. The existence of a substantial boundary point

Let $\zeta \in \Gamma$. Then the local dilatation $H^I_{\zeta}$ of $F|_{\partial D}$ in $\zeta$ with respect to $I$ is defined by

$$H^I_{\zeta} = \inf \{ K[G]|G: U(\zeta) \xrightarrow{\varepsilon} U(F(\zeta)), \quad G|_{\partial D} = F|_{\partial D} \},$$

where the inf is taken over all such mappings $G$ and all open neighbourhoods $U(\zeta)$ of $\zeta$ and $U(F(\zeta))$ of $F(\zeta)$ in $\overline{D}$. For this definition and notation we refer to [4, 8, 1]. If $K_0 = H^I_{\zeta}$, then $\zeta$ is called a substantial boundary point. Since the function $\zeta \mapsto H^I_{\zeta}$ is upper semicontinuous, there always exists a point $\zeta_0 \in \Gamma$ with $H^I_{\zeta_0} = \max_{\zeta \in \Gamma} H^I_{\zeta}$.

We assume now that $F$ fulfills the general necessary condition (1). It is possible that there is a sequence $\varphi_n \in \mathcal{B}$ with $\|\varphi_n\| = 1$ such that $\varphi_n$ tends to zero locally uniformly in $D$ and

$$\int_D \int_D \chi \varphi_n \, dx \, dy \to \|\chi\|_\infty, \quad n \to \infty.$$ 

We recall that $\chi = f_z|f_z$ in $F(D - T)$ and $\chi = 0$ in $D - F(D - T)$. Let $f^*$ be the quasiconformal selfmapping of $D$ with complex dilatation $\chi$ in $D$ and $f^*(1) = 1$, $f^*(i) = i$, $f^*(-1) = -1$. By [1] we conclude that $f^*$ has a substantial boundary point on $F(\Gamma)$ and is hence extremal for its boundary values on $F(\Gamma)$. Since $f$ and $f^*$ have the same complex dilatation in $F(D - T)$, there is a conformal mapping $h$ in $D - T$ such that $h \circ f = f^*$ in $F(D - T)$. But because $K[f] = K[f^*]$ and because local dilatations are conformally invariant, this point is a substantial boundary point for $f^*$ too, and we conclude that $F$ has a substantial boundary point on $\Gamma$, i.e. $K[F] = \max_{\zeta \in \Gamma} H^I_{\zeta}$. We remark, that this forces $F$ to be absolutely extremal, since $K_0 = \max_{\zeta \in \Gamma} H^I_{\zeta}$ clearly holds, hence $K_0 = K[F]$. Therefore the general necessary condition (1) together with the existence of a degenerating sequence $\varphi_n$ as above is a sufficient condition for $F$ to be absolutely extremal.

4. A second necessary condition in the case without substantial boundary point

We consider the case where $K_0 > \max_{\zeta \in \Gamma} H^I_{\zeta}$ and $F$ is absolutely extremal, hence the condition (1) is fulfilled. Since there is no substantial boundary point, every sequence $\varphi_n \in \mathcal{B}$, $\|\varphi_n\| = 1$ with

$$\int_D \int_D \chi \varphi_n \, dx \, dy \to \|\chi\|_\infty = k_0, \quad n \to \infty,$$

contains a subsequence which tends to a function $\varphi \in \mathcal{B}$, $0 < \|\varphi\| \leq 1$. It is known that then

$$\int_D \int_D \chi \varphi \, dx \, dy = k_0 \|\varphi\|.$$
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Therefore we conclude that \( z = k_0 \bar{\varphi}/|\varphi| \) a.e. in \( D \). Since \( k_0 > 0 \) we conclude that the measure of \( D - F(D - T) \) is zero (because \( z = 0 \) there) and that \( F \) is a Teichmüller mapping.

But as can be seen by an example these conditions are not sufficient. We derive a second necessary condition: We claim that every component \( A \) of \( D \setminus F(D - T) \) is a subarc of a vertical trajectory of \( \varphi \) or a connected subset of the vertical critical graph of \( \varphi \). To prove this we may assume that \( A \) contains at least two points. We consider again the quasiconformal mapping \( f^*: D \to D \) with complex dilatation \( \kappa^* = k_0 \bar{\varphi}/|\varphi| \) in \( D \) and \( f^*(1) = 1, f^*(i) = i, f^*(-1) = -1 \). The mapping \( f^* \circ F \) is conformal in \( D - T \). Let us consider the class of mappings from \( D - A \) onto \( D - f^*(A) \) with the boundary values of \( f^* \) on \( F(\Gamma) \). The mapping \( f^* \) must be extremal in this class, otherwise we could replace \( F^*: F^* \) by a quasiconformal mapping \( G^*: D - f^*(A) \to D - A \) with \( K[G^*] < K_0 \) and boundary values as \( f^*-1 \). The mapping \( G^* \circ f^* \circ F \) would contradict the absolute extremality of \( F \).

Therefore \( f^* \) is extremal in this class of mappings between these two ring domains. We have already seen that the lack of a substantial boundary point on \( \Gamma \) for \( F \) implies a lack of a substantial boundary point on \( F(\Gamma) \) for \( f^* \). We conclude by [1, 2] that \( \varphi \) is real along \( A \), i.e. by transformation of the ring domains \( D - A \) and \( D - f^*(A) \) onto annuli, the induced quadratic differential must be real along the interior boundary component. But \( \Phi = \int V \bar{\varphi} \, dz \) is conformal in neighbourhoods of points \( z_0 \in A \) where \( \varphi(z_0) \neq 0 \), therefore \( A \) must consist of horizontal and vertical arcs of \( \varphi \) including zeroes. We show that horizontal arcs do not occur by using the following lemma.

**Lemma.** Let \( R \) be the square \( \{ x + iy | -1/2 < x < 1/2, 0 < y < 1 \} \), \( s \) its horizontal side \( \{ y = 0 \} \) and \( A_K \) denote the affine stretch \( A_K(x + iy) = Kx + iy, K > 1 \). Then there is a quasiconformal mapping \( F_0 \) defined in \( A_K(R) \), such that \( F_0(A_K(R)) \subset R \), \( F_0 \) agrees with \( A_K^{-1} \) on \( \partial A_K(R) - A_K(s) \), and the maximal dilatation of \( F_0 \) is less than \( K \).

**Proof.** We consider the right half \( R^+ = R \cap \{ x + iy | x > 0 \} \) of the square \( R \). We choose \( \Gamma = \partial A_K(R^+) \cap \{ x + iy | x = K/2 \text{ or } y = 1 \} \). Let \( F_0 \) be an extremal mapping from \( A_K(R^+) \) onto \( R^+ \) which agrees with \( A_K^{-1} \) on \( \Gamma \). If \( F_0 \) has a substantial boundary point with respect to \( \Gamma \), then its maximal dilatation is less than \( K \). Otherwise, if there is no substantial boundary point, then \( F_0 \) is uniquely determined and it is a Teichmüller mapping with associated quadratic differential of finite norm, which is real along \( \partial A_K(R^+) - \Gamma \). Therefore \( A_K^{-1} \neq F_0 \) since the quadratic differential \(-1\) of \( A_K^{-1} \) has a pole of first order at the corner \((0,0)\). We conclude that the maximal dilatation of \( F_0 \) is less than \( K \).

Next we consider \( F_0(0) \). This point must be on the interior of the vertical side \( \{ iy | 0 < y < 1 \} \) of \( R^+ \), since otherwise the ratio of the moduli of the rectangles \( A_K(R^+)(0, K/2, K/2 + i, i) \) and \( R^+(F_0(0), 1/2, 1/2 + i, i) \) would be larger or equal to \( K \). By reflection we extend \( F_0 \) to a mapping from \( A_K(R) \) onto the slit rectangle \( R - \{ iy | 0 < y < 1 \} \) and this lemma is proved.
We assume now that \( \Lambda \) contains a horizontal arc of \( \varphi \). Since \( f^* \) is locally equal to a conformal mapping \( \Phi \) followed by an affine stretch \( A_{K_0} \) and again by a conformal mapping \( \Psi^{-1} \), we can choose a \( \varphi \)-rectangle \( R^* \), which is mapped by \( \Phi = \int \sqrt{\varphi} \, dz \) onto a square \( R = \{ x + iy \mid -1/2 < x < 1/2, 0 < y < 1 \} \) such that a horizontal arc of \( \varphi \) on \( \Lambda \) is mapped onto \( -1/2 < x < 1/2 \). We apply the Lemma on \( R \) and \( A_{K_0} \). Therefore we can replace \( F^* = f^*-1 \) in \( f^*(R^*) \) by \( \Phi^{-1} \circ F_0 \circ \Psi \), i.e. we define

\[
\overline{F^*} = \begin{cases} 
F^* \text{ in } f^*(D - \Lambda - R^*), \\
\Phi^{-1} \circ F_0 \circ \Psi \text{ in } f^*(R^*). 
\end{cases}
\]

Because \( F^* \) and \( \Phi^{-1} \circ F_0 \circ \Psi \) agree on those three sides of \( f^*(R^*) \) which are contained in \( f^*(D - \Lambda) \), \( \overline{F^*} \) is well-defined in \( f^*(D - \Lambda) \), \( K_0 \)-quasiconformal and not a Teichmüller mapping since its dilatation is not constant. In the class \( Q_F \) the mapping \( \overline{F^*} \circ f^* \circ F \) is absolutely extremal but not a Teichmüller mapping. This contradicts the first conclusion of §4, that an absolutely extremal mapping without substantial boundary point must be a Teichmüller mapping. We have proved the

**Theorem 2.** If \( K_0 \geq \max_{\xi \in F} H_\xi \), then an absolutely extremal mapping \( F \) is a Teichmüller mapping and the holomorphic quadratic differential \( \varphi \) of the inverse mapping \( f = F^{-1} \) is in \( B = B_{F(I)} \). The set \( D - F(D - T) \) has area-measure zero and each component of it is a subarc of a vertical trajectory of \( \varphi \) or a connected subset of the vertical critical graph of \( \varphi \).

### 5. Sufficient conditions

We have already seen that condition (1) together with the existence of a degenerating sequence \( \varphi_n \) as described in §3 is a sufficient condition for \( F \) to be absolutely extremal. Now we want to show that with a slight restriction the necessary conditions of Theorem 2 are sufficient for absolute extremality.

**Theorem 3.** With \( F \) given as above, let \( f = F^{-1} \) be a Teichmüller mapping with associated quadratic differential \( \varphi \in B = B_{F(I)} \), and let the following conditions be fulfilled:

a) \( D - F(D - T) \) has area-measure zero,

b) the components of \( D - F(D - T) \) are subarcs of vertical trajectories of \( \varphi \) or connected subsets of the vertical critical graph of \( \varphi \),

c) at most denumerably many components of \( D - F(D - T) \) are points. Then \( F \) is uniquely absolutely extremal.

---

1) In the set of positive measure which is mapped by \( F \) onto \( F(D - T) \cap R^* \), the maximal dilatation is less than \( K_0 \).
This theorem is a consequence of the Main Inequality of Reich and Strebel [7], stated in the following form:

Main Inequality. Let $\varphi \in \mathcal{R}$ and $L$ be a compact set in $D$, such that $D - L$ is a domain and each component $\Lambda$ of $L$ is a subarc of a vertical trajectory of $\varphi$ or a connected subset of the vertical critical graph of $\varphi$. Furthermore, suppose that the set of all vertical trajectories of $\varphi$ in $D - L$ which meet $L$ has area-measure zero. Then a quasiconformal mapping $g$ with complex dilatation $\kappa$ which maps $D - L$ into $D$, $\delta D$ onto itself and keeps the points of $\Gamma$ pointwise fixed fulfills the inequality

$$\|\varphi\| \equiv \int_D \int |\varphi| \left| \frac{1 - \kappa \varphi}{\varphi} \right|^2 \frac{1}{1 - |\kappa|^2} dxdy.$$  

Proof. We consider non-critical vertical trajectories $\beta$ of $\varphi$. If $\beta$ is contained in $D - L$, we have the length-inequality (see [9])

$$\int_\beta |\varphi(z)|^{1/2} dz \equiv \int_{g(\beta)} |\varphi(z)|^{1/2} dz.$$  

As in the proof of the Main Inequality in [7], we consider each vertical strip $S$ of $\varphi$ in $D$. By our assumption all vertical trajectories $\beta$ of $\varphi$ up to a set of area-measure zero fulfill the length-inequality (4). Therefore the length-area method applied to each strip and then summed up yields

$$\left( \int_D \int |\varphi(z)| dxdy \right)^2 \equiv \int_D \int |\varphi(z)| dxdy \cdot \int_D \int |\varphi(z)| \left| \frac{1 - \kappa(z) \varphi(z)}{\varphi(z)} \right|^2 \frac{1}{1 - |\kappa(z)|^2} dxdy.$$  

Using $g(D - L) \subset D$ and the fact that $L$ necessarily has area-measure zero gives (3). (The intersection of $L$ with each strip must have $\varphi$-area zero!)

Proof of Theorem 3

Let $G$ be a mapping in $QF$. We apply inequality (3) for $g = G \circ F^{-1} = G \circ f$, where $\Gamma$ is replaced by $F(\Gamma)$. The mapping $G \circ f$ is defined in $F(D - T)$ and keeps $F(\Gamma)$ pointwise fixed, and $L = D - F(D - T)$. By assumption c), there are at most denumerably many vertical trajectories $\beta$ which meet components of $L$ which are points, so these trajectories cover only a set of area-measure zero. $L$ has area-measure zero by assumption a), so the vertical trajectories $\beta$ which meet components of $L$ which are vertical subarcs of positive length can only cover a horizontal length-measure zero in each strip. Therefore these trajectories cover only a set of area-meas-
ure zero too, and we may apply the Main Inequality:

$$
\| \varphi \| \leq \int_D \int |\varphi| \left| \frac{1 - x}{x} \varphi \right|^2 E[\varphi, F, G] dxdy.
$$

Here

$$
E[\varphi, F, G](z) = \frac{\left| 1 + x(z) \frac{z_1(w)}{z(w)} \varphi(z) \right|^2}{1 - \left| z_1(w) \right|^2},
$$

and \( w = f(z), x_1 = G_w / G_w, \hat{x} = F_w / F_w, x = f / f_z \). Then we use \( x = \frac{\varphi}{|\varphi|} \) and \( E = K[G] \) and get

$$
K[F] \subseteq K[G],
$$

i.e., \( F \) is absolutely extremal.

By the procedure of [6] one concludes from (5) that if \( K[G] = K[F] \), then

$$
x_1 = \hat{x} \quad \text{a.e. in } D - T,
$$

and therefore \( G \circ f \) is conformal.

Because \( G \circ f \) keeps \( F(D) \) pointwise fixed, we conclude at once: If \( \Gamma \) does not only consist of single points, then \( G \circ f \) is the identity. So then \( F \) is uniquely absolutely extremal. We can see this also in the general case where \( \Gamma \) contains at least three points.\(^2\) We assume \( F \) and \( G \) to be absolutely extremal mappings and necessarily \( f = F^{-1} \) and \( g = G^{-1} \) to be Teichmüller mappings with quadratic differentials \( \varphi \) and \( \psi \) in \( \mathcal{B} \). So the conformal mapping \( G \circ f \) consists of two Teichmüller mappings, and there is a quadratic differential \( \varphi_0 \) of finite norm in \( D - T \), such that

$$
x_1 = \hat{x} = k_0 \frac{\varphi_0}{|\varphi_0|}.
$$

We consider a component \( \Lambda \) of \( D - F(D - T) \), which necessarily is a subarc of a vertical trajectory or a connected subset of the vertical graph of \( \varphi \). We map \( D \setminus f(\Lambda) \) conformally onto an annulus (without loss of generality a punctured disk can be excluded), and in the conformal image of \( D - T \) we get an induced quadratic differential \( \hat{\varphi}_0 \). Along one boundary circle of this annulus \( \hat{\varphi}_0 \) is real and the zeroes of \( \varphi \) and the \( \varphi \)-length of the subarcs of \( \Lambda \) determine the zeroes of \( \hat{\varphi}_0 \) on this circle. But the same can be done with the corresponding component of \( D - G(D - T) \) and the quadratic differentials \( \psi \) and \( \varphi_0 \). Therefore, corresponding slits of \( D \setminus F(D - T) \) and \( D - G(D - T) \) have the same length in the metric of the quadratic differentials and the same configuration. Hence the conformal mapping \( G \circ f \) can be extended homeomorphically in all of \( D \). Therefore it must be the identity.

\(^2\) We exclude the conformal case where non-uniqueness may occur.
6. An application: $H = \text{Max } H_\zeta$

We consider the special case, where $\Gamma = \delta D$, $T = \{w \mid |w| \leq r\}$ for some $r, 0 \leq r < 1$, and the boundary homeomorphism of \(\delta D\) onto itself is called $h$. The local dilatations of $h$ are denoted by $H_\zeta$. If $\text{Max}_{\zeta \in \partial D} H_\zeta$ is finite, it is known that $h$ is quasisymmetric, i.e. quasiconformally extendable in $D$ [1]. Then the dilatation $H$ of $h$ is defined in [8] by

$$H = \inf \{K[G] \mid G: U(\delta D) \to U'(\delta D), G|_{\partial D} = h\}$$

where the inf is taken over all such mappings $G$ and all open neighborhoods $U(\delta D)$ and $U'(\delta D)$ of $\delta D$ in $\overline{D}$. The class of all extensions of $h$ in $D - T$ which map $D - T$ into $D$ is denoted by $Q_r$, in view of the dependence on $r$. For the same reason we call the absolutely extremal mapping in this class $F_r$ and its maximal dilatation $K_r$. Evidently we then have $H = \lim_{r \to 1} K_r$.

The function $r \to K_r$ is strictly monotonic decreasing as long as $K_r = \text{Max}_{\zeta \in \partial D} H_\zeta$. This can be seen by the preceding result: For each number $r, 0 < r < 1$, where $K_r = \text{Max } H_\zeta$, $F_r$ is a uniquely determined Teichmüller mapping, and the associated quadratic differential $\varphi_r$ of its inverse mapping $f_r = F_r^{-1}$ is defined in all of $D$. Moreover, $D \setminus F_r(D_r)$ ($D_r = \{w \mid |w| < 1\}$) consists of a subarc of a vertical trajectory of $\varphi_r$ or of a connected subset of the vertical critical graph of $\varphi_r$.

We remark that the Main Inequality (5) holds for $F = F_r$, $\varphi = \varphi_r$ and $G$, if $G$ is an extension of $h$ and if it is defined in a domain which contains $D_r$. We prove the

**Theorem 4.** The dilatation $H$ of a quasisymmetric mapping $h: \delta D \to \delta D$ is equal to $\text{Max}_{\zeta \in \partial D} H_\zeta$.

**Proof.** We may assume $K_r = \text{Max}_{\zeta \in \partial D} H_\zeta$, $0 < r < 1$. Let $\varphi_r$ and $\psi_r$ denote the quadratic differentials associated with the Teichmüller mapping $f_r: F_r(D_r) \to D_r$, which are normalised by

$$\|\varphi_r\| = \iint_D |\varphi_r(z)| \, dx \, dy = 1, \quad \|\psi_r\|_{D_r} = \iint_{D_r} |\psi_r(w)| \, du \, dv = K_r \quad (w = u + iv).$$

Then we have locally

$$f_r = \Psi_r^{-1} \circ A_{K_r} \circ \Phi_r,$$

where $\Phi_r(z) = \int \varphi_r(z) \, dz$, $\Psi_r(w) = \int \psi_r(w) \, dw$ and $A_{K_r}(z + i\eta) = K_r \cdot z + i\eta$.

For every measurable subset $E \subset F_r(D_r)$ we have

$$K_r \iint_{f_r(E)} |\varphi_r(z)| \, dx \, dy = \iint_{f_r(E)} |\psi_r(w)| \, du \, dv. \quad (6)$$

We extend the functions $\psi_r$ in $T$ by putting

$$\psi_r(w) = 0 \quad w \in D - D_r.$$

Then $\psi_r$ are measurable functions in $D$ with finite $L_1$-norm $\|\psi_r\| = \int_D |\psi_r(w)| \, du \, dv$, i.e. $\psi_r \in L_1(D)$. If $r \to 1$, then $\psi_r$ tend to zero locally uniformly in $D$ and their
$L_1$-norms $\|\psi_r\|=K_r$ are bounded ($K_r \to H!$), hence the assumptions of Lemma 4.1 of [1] are fulfilled.

Let $I$ denote an open interval on $\delta D$ with endpoints $w_1, w_2$ and $\arg w_1 < \arg w < \arg w_2$ for $w \in I$. We then write $I = w_1w_2$, and define 
$$S_I := \{w \in D | \arg w_1 < \arg w < \arg w_2\}.$$

For a given sequence $r_n \to 1$ we put 
$$\theta(I) := \lim_{n \to \infty} \int_{S_I} |\psi_{r_n}(w)| \, dudv$$
and finally 
$$\theta(\zeta) := \inf \{\theta(I) | \zeta \in I, \ I \text{ an open interval on } \partial D\}$$
for every $\zeta \in \partial D$.

By Lemma 4.1 of [1] we can choose for given numbers $e > 0$, $l > 0$ a subdivision 
$$\{w_1, \ldots, w_N\}$$
of $\partial D$ and a sequence $r_n \to 1$ such that 
$$\sum_{i=1}^N \theta(w_i) < e \quad \text{and} \quad |w_{i-1}w_i| < l.$$

(Here $|w_{i-1}w_i|$ denotes the arc length of $w_{i-1}w_i$.)

We apply the method for the proof of Theorem 4.1 in [1]. Let $H' = \max_{\zeta \in \partial D} H_\zeta$. There is an $l > 0$ such that the restriction of $h$ to an arbitrary interval on $\partial D$ with length less than $l$ can be extended $H'$-quasiconformally in a neighbourhood in $\overline{D}$ of this interval. We apply Lemma 4.1 of [1] on $e$ and $l$ and Corollary 4.1 of [1] on $e$, $l$ and the sequence $\psi_{r_n}$. Therefore we can cut off some neighborhoods $G_i(i \equiv N)$ of subintervals in $w_{i-1}w_i$ by Jordan arcs $\gamma_i$, such that $h$ can be extended $H'$-quasiconformally in the $G_i$, and for $D_\varepsilon = D - \bigcup_{i=1}^N G_i$ we have

\begin{equation}
\lim_{n \to \infty} \int_{D_\varepsilon} |\psi_{r_n}(w)| \, dudv \equiv \varepsilon.
\end{equation}

The construction in [1] yields a quasiconformal extension $h_\varepsilon$ of $h$ in $\{w | w \in 1\}$ which is $H'$-quasiconformal in $\bigcup_i G_i$ and $\tilde{H}$-quasiconformal in a neighborhood $U(\partial D)$ of $\delta D$, where $\tilde{H}$ does not depend on $\varepsilon$ (only on $\max_{\zeta \in \partial D} H_\zeta$).

We apply the Main Inequality (5) for $F = F_r$, $\phi = \phi_r$ and $G = h_\varepsilon$. Then 
$$\frac{1 - \varepsilon}{1 - |\varphi_r|^2} = \frac{1}{K_r},$$
and $E[\phi_r, F_r, h_\varepsilon](z) \equiv D_h (f_r(z))$, where $D_h$ denotes the dilatation of $h_\varepsilon$. If $r$ is close to one, the image of $f_r$ is contained in $U(\partial D)$. Hence $D_h$ is bounded by $\tilde{H}$, and in $G_i' := f_r^{-1}(G_i)$ we have $D_h \equiv H'$. Therefore, (5) yields 
$$K_r \equiv H' \int \int_{G_i'} |\phi_r(z)| \, dxdy + \tilde{H} \int \int_{D_\varepsilon - \bigcup_{i=1}^N G_i'} |\phi_r(z)| \, dxdy.$$
Because \( f_r(F_r(D_r) - \bigcup_{i=1}^{N} G_i) = D_r - \bigcup_{i=1}^{N} G_i \), we have by (6) and the fact that \( D - F_r(D_r) \) has measure zero,

\[
K_r \iint_{D - \bigcup_{i=1}^{N} G_i} |\varphi_r(z)| \, dx \, dy = \iint_{D - \bigcup_{i=1}^{N} G_i} |\psi_r(w)| \, du \, dv.
\]

Since \( D_r - \bigcup_{i=1}^{N} G_i \subseteq D_\varepsilon \) and \( \iint_{\bigcup_{i=1}^{N} G_i} |\varphi_r(z)| \, dx \, dy \equiv \|\varphi_r\| = 1 \), it follows that

\[
K_r \equiv H' + \frac{\hat{H}}{K_r} \iint_{D_\varepsilon} |\psi_r(w)| \, du \, dv.
\]

Putting \( r = r_n \) and letting \( r_n \) tend to one we get because of (7)

\[
H \equiv H' + \frac{\hat{H}}{H} \varepsilon.
\]

Since \( \hat{H} \) does not depend on \( \varepsilon \), we conclude that \( H \equiv H' \). \( H' \) was arbitrarily close to \( Max_{\zeta \in \partial D} H_{\zeta} \), and so

\[
H \equiv Max_{\zeta \in \partial D} H_{\zeta},
\]

which finishes the proof.

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References


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