TOPOLOGICALLY, QUASICONFORMALLY OR LIPSCHITZ LOCALLY FLAT EMBEDDINGS IN CODIMENSION ONE

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1. Introduction

We consider locally C-flat embeddings in codimension one in three categories C: TOP (continuous maps), LQC (locally quasiconformal maps), and LIP (locally Lipschitz maps). The latter two categories are our main subjects; the first one is in fact only auxiliary. In $[TV_1]$, $[V_3]$, $[LT]$ the category LQC is extended to a category LQS of locally quasisymmetric maps: LQS manifolds and their LQS homeomorphisms are the same thing as LQC manifolds and their LQC homeomorphisms, but LQS embeddings are defined in every codimension. We consider LQS or LIP embeddings in codimension one (which need not be locally TOP flat), too. We refer to Section 2 for terminology. The prefix TOP will usually be omitted.

Sullivan [S] proved recently for $n \neq 4$ the existence and Hauptvermutung of C-structures, $C \in \{LQC, LIP\}$, on every n-manifold without boundary. Tukia and Väisälä $[TV_3]$ then extended for $n \neq 4, 5$ these results to all n-manifolds which may have a boundary. In the LQC case they made use of the extension of a quasi-conformal selfhomeomorphism of R^n to one of R^{n+1}_+, which they proved in $[TV_2]$ by the aid of [S].

In Section 3 we apply the basic result of $[TV_3]$ on approximation of homeo-morphisms by C-homeomorphisms, $C \in \{LQC, LIP\}$, to generalize it so as to give it the following form: The given homeomorphism f is supposed to carry, either homeomorphically or C-homeomorphically, a given closed locally C-flat submanifold M of codimension one onto a submanifold of the same kind, and the approximating C-homeomorphism is claimed either to carry M onto fM or to coincide with f on M, respectively.

In Section 4 we prove, in a relative form, for all $C \in \{TOP, LQC, LIP\}$ and $n \neq 4$ that if two locally C-flat embeddings of an $(n-1)$-dimensional C-manifold into an n-dimensional C-manifold without boundary are sufficiently close to a given closed embedding, then one of them is carried onto the other by a small ambient isotopy consisting of C-homeomorphisms. For $C = TOP$ this absolute form is due to Craggs [Cr] if $n = 3$ and to Price and Seebeck [PS] if $n \geq 5$. We prove the

relative form by applying [Cr] and [PS]. For $C \neq \text{TOP}$ we use the TOP result and Theorem 3.1.

Section 5 contains our main theorem: If $C \in \{\text{TOP, LQC, LIP}\}$ and $n \neq 4$, every embedding of an $(n-1)$-dimensional C-manifold into an n-dimensional C-manifold without boundary can be relatively approximated by locally C-flat embeddings. The absolute case for $C = \text{TOP}$, $n \geq 5$ is a recent deep result of Ancel and Cannon [AC]. The relative case follows from this result and Theorem 4.1. For $C = \text{TOP}$, $n = 3$ the theorem is due to Bing. For $C \neq \text{TOP}$ we apply the TOP case and results of Section 3. Our theorem has also an analogue for immersions.

In Section 6 we prove that if either $Y \subset X \subset R^{n-1}$ with Y open in R^{n-1} or $Y \subset X \subset R_+^{n-1}$ with Y open in R_+^{n-1} and if $n \neq 4$, then every embedding $f : X \to R^n$ which is quasisymmetric or bilipschitz, respectively, can be approximated on Y by locally LIP flat embeddings which can be extended by $f | X \setminus Y$ to quasisymmetric embeddings or bilipschitz embeddings, respectively, of X. The proof is based on Theorem 5.3 on extending locally LIP flat approximations. An analogue of this result is proved in [LT]. There the dimension pair $(n-1, n)$, $n \neq 4$, is replaced by a pair (n, q) such that either $q \geq n = 1$ or $2 \leq n \leq q \leq 3$ or $n \geq 2$, $q \geq n+3$ and PL (piecewise linear) embeddings are used in place of locally LIP flat embeddings.

In Section 7 we prove, using theorems of the two previous sections, that if $C \in \{\text{LQS, LIP}\}$ and $n \neq 4$, then every embedding (or immersion, respectively) of an $(n-1)$-dimensional C-manifold into an n-dimensional C-manifold without boundary can be relatively approximated by C-embeddings (or C-immersions, respectively). This was proved in [LT, Theorem 4.4] for the dimension pairs (n, q) of the previous paragraph.

In Section 8 we show that if M is an n-manifold, $n \neq 4$, and if $C \in \{\text{LQC, LIP}\}$, every C-manifold structure on the boundary ∂M of M can be extended over M. This implies that every manifold carries a C-structure if every 4-manifold carries a C-structure.

In the last section we show that for every n, every LQC selfhomeomorphism of R^n can be extended to an LQC selfhomeomorphism of R_+^{n+1}.

One of our starting-points was the paper [G₂] by Gauld, who applied [S]. We give both of his two theorems either as a corollary (3.2.2) or with a short proof (9.4).

Relatively little is known about the results of Sections 3, 4, and 5 for LQC or LIP in codimensions ≥ 2. The theorems of Section 3 are open in these codimensions; see, however, [GV] and [G₁] (in [G₁, Theorems 1 and 2] one can omit the assumption $n \neq 5$ using LIP straightening of TOP handles instead of PL straightening). In codimension two the equivalence and approximation theorems of Sections 4 and 5 do not hold as shown by the following facts: There are knotted locally PL flat embeddings $f : S^{n-2} \to R^n$ $(n \geq 3)$ arbitrarily close to id, and there are embeddings $f : S^1 \times S^{2n-1} \to R^{2n+2}$ $(n \geq 3)$ that cannot be approximated by PL embeddings or locally flat embeddings [M, Theorem 5.3] (here $S^i = \partial I^{i+1}$). However, by [Ve] every

embedding of I^n into a PL $(n+2)$-manifold can be approximated by locally flat PL embeddings. In codimensions $\geqq 3$ the results of Sections 4 and 5 are (essentially) known for TOP and PL (cf. [Mi], [Ma], [E]) but open for LQC and LIP. However, the author has partial results. The LQS and LIP approximation theorems of Section 7 are open in codimension two even in the absolute case.

One has considered C-embeddings and locally C-flat embeddings, $C \in \{$LQS, LIP$\}$, separately. This is in order because of the following facts: In codimensions 0, 1, 2 there are LIP embeddings which are not locally flat [LuV, Examples 3.10 (1) and (5)] and others which are locally flat but not locally LQC flat [T_2, Section 17]. It is known that LIP embeddings in codimensions $\geqq 3$ are locally flat [LT, Theorem 1.14] but not whether they are locally LIP flat or at least locally LQC flat. In every codimension there are LQS embeddings which are not locally flat [V_3, Theorem 6.3] and others which are locally flat but not locally LQC flat [V_3, Remark 6.4.3].

2. Terminology and preliminary results

In this section we let X and Y denote metric spaces.

The letter n denotes an integer which is positive except when n is explicitly stated to be only non-negative. We identify R^{n-1} with $\{x \in R^n | x_n = 0\}$. We set $R^n_+ = \{x \in R^n | x_n \geqq 0\}$, and for $n \geqq 0$ and $r > 0$, $I^n = [-1, 1]^n$, $I^n(r) = rI^n$, $J^n = (-1, 1)^n$, $J^n(r) = rJ^n$, $B^n(x, r) = \{y \in R^n : |x - y| < r\}$ if $x \in R^n$, $B^n(r) = B^n(0, r)$, $\bar{B}^n(r) = \text{cl } B^n(r)$, $B^n = B^n(1)$, $S^n = \partial B^{n+1}$. If $A \subset R^n$ (and int $A \neq \emptyset$), we write $A_+ = A \cap R^n_+$. The standard orthonormal basis of R^n is written as $e_1, ..., e_n$. We set $I = [0, 1]$ and $N = \{1, 2, ...\}$. We let id denote an inclusion map. Let $C_+(X) = \{f | f : X \to (0, \infty)$ continuous$\}$.

All manifolds are separable metric. Every metric, except the Euclidean one, is denoted by d. In a metric space, $d(A, B)$ denotes the distance between two sets A, B, with $d(\{x\}, B) = d(x, B)$, and $d(A)$ denotes the diameter of A.

Suppose that $f, g : X \to Y$ are continuous maps. We write

$$d(f, g) = \sup \{d(f(x), g(x)) | x \in X\}.$$

If $\varepsilon \in C_+(X)$, we let $U(f, \varepsilon)$ denote the set of all such g that $d(f(x), g(x)) < \varepsilon(x)$ for each $x \in X$. For some facts we need about these ε-neighborhoods $U(f, \varepsilon)$, we refer to [LT, 1.6].

If $A \subset X$, we let "near A" mean "on an open neighborhood of A".

An isotopy H of X onto Y is a continuous map $H : X \times I \to Y$ such that for each $t \in I$ the map $H_t : X \to Y$, $x \mapsto H(x, t)$, is a homeomorphism. As known, there are other definitions of an isotopy of X onto Y, equivalent to this one if

X and Y are locally compact. If $A \subset X$ and $H_t|A = H_0|A$ for each t, we say that H is rel A. If $\varepsilon \in C_+(X)$ and $H_t \in U(H_0, \varepsilon)$ for each t, we say that H is an ε-isotopy. If $X = Y$ and $H_0 = \mathrm{id}$, we say that H is an ambient isotopy of X. If A, ε are as above, the ε-neighborhood of A in X is the set $N(A, \varepsilon) = \{x \in X \mid \exists y \in A, \ d(x, y) < \varepsilon(y)\}$. If H is an ambient ε-isotopy of X rel $X \setminus N(A, \varepsilon)$, we say that the homeomorphism $H_1: X \to X$ is an ε-push of (X, A). The combination of two isotopies G and H of X onto Y with $H_0 = G_1$ is the isotopy K of X onto Y defined by $K_t = G_{2t}$ if $0 \leq t \leq 1/2$ and by $K_t = H_{2t-1}$ if $1/2 \leq t \leq 1$. Given an isotopy G of X onto Y and an ambient isotopy H of Y, the isotopy $H * G$ of X onto Y is defined by $(H * G)_t = G_{2t}$ if $0 \leq t \leq 1/2$ and by $(H * G)_t = H_{2t-1} G_1$ if $1/2 \leq t \leq 1$; $H * G$ is ambient if and only if G is ambient.

We will often need the fact that if $U \subset X$ is open and $f: X \to Y$, $g: U \to Y$ are embeddings such that $d(f(x), g(x)) \leq d(f(x), f[X \setminus U])/2$ for each $x \in U$, then $h = g \cup (f|X \setminus U): X \to Y$ is an embedding. This follows from

$$\frac{1}{2} d\big(f(x), f(y)\big) \leq d\big(h(x), h(y)\big) \leq \frac{3}{2} d\big(f(x), f(y)\big) \quad \text{for} \quad x \in U, \quad y \in X \setminus U.$$

An immersion $f: X \to Y$ is a continuous map which embeds some neighborhood of each point of X.

An embedding $f: X \to Y$ is called $(L\text{-})$ bilipschitz if, for some $L \geq 1$,

$$d(x, y)/L \leq d\big(f(x), \ f(y)\big) \leq L d(x, y) \quad \text{for all} \quad x, y \in X.$$

If f is only a function satisfying the right inequality, f is called Lipschitz. A LIP map $f: X \to Y$ is one which is locally Lipschitz. Every PL map between polyhedra in Euclidean spaces is LIP. An embedding $f: X \to Y$ is called $(\eta\text{-})$ quasisymmetric if, for some homeomorphism $\eta: R^1_+ \to R^1_+$,

$$d\big(f(a), \ f(x)\big)/d\big(f(b), \ f(x)\big) \leq \eta\big(d(a, x)/d(b, x)\big) \quad \text{for} \quad a, b, x \in X, \quad b \neq x.$$

Like bilipschitz embeddings, quasisymmetric embeddings form a category; cf. $[TV_1]$. Every bilipschitz embedding is quasisymmetric. An embedding $f: X \to Y$ is said to be a LIP embedding or an LQS embedding if f is, locally, bilipschitz or quasisymmetric, respectively. Every LIP or LQS embedding of a compact space is bilipschitz or quasisymmetric $[TV_1$, Theorem 2.23$]$, respectively. For increasing embeddings $f: A \to R^1$, where $A \subset R^1$ is an interval, our definition of quasisymmetry is equivalent to the one in $[LV_2$, II. 7.1$]$. Hence, by $[LV_2$, II, Lemma 7.1 and (7.2)$]$, every quasisymmetric homeomorphism $f: [0, 1) \to [0, 1)$ can be extended to a quasisymmetric homeomorphism $F: J^1 \to J^1$ by reflection, i.e., if we set $F(x) = -f(-x)$ for $x < 0$.

Let $n \geq 2$, let $A \subset R^n$ be a set such that $A \subset \mathrm{cl} \, \mathrm{int} \, A$, and let $f: A \to R^n$ be an embedding. If there is $K \geq 1$ such that the restrictions of f to all components of $\mathrm{int} \, A$ are K-quasiconformal in the sense of $[V_1]$, f is called $(K\text{-})$ quasiconformal. If each point of A has an open neighborhood in A on which f is quasiconformal,

f is said to be LQC. The sets A and the LQC embeddings $f: A \to R^n$ (observe that $fA \subset \mathrm{cl\ int} fA$) form a category. We will freely use Theorem 35.1 of $[V_1]$ about the piecewise construction of quasiconformal embeddings. In particular, an LQC embedding of a compact set is quasiconformal. Further, if $p: R^n \to R^n$ is the orthogonal reflection in R^{n-1}, we can extend every open quasiconformal (or LQC) embedding $f: U \to R^n_+$, where $U \subset R^n_+$ is open, to a quasiconformal (or LQC, respectively) embedding $F: U \cup pU \to R^n$ by reflection, i.e., by setting $F = pfp$ on pU; cf. $[V_1$, Theorem 35.2]. It follows from this and $[V_3$, Theorems 2.3 and 2.4] that if $U \subset R^n_+$ is open, an open embedding $f: U \to R^n_+$ is LQC if and only if f is LQS. For embeddings $f: A \to R^1$, where $A \subset R^1$, we let "quasiconformal" and "LQC" mean "quasisymmetric" and "LQS", respectively. Also $\mathrm{id}: R^0 \to R^0$ is called LQC. It follows that if $n \geq 1$, $U \subset R^n_+$ is open, and $f: U \to R^n_+$ is an open LQC embedding, $f | U \cap R^{n-1}: U \cap R^{n-1} \to R^{n-1}$ is LQC.

Let C denote either LQS, LQC, or LIP. An atlas on an n-manifold, $n \geq 0$, is called a C-atlas if its coordinate changes are C-homeomorphisms (cf. [LT, 1.3]). A C-structure is a maximal C-atlas, and a manifold equipped with a C-structure is called a C-manifold. Observe that LQC manifolds are the same as LQS manifolds. However, we prefer to use the term LQC rather than LQS whenever possible. We define C-embeddings of C-manifolds (of the same dimension if $C = $ LQC) by using charts. A C-immersion is a continuous map which is locally a C-embedding. The restrictions of the charts of the C-structure $\mathscr{A}$ of a C-manifold M to the boundary ∂M of M form a C-atlas $\mathscr{A} | \partial M$ on ∂M. In fact, $\mathscr{A} | \partial M$ is a C-structure, because by $[TV_3$, Theorem 4.10 and Remark 4.11] every C-embedding $f: \partial N \to \partial Q$, where N, Q are n-dimensional C-manifolds, can be extended to a C-embedding of a neighborhood of ∂N. If M, Q are C-manifolds (of the same dimension in the case $C = $ LQC), M is a subset of Q, and the inclusion $M \to Q$ is a C-embedding, then M is called a C-submanifold of Q.

Let $C \in \{$TOP, LQS, LQC, LIP$\}$ and let Q be an n-dimensional C-manifold. Every discrete subspace of Q is defined to be a locally C-flat 0-submanifold of Q. For $1 \leq k \leq n$, we want every locally C-flat k-submanifold of Q to meet ∂Q transversally. For this purpose we introduce the notation $R^{n,k}_{++} = \{x \in R^n_+ | x_1 = \ldots = x_{n-k} = 0, x_{n-1} \geq 0\}$ for $k \geq 2$ and $R^{n,1}_{++} = \{x \in R^n_+ | x_1 = \ldots = x_{n-1} = 0, x_n \in I\}$. We say that a subset N of Q is a *locally C-flat k-submanifold* of Q ($k \geq 1$) if for each point $x \in N$ there are an open neighborhood U of x in Q and an open C-embedding $h: U \to R^n_+$ such that $h[U \cap N] = hU \cap R^{n,k}_{++}$. Since $R^{n,k}_{++}$ for $k \geq 2$ is PL homeomorphic to R^k_+, it is easy to see that N is then a C-manifold in a natural way (a C-submanifold of Q except if $C = $ LQC and $k \neq n$). Clearly $N \cap \partial Q$ is a locally C-flat $(k-1)$-submanifold of ∂Q and of ∂N. Let M be a k-dimensional C-manifold and $f: M \to Q$ an embedding. If $k = 0$, we call f locally C-flat. For $k \geq 1$, we say that f is *locally C-flat* if for each $x \in M$ there are an open neighborhood $U \subset M$ of x, an open subset V of R^n_+, a C-homeomorphism $h: U \to V \cap R^{n,k}_{++}$, and an open C-embedding $g: V \to Q$ such that $g | hU = fh^{-1}$. Then fM is a locally C-flat sub-

manifold of Q and f gives a C-homeomorphism $M \to fM$. Conversely, if fM is a locally C-flat submanifold of Q and f gives a C-homeomorphism $M \to fM$, it is easy to see that f is locally C-flat. An immersion $f: M \to Q$ is said to be locally C-flat if it is locally a locally C-flat embedding. We need the easily proved fact that if M, Q are C-manifolds, $f: M \to Q$ is a locally C-flat embedding, and N is a locally C-flat submanifold of M, then $f|N$ is locally C-flat, provided that either $f^{-1}\partial Q = \emptyset$ or $f^{-1}\partial Q = \partial M$. It is useful in this paper to have the concept of local LQS flatness, too, though it is equivalent to local LQC flatness.

Local PL flatness is defined in an analogous way. We need the fact that if A is a closed subset of a PL manifold M and $U \subset M$ is a neighborhood of A, there is a closed neighborhood P of A such that $P \subset U$ and P is a locally PL flat PL submanifold of M. To construct such a P, choose a sufficiently fine triangulation K of M, let $N(A, K)$ denote the union of all (closed) simplexes of K which meet A, and set $P = N(N(A, K), K'')$, where K'' is the second barycentric subdivision of K; cf. [H, pp. 65, 67], [RS, pp. 32—35], [Ru, p. 20]. We also need the fact that a PL n-ball B in R^n_+ is locally PL flat if $B \cap R^{n-1}$ is either empty [RS, Corollary 3.13] or a PL $(n-1)$-ball [H, Lemma 2.14].

The following lemma is needed in 3.1.

2.1. Lemma. *Let* $I_j = Ie_j \subset R^3$, $j = 1, 2, 3$, *let* $T = I_1 \cup I_2 \cup I_3$, *and let* $f: T \to T$ *be a homeomorphism.*

(a) *If* $f|I_i \cup I_j$ *is quasisymmetric for all* i, j, f *is quasisymmetric.*

(b) *If* $f|I_1 \cup I_2$ *and* $f|I_1 \cup I_3$ *are quasisymmetric,* f *is quasisymmetric.*

(c) *If* $f|I_1$ *is quasisymmetric, there is a quasisymmetric homeomorphism* $F: T \to T$ *such that* $F = f$ *on* $I_1 \cup \{e_2, e_3\}$ *and* F *is PL near* $\{e_2, e_3\}$.

(d) *If* $f|I_1 \cup I_2$ *is quasisymmetric, there is a quasisymmetric homeomorphism* $F: T \to T$ *such that* $F = f$ *on* $I_1 \cup I_2 \cup \{e_3\}$ *and* F *is PL near* e_3.

Proof. We first observe that if $f|I_1 \cup I_2$ is η-quasisymmetric and $H = \eta(1)$ ($\geqq 1$), then, since $f(0) = 0$, $|f(x_1)| \leqq H|f(x_2)|$ and, hence, $|f(x_1) - f(x_3)| \leqq H|f(x_2) - f(x_3)|$ whenever $x_j \in I_j \setminus 0$, $|x_1| = |x_2|$. By the aid of this fact it is easy to prove (a) and reduce (b) to (a) (we do not need (b)). To prove (c), we first extend $f|I_1 \cup \{e_2, e_3\}$ to a homeomorphism $F_0: T \to T$ by setting $F_0(te_j) = |f(te_1)|f(e_j)$ for $t \in I$, $j = 1, 2, 3$. By the reflection principle and (a), F_0 is quasisymmetric. The existence of F now easily follows from the fact ([LV₁, p. 13]) that every increasing quasisymmetric homeomorphism $\varphi: I \to I$ can be extended to a quasisymmetric homeomorphism $\varphi': 2I \to 2I$ which is PL near 2. The proof of (d) is similar. $\square$

3. LQC or LIP approximation of homeomorphisms

Consider the approximation result which is contained in [TV₃, Theorem 4.8] and which generalizes [TV₃, Theorem 4.4]. A slight modification of [TV₃, Remark 4.9] (where s_1 should have been chosen so that $g'(s_1) \neq 0$) implies that this result also holds for $n=1$. Moreover, the condition "$f|\partial M$ is a C-embedding" can be replaced, as the proof shows, by the more general condition "$f|\partial M$ is a C-embedding near $B \cap \partial M$". In this section we apply this theorem to prove the more general Theorems 3.1 and 3.3, whose special case of empty M_1 and M_2 it is. In the proof of 3.3 we also use 4.1.

3.1. **Theorem.** *Let* $C \in \{$LQC, LIP$\}$, *let* Q_1, Q_2 *be n-dimensional C-manifolds, let* $A, B \subset Q_1$ *be closed and* $B' \subset Q_1$ *a neighborhood of* B, *let* M_i *be a closed locally C-flat $(n-1)$-submanifold of* Q_i, $i=1,2$, *such that* $(M_1 \backslash A) \cap \partial Q_1 = \partial M_1 \backslash A$, *let* $f: Q_1 \to Q_2$ *be a homeomorphism such that* f *is a C-embedding near* A *and* $fM_1 = M_2$, *and let* $\varepsilon \in C_+(Q_1)$.
   (a) *Provided* $n \neq 4, 5, 6$, *there is an ε-isotopy* H *of* Q_1 *onto* Q_2 *such that*
   (1) $H_0 = f$,
   (2) $H_t = f$ *for each* $t \in I$ *near* A *and on* $Q_1 \backslash B'$,
   (3) $H_t M_1 = M_2$ *for each* $t \in I$,
   (4) H_1 *is a C-embedding near* $A \cup B$.
   (b) *If* $X \in \{\partial M_1, \partial Q_1, M_1\}$ *and if* $f|X$ *is a C-embedding near* $B \cap X$, *then* (a) *is also true for* $n=6$, *and* H *can be chosen to be* rel X.
   (c) *Suppose that* $f|X$ *is a C-embedding near* $B \cap X$ *for each* $X \in \{\partial Q_1, M_1\}$. *If* $C =$ LQC *and* $n=2$, *suppose in addition that* $f|\partial Q_1 \cup M_1$ *is LQS near* $\partial M_1 \cap (B \backslash A)$. *Then* (a) *is also true for* $n=5$ *or* 6, *and* H *can be chosen to be* rel $\partial Q_1 \cup M_1$.

Proof. Note that $f|\partial M_1$ is necessarily a C-embedding near $B \cap \partial M_1$ in (b) and (c). As in the proof of [TV₃, Theorem 4.4], the general case can be reduced to the special case where B is compact and either $(Q_1, M_1) = (Q_2, M_2) \in \{(R^1, \emptyset),$ $(R^1_+, \emptyset), (R^1, \{0\})\}$ if $n=1$ or, if $n \geq 2$, Q_1, Q_2 are open subsets of R^n_+ and $M_i = Q_i \cap L_0$, $i=1,2$, where $L_0 = R^{n-2} \times \{0\} \times R^1_+ \subset R^n_+$. For $n=1$, the case $(Q_i, M_i) = (R^1, \emptyset)$, $i=1,2$, is proved in [TV₃, Remark 4.9]; the remaining cases can be proved similarly.

Let now $n \geq 2$. To prove the special case, we may additionally assume that B' is compact and Q_1 connected. Choose an open neighborhood A' of A such that $f|A'$ is a C-embedding. Let $Q_i^j = Q_i \cap (R^{n-2} \times jR^1_+ \times R^1_+)$, $j = -1, 1$. Then $Q_i = Q_i^1 \cup Q_i^{-1}$ and $Q_i^1 \cap Q_i^{-1} = M_i$. It is easy to see that fQ_1^1 is either Q_2^1 or Q_2^{-1}. Hence we may assume that $fQ_1^j = Q_2^j$, $j = -1, 1$. Let $L_j = R^{n-2} \times jR^1_+ \times \{0\} \subset R^n_+$, $j = -1, 1$. Then $f[Q_1 \cap L_j] = Q_2 \cap L_j$. Note that there is a PL homeomorphism of $R^{n-2} \times jR^1_+ \times R^1_+$ onto R^n_+ which is the identity on L_j, $j = -1, 1$.

By Theorem 4.8 of [TV₃] and its proof, for all $j = -1, 0, 1$ and $\delta > 0$ there is a δ-isotopy G^j of $Q_1 \cap L_j$ onto $Q_2 \cap L_j$ with the following properties: $G_0^j =$

$f|Q_1 \cap L_j$, G^j is rel $(Q_1 \cap L_j) \setminus D$ for some compact neighborhood D (independent of j) of $B \setminus A'$ in $\mathrm{int}_{Q_1} B' \setminus A$, and G_1^j is a C-embedding near $B \cap L_j$; $G_1^1 = G_t^0 = G_t^{-1}$ on ∂M_1 for each $t \in I$; G^j is rel ∂M_1 whenever $f|\partial M_1$ is a C-embedding near $B \cap \partial M_1$, and finally, G^j is rel $X \cap L_j$ whenever $X \in \{\partial Q_1, M_1\}$ and $f|X$ is a C-embedding near $B \cap X$. In the part (a) we needed $n \neq 5, 6$, in the part (b) $n \neq 5$. For $j = -1, 1$ define an isotopy $K^j = G^0 \cup G^j$ of $\partial Q_1^j = Q_1 \cap (L_0 \cup L_j)$ onto ∂Q_2^j. Then K_1^j is a C-embedding near $B \cap \partial Q_1^j$ if $(C, n) \neq (\mathrm{LQC}, 2)$. To obtain this claim for $(C, n) = (\mathrm{LQC}, 2)$, too, additional care is needed. We may assume that $\{0\} = \partial M_1 \subset B \setminus A$. The part (c) is clear, since $K_1^j = f|\partial Q_1^j$ is LQS near 0 by our particular assumption. In the part (a), [TV_3, Remark 4.9] shows that the isotopies G^i, $i = -1, 0, 1$, can be chosen in such a way that G_1^i is PL near 0, in which case the claim holds. Consider the subcase of (b) where $f|M_1$ is LQS near $B \cap M_1$ (implying $G_1^0 = f|M_1$). We may assume that $G_1^1 \cup G_1^{-1}$ is PL near 0. Hence, modifying $G^1 \cup G^{-1}$ by combining it with a new isotopy, we may assume by 2.1 (c) that $G_1^1 \cup G_1^0 \cup G_1^{-1}$ is LQS near 0, whence the claim. Finally, the treatment of the subcase of (b) where $f|\partial Q_1$ is LQS near $B \cap \partial Q_1$ is similar; we now use 2.1(d).

Again by Theorem 4.8 of [TV_3] and its proof, if the above δ is sufficiently small, for each $j = -1, 1$ there is an $(\varepsilon | Q_1^j)$-isotopy H^j of Q_1^j onto Q_2^j which extends K^j such that $H_0^j = f|Q_1^j$, $H_t^j = f$ for each $t \in I$ near $A \cap Q_1^j$ and on $Q_1^j \setminus B'$, and H_1^j is a C-embedding near $B \cap Q_1^j$. Then $H = H^1 \cup H^{-1}$ is the desired isotopy. $\square$

3.2. Remarks. 1. In the case $n = 1$, Theorem 3.1 also holds if we omit the assumption $(M_1 \setminus A) \cap \partial Q_1 = \partial M_1 \setminus A$ $(= \emptyset)$. In fact, the isotopy H given by 3.1 with M_i replaced by $M_i \setminus \partial Q_i$ is then the desired isotopy, since it is rel $\partial Q_1 \cup M_1$.

2. The following special case of 3.1 is [G_2, Theorem 1]: If $C \in \{\mathrm{LQC}, \mathrm{LIP}\}$, $n \neq 4, 5$, and M_1, M_2 are compact locally C-flat $(n-1)$-submanifolds of S^n without boundary such that (S^n, M_1) is homeomorphic to (S^n, M_2), then (S^n, M_1) is C-homeomorphic to (S^n, M_2).

In the case $\partial M_1 \subset A$, the next theorem also follows from 3.1.

3.3. Theorem. *Let* $C, Q_i, A, B, B', M_i, f, \varepsilon$ *be as in* 3.1 *but such that* $M_1 \subset A \cup B$ *and that the assumption* $(M_1 \setminus A) \cap \partial Q_1 = \partial M_1 \setminus A$ *is replaced by the assumption* $M_1 \setminus A \subset \mathrm{int}\, Q_1$.

(a) *The part* (a) *of* 3.1 *holds with* (3) *replaced by the condition* $H_1 M_1 = M_2$.

(b) *If* $X \in \{\partial M_1, M_1\}$ *and if* $f|X$ *is a C-embedding, then* (a) *is also true for* $n = 6$, *and* H *can be chosen such that* $H_1|X = f|X$.

(c) *If* $f|\partial Q_1$ *is a C-embedding near* $B \cap \partial Q_1$, *then* (b) *with* $X = M_1$ *is also true for* $n = 5$ *or* 6, *and* H *can be chosen to be* rel ∂Q_1 *in* (a) *and* (b).

Proof. Choose open sets $A_1, A_2, B_1 \subset Q_1$ such that $A \subset A_2 \subset \bar{A}_2 \subset A_1$, f is a C-embedding near $\bar{A}_1$, $B \subset B_1 \subset \bar{B}_1 \subset \mathrm{int}_{Q_1} B'$, and $f|\partial Q_1$ is a C-embedding near $\bar{B}_1 \cap \partial Q_1$ in (c). Then $N = f[B' \setminus A_2] \cap \mathrm{int}\, Q_2$ is a neighborhood of $f[M_1 \setminus A_1]$

in Q_2. Choose $\eta \in C_+(Q_2)$ with $U(\mathrm{id}_{Q_2}, \eta) \circ U(f, \eta f) \subset U(f, \varepsilon)$. Then choose $\eta_1 \in C_+(Q_1)$, $\eta_1 \leqq \eta f$, such that $\delta = \eta_1 | M_1$ is the function given by 4.1 with the substitution $C \mapsto C$, $n \mapsto n$, $M \mapsto M_1$, $Q \mapsto Q_2$, $A, B \mapsto M_1$, $U \mapsto A_1 \cap M_1$, $f \mapsto f | M_1$, $N \mapsto N$, $\varepsilon \mapsto \eta | N$. By [TV$_3$, Theorem 4.8] there is an η_1-isotopy G of Q_1 onto Q_2 rel $A_1 \cup (Q_1 \setminus B')$ such that $G_0 = f$, $G_1 | A_1 \cup B_1$ is a C-embedding, and G is rel ∂Q_1 in (c). Then $f_1 = G_1 | M_1$ is a locally C-flat embedding in $U(f | M_1, \delta)$ with $f_1 = f$ on $A_1 \cap M_1$. Again by [TV$_3$, Theorem 4.8], there is a C-homeomorphism $f_2 \colon M_1 \to M_2$ in $U(f | M_1, \delta)$ such that $f_2 = f$ on $A_1 \cap M_1$ and that $f_2 | X = f | X$ whenever $X \in \{\partial M_1, M_1\}$ and $f | X$ is a C-embedding. Then f_2 is locally C-flat as an embedding into Q_2. Hence, by the definition of δ, there is an ambient η-isotopy K of Q_2 rel $Q_2 \setminus N$ such that K_1 is a C-homeomorphism and $K_1 f_1 = f_2$. It follows that $H = K * G$ is the desired isotopy. $\square$

3.4. **Remark.** For $n = 2$, the modification of 3.3 holds where the assumption $M_1 \setminus A \subset \mathrm{int}\, Q_1$ and the assertion (c) are omitted and where $H_1 | \partial M_1 = f | \partial M_1$. This result can be proved as 3.3 by the aid of 4.2.

3.5. **Example.** For every $C \in \{\mathrm{LQC}, \mathrm{LIP}\}$ and for every $n \geqq 1$, each of the parts (a), (b), and (c) of 3.1 and of 3.3 is false without the assumption that M_i is closed in Q_i. We show this by constructing a locally LIP flat $(n-1)$-submanifold M of R^n without boundary and a homeomorphism $f \colon R^n \to R^n$ which is a LIP embedding near M such that fM is not the image of M under any LQC homeomorphism $h \colon R^n \to R^n$. Suppose first $n \geqq 2$. We modify [V$_3$, Example 5.11]. Let $M = R^{n-1} \setminus 0$. Set $\varphi(t) = t \sin(1/t)$ for $t > 0$ and $\varphi(0) = 0$, and define $f(x, t) = (x, t + \varphi(|x|))$ for $x \in R^{n-1}$, $t \in R^1$. If now $h \colon R^n \to R^n$ were an LQC homeomorphism such that $hM = fM$, then $hR^{n-1} = fR^{n-1}$, which is impossible, since no neighborhood of the origin in fR^{n-1} is of bounded turning; cf. [TV$_1$, Theorem 2.11]. Consider $n = 1$. We let $M = \{\pm 1/j \mid j \in \mathbf{N}\}$ and define $f(x) = x^2$ for $x \geqq 0$ and $f(x) = x$ for $x < 0$. Suppose that there is an LQS homeomorphism $h \colon R^1 \to R^1$ such that $hM = fM$. We may assume that h is increasing. Then $h | \overline{M} = f | \overline{M}$, which is impossible, since $f | \overline{M}$ is not locally quasisymmetric at the origin.

4. Equivalence of close locally C-flat embeddings

In this section we prove the following two theorems. For a discussion on them, see 4.3.

4.1. **Theorem.** *Let* $C \in \{\mathrm{TOP}, \mathrm{LQC}, \mathrm{LIP}\}$ *and* $n \neq 4$, *let* M *be an* $(n-1)$- *and* Q *an* n-*dimensional* C-*manifold, let* $A \subset M$ *be closed,* $B \subset M$ *a neighborhood of* A, *and* $U \subset M$ *open, let* $f \colon M \to Q$ *be an embedding with* $f[A \setminus U] \subset \mathrm{int}\, Q$, *let* $N \subset Q$ *be a neighborhood of* $f[A \setminus U]$, *and let* $\varepsilon \in C_+(N)$. *If* $C \neq \mathrm{TOP}$ *and* $n \geqq 2$, *suppose that* $f[A \setminus U]$ *is closed in* Q. *Then there is* $\delta \in C_+(M)$ *with the*

following property: If $f_1, f_2: M \to Q$ *are locally C-flat embeddings such that* $d(f_i(x), f(x)) < \delta(x)$ *for each* $x \in B$, $i = 1, 2$, *and* $f_1|U = f_2|U$, *there is an ambient isotopy* H *of* Q *rel* $Q \setminus N$ *such that* $d(H_t(x), x) < \varepsilon(x)$ *for all* $x \in N$ *and* $t \in I$, H_t *is a C-homeomorphism for each* $t \in I$, *and* $H_1 f_1 = f_2$ *on* $N(A, \delta)$.

4.2. Theorem. *For* $n \leq 3$, *Theorem 4.1 also holds with the following modifications:* 1) *The assumption* $f[A \setminus U] \subset \text{int } Q$ *is replaced by the following assumption. Let* $T \subset M$ *be an open neighborhood of* $A \setminus U$. *Suppose that* $T \cap f^{-1} \partial Q$ *is an* $(n-2)$-*submanifold of* ∂T *whenever either* $n = 2$ *or* $C = \text{TOP}$, $n = 3$ *and that* $T \cap f^{-1} \partial Q = \partial T$ *whenever* $C \neq \text{TOP}$, $n = 3$. 2) *The embeddings* f_1, f_2 *are assumed to have the additional property that either* $T \cap f_i^{-1} \partial Q = T \cap f^{-1} \partial Q$, $i = 1, 2$, *whenever* $n \geq 2$ *or* $T \cap f_1^{-1} \partial Q = T \cap f_2^{-1} \partial Q$ *whenever* $n = 1$. 3) *Only* H_1 *is claimed to be a C-homeomorphism, not* H_t *for each* $t \in I$.

4.3. Remarks. 1. For $C = \text{TOP}$ and $n \leq 3$, 4.2 generalizes 4.1.

2. The forms of 4.1 and 4.2 where N is more generally a neighborhood of $f[A \setminus \text{int}_M \overline{U}]$ only follow trivially.

3. Consider the (uninteresting) case $n = 1$. In 4.2 the isotopy H can, as in 4.1, be chosen such that H_t is a C-homeomorphism for each $t \in I$ also if $C \neq \text{TOP}$. We omit the proofs of 4.1 and 4.2 because they differ from the proofs in the case $n \geq 2$. We only remark that these results reduce to the special case of 4.2 where $C = \text{LIP}$, $A = M$, $U = \emptyset$, for which there exists an easy direct proof.

4. For all $C \neq \text{TOP}$ and $n \geq 2$, the assumption in 4.1 and 4.2 that $f[A \setminus U]$ is closed is essential. For example, for every $\delta \in C_+(B^{n-1})$, where $n \geq 2$, the δ-neighborhood of the inclusion $B^{n-1} \to R^n$ contains a locally LIP flat embedding which is not quasisymmetric, and which thus is not a restriction of any LQC homeomorphism $R^n \to R^n$.

5. For $C = \text{TOP}$, 4.1 and 4.2 are known in the absolute case $A = M$, $U = \emptyset$ (we may now assume that f is closed). For $n \geq 5$ this is due to Price and Seebeck [PS, Theorem 3]. The case $n = 3$ follows from Craggs [Cr, Theorem 8.2] (cf. [B₁, Theorem 9]); this result of [Cr] also implies for $n = 2$ the absolute case of 4.1. Of course, 4.2 for $C = \text{TOP}$, $n = 2$ is well-known, but it seems that it does not exist in the literature.

6. The first of the three stages of the proofs of 4.1 and 4.2 consists of handle lemmas. For $C = \text{TOP}$ these lemmas base on the handle lemma [PS, Theorem 2] if $n \geq 5$, on [Cr, Theorem 8.1] if $n = 3$ (cf. the paragraph just before Theorem 8.2 in [Cr]), and on the Schoenflies theorem if $n = 2$. The handle lemmas for $C \neq \text{TOP}$ follow from the TOP handle lemmas and 3.1.

We begin with $C = \text{TOP}$, $n = 3$.

4.4. Lemma. *Let* M *be a* PL 2-*manifold,* Q *a* PL 3-*manifold,* $f: M \to Q$ *a closed locally flat embedding,* $A \subset Q$ *closed with* f *being* PL *near* $f^{-1}A$, *and* $\varepsilon \in C_+(Q \setminus A)$. *Then there is an ambient isotopy* H *of* Q *rel* A *such that* $d(H_t(x), x) < \varepsilon(x)$ *for all* $x \in Q \setminus A$ *and* $t \in I$ *and that* $H_1 f$ *is* PL.

Proof. Choose a closed polyhedral neighborhood N of A in Q such that f is PL near $f^{-1}N$. Obviously, there is a triangulation K of the space $S = fM \cup N$ such that f is PL, N a polyhedron, and the inclusion map $N \to Q$ PL with respect to K. Then $S_0 = S \cap \partial Q$ is a polyhedron under K. We may assume that a subcomplex K_0 of K triangulates S_0. We conclude, by the analogue of [B$_1$, Theorem 8'] for 2-manifolds without boundary (this is obviously true) or directly, that a subdivision of K_0, which we may assume to be K_0 itself, can be extended to a triangulation K' of the space ∂Q. Then by [B$_1$, Theorem 8'] a subdivision of the triangulation $K \cup K'$ of $S \cup \partial Q$, which we may assume to be $K \cup K'$ itself, can be extended to a triangulation K_1 of the space Q. By [B$_1$, Theorem 4], for every $\delta \in C_+(Q \setminus A)$ there is a PL homeomorphism $g : (Q, K_1) \to Q$ such that $g|A = \mathrm{id}$ and $d(g(x), x) < \delta(x)$ for each $x \in Q \setminus A$. Then gf is PL. If δ is small enough, by [Ks, Theorem] there is an ambient isotopy H of Q rel A such that H is an ε-isotopy on $Q \setminus A$ and $H_1 = g$. $\square$

4.5. Lemma. *Theorem 4.2 holds if* $C = \mathrm{TOP}$, $n = 3$, *and* $A = B = T = M$.

Proof. We first reduce the lemma to the case where f is closed. Since fM is locally compact, it has an open neighborhood Q_0 in Q where it is closed. We may assume that N is open, $N \subset Q_0$, and $\varepsilon(x) \leqq d(x, Q \setminus N)/2$ for each $x \in N$. Let δ with $\delta(x) \leqq d(f(x), Q \setminus Q_0)$ for each $x \in M$ be the function given by the lemma if Q is replaced by Q_0. Consider f_1, f_2 as in the lemma; then $f_i M \subset Q_0$. Let H^0 be the ambient isotopy of Q_0 thus obtained. Extending H_t^0 for each $t \in I$ by the identity of $Q \setminus Q_0$ we get the desired ambient isotopy H of Q.

Thus we may assume that f is closed. Then we may also assume that there is $\varepsilon' \in C_+(Q)$ extending ε. We may further assume that N is open and that M and Q are PL manifolds. Then $L = f^{-1}\partial Q$ is a PL 1-submanifold of ∂M. Choose closed locally PL flat submanifolds $M_1, M_2 \subset U$ of M such that M_1 is a neighborhood of $M \setminus f^{-1}N$, M_2 is a neighborhood of M_1, and $M_2 \cap L$ is a 1-manifold. Choose an open set $N_1 \subset Q$ such that $\overline{N}_1 \subset N$ and

(4.6) $f\overline{M \setminus M_2} \subset N_1, \quad fM_1 \cap \overline{N}_1 = \emptyset.$

Then choose an open neighborhood N_2 of $\overline{N}_1 \cup f\overline{M \setminus M_1}$ in Q with $\overline{N}_2 \subset N$. Choose $\eta_0 \in C_+(Q)$ such that if $\varphi_1, \ldots, \varphi_5 : Q \to Q$ are homeomorphisms in $U(\mathrm{id}, \eta_0)$, then $\varphi_1^{-1} \varphi_2^{-1} \varphi_3 \varphi_4 \varphi_5 \in U(\mathrm{id}, \varepsilon')$. Define a continuous $\mu : Q \to R_+^1$ by $\mu(x) = \min(\eta_0(x)/2, d(x, Q \setminus N_2))$. Let $\nu \in C_+(M)$ be the function given by [Cr, Theorem 8.1] with the substitution $M \mapsto Q$, $K \mapsto M$, $K_a \mapsto L$, $L \mapsto M_1$, $f \mapsto f$, $\mu \mapsto \mu$. That is, if $g_1, g_2 : M \to Q$ are PL embeddings in $U(f, \nu)$ such that $g_1 = g_2$ on M_1 and $g_i^{-1}\partial Q = L$, there is an ambient (PL) isotopy G of Q such that $G_1 g_1 = g_2$ and $d(G_t(x), x) \leqq \mu(x)$ for all $x \in Q$ and $t \in I$. Choose $\delta \in C_+(M)$ and $\eta \in C_+(Q)$, $\eta \leqq \eta_0$, such that if $\varphi \in U(f, \delta)$ is an embedding and $\varphi_1, \varphi_2 \in U(\mathrm{id}_Q, \eta)$, then φ is closed, $\varphi_1[Q \setminus N] \subset Q \setminus N_2$, (4.6) holds with f replaced by $\varphi_1 \varphi$, and $\varphi_2 \varphi_1 \varphi \in U(f, \nu)$.

We show that this δ is the desired function. Let f_1, f_2 be as in 4.5. Then, since $(f_i|M_2)^{-1}\partial Q = M_2 \cap L$ is a 1-submanifold of ∂M_2, the embedding $f_1|M_2 = f_2|M_2$ is locally flat, and, hence, by 4.4 there is a homeomorphism $g: Q \to Q$ in $U(\mathrm{id}, \eta)$ such that $f_i' = g f_i$, $i = 1, 2$, is PL on M_2. Since $(f_i')^{-1}[Q \setminus N_1] \subset \mathrm{int}_M M_2$, Lemma 4.4 gives an ambient η-isotopy G^i of Q rel $Q \setminus N_1$ such that $g_i = G_1^i f_i'$ is PL. We have $g_i \in U(f, v)$, $g_1|M_1 = g_2|M_1$, and $g_i^{-1}\partial Q = L$. Thus the second paragraph of the proof gives a certain isotopy G. Then by defining $H_t = g^{-1}(G_t^2)^{-1}G_t G_t^1 g$ for $t \in I$ we obtain an ambient ε'-isotopy H of Q rel $Q \setminus N$ such that $H_1 f_1 = f_2$. $\square$

The next lemma is needed for $n = 2$.

4.7. Lemma. *Let $N \subset R^2$ be a neighborhood of I and let $\varepsilon > 0$. Then there is $\delta > 0$ with the following property: If $f_1, f_2: [-1, 2] \to R^2$ are embeddings with $d(f_i, \mathrm{id}) < \delta$, there is a homeomorphism $h: R^2 \to R^2$ such that $d(h, \mathrm{id}) < \varepsilon$, $h|R^2 \setminus N = \mathrm{id}$, and $h f_1 = f_2$ on I. Moreover, if*

$$(4.8) \qquad\qquad f_1 = f_2 \quad on \quad [-1, 2] \setminus I,$$

h can be chosen such that $h f_1 = f_2$ on $[-1, 2]$.

Proof. Choose an integer $p \geq 2$ such that if $\eta = 1/p$, then $\eta < \varepsilon/8$ and $(-2\eta, 1 + 2\eta) \times (-\eta, \eta) \subset N$. Let $\delta = \eta/4$. Let $f_1, f_2: [-1, 2] \to R^2$ be embeddings with $d(f_i, \mathrm{id}) < \delta$. Choose arcs A_{ijk}, $i \in \{1, 2\}$, $j \in \{-1, 0, \ldots, p+1\}$, $k \in \{0, 1\}$, with endpoints $f_i(j\eta)$ and $(j\eta, (-1)^k \eta)$ such that

$$\mathrm{int}\, A_{ijk} \subset ((j-1/2)\eta, (j+1/2)\eta) \times (-\eta, \eta) \setminus f_i[-1, 2],$$

$$A_{1jk} = A_{2jk} \quad \text{for} \quad j \in \{-1, p+1\} \quad \text{in the case (4.8).}$$

Then the open arcs $\mathrm{int}\, A_{ijk}$ are disjoint. The arcs A_{ijk} can be found, for example, as follows. Let $f_i(a_{jk})$ be the point of $f_i[-1, 2] \cap \{j\eta\} \times [-\eta, \eta]$ closest to $(j\eta, (-1)^k \eta)$. We have $|a_{jk} - j\eta| < \delta$ and $f_i J_{jk} \subset (j\eta - 2\delta, j\eta + 2\delta) \times (-\delta, \delta)$, where $J_{jk} = [a_{jk}, j\eta]$ (or $[j\eta, a_{jk}]$). The arc A_{ijk} is now the union of a subarc of the segment $\{j\eta\} \times [-\eta, \eta]$ and an arc close to $f_i J_{jk}$.

There are Jordan domains

$$G_{ijk} \subset ((j-3/2)\eta, \quad (j+1/2)\eta) \times (-\eta, \eta)$$

for $i \in \{1, 2\}$, $j \in \{0, \ldots, p+1\}$, $k \in \{0, 1\}$ such that

$$\partial G_{ijk} = [(j-1)\eta, j\eta] \times \{(-1)^k \eta\} \cup A_{i(j-1)k} \cup f_i[(j-1)\eta, j\eta] \cup A_{ijk},$$

if (4.8) holds or if $j \notin \{0, p+1\}$, or such that

$$G_{ij0} = G_{ij1} \quad \text{for} \quad j \in \{0, p+1\},$$

$$\partial G_{i00} = [-\eta, 0] \times \{-\eta, \eta\} \cup \{-\eta\} \times [-\eta, \eta] \cup A_{i00} \cup A_{i01},$$

$$\partial G_{i(p+1)0} = [1, 1+\eta] \times \{-\eta, \eta\} \cup \{1+\eta\} \times [-\eta, \eta] \cup A_{ip0} \cup A_{ip1},$$

if (4.8) does not hold.

By the Schoenflies theorem there is a homeomorphism $h: R^2 \to R^2$ such that

$$h\bar{G}_{1jk} = \bar{G}_{2jk} \quad \text{for} \quad j \in \{0, ..., p+1\}, \quad k \in \{0, 1\},$$

$$h = \text{id} \quad \text{on} \quad R^2 \backslash \text{int}(\cup \{\bar{G}_{1jk} | j \in \{0, ..., p+1\}, \quad k \in \{0, 1\}\}),$$

$$hf_1 = f_2 \quad \text{on} \quad I,$$

$$hf_1 = f_2 \quad \text{on} \quad [-1, 2] \quad \text{in the case (4.8).}$$

Then $h|R^2 \backslash N = \text{id}$. Finally, since $\bar{G}_{1jk} \cap \bar{G}_{2jk} \neq \emptyset$ and $d(\bar{G}_{ijk}) < \varepsilon/2$ for all i, j, k, we have $d(h, \text{id}) < \varepsilon$. $\square$

The next two lemmas are the handle lemmas needed for 4.1.

4.9. **Lemma.** *Let $C \in \{\text{TOP}, \text{LQC}, \text{LIP}\}$, let $n \neq 4$, let $k, m \geq 0$ be integers with $k+m = n-1$, let $f: I^{n-1} \to R^n$ be an embedding, and let $\varepsilon > 0$. Then there is $\delta > 0$ with the following property: If $f_1, f_2: I^{n-1} \to R^n$ are locally C-flat embeddings such that $d(f_i, f) < \delta$ and $f_1 = f_2$ on $(I^k \backslash I^k(1/2)) \times I^m$, there is an ambient ε-isotopy H of R^n rel $R^n \backslash N(fI^{n-1}(1/2), \varepsilon)$ such that H_t is a C-homeomorphism for each $t \in I$ and $H_1 f_1 = f_2$ on $I^k \times I^m(1/2)$.*

Proof. The case $n = 1$ being trivial, we may assume $n \geq 2$. Consider first $C = \text{TOP}$. The case $n \geq 5$ is [PS, Theorem 2]. The case $n = 3$ follows from 4.5 (apply it to $f|I^k \times I^m(1/2)$). Let now $n = 2$. Since f can be extended to a homeomorphism $R^2 \to R^2$, we may assume that f is the inclusion map. Then 4.7 and Alexander's construction of isotopies (cf. [RS, Proposition 3.22]) give the lemma.

Consider now $C \neq \text{TOP}$. By [TV$_3$, Theorem 3.4] it suffices to construct $\delta > 0$ with the following weaker property (this reduction also holds for $C = \text{TOP}$): If f_1, f_2 are as in the lemma, there is a C-homeomorphism $h: R^n \to R^n$ such that $d(h, \text{id}) < \varepsilon$, $h = \text{id}$ on $Y = R^n \backslash N(fI^{n-1}(1/2), \varepsilon)$, and $hf_1 = f_2$ on $I^k \times I^m(1/2)$. We apply the case $C = \text{TOP}$ of the lemma and 3.1. We first choose $\eta \in (0, 1/2)$ and an open neighborhood Y_1 of Y such that $\bar{Y}_1 \cap fI^{n-1}(1/2+\eta) = \emptyset$. By the TOP case, there is $\delta > 0$ with the following property: If $f_1, f_2: I^{n-1} \to R^n$ are locally flat embeddings such that $d(f_i, f) < \delta$ and $f_1 = f_2$ on $(I^k \backslash I^k(1/2)) \times I^m$, there is a homeomorphism $g: R^n \to R^n$ such that $d(g, \text{id}) < \varepsilon/2$, $g = \text{id}$ near $\bar{Y}_1$, and $gf_1 = f_2$ on $U = J^k \times J^m(1/2+\eta)$. We may assume that ε and δ are so small that $f_1[I^{n-1} \backslash I^{n-1}(3/4)] \subset Y$.

We show that this δ is the desired number. Suppose that the last-mentioned f_1, f_2 are, in fact, locally C-flat. Choose an open set $V \subset R^n$ such that $f_1 U$ is a closed subset of V; then $f_2 U$ is closed in the open set gV. Define $\varepsilon_1 \in C_+(V)$ by $2\varepsilon_1(x) = \min(\varepsilon, d(g(x), R^n \backslash gV))$. We apply 3.1 (c) with the substitution $C \mapsto C$, $Q_1 \mapsto V$, $Q_2 \mapsto gV$, $A \mapsto \bar{Y}_1 \cap V$, $B, B' \mapsto V$, $M_i \mapsto f_i U$, $f \mapsto g|V$, $\varepsilon \mapsto \varepsilon_1$. This is possible since g extends the C-homeomorphism $f_2 f_1^{-1}: f_1 U \to f_2 U$. We obtain a C-homeomorphism $g': V \to gV$ such that $g' \in U(g|V, \varepsilon_1)$, $g' = g = \text{id}$ on $Y_1 \cap V$, and $g' = g$ on $f_1 U$. Extending g' by $g|R^n \backslash V$ we obtain a homeomorphism $g_1: R^n \to R^n$ such that

$d(g_1, \mathrm{id}) < \varepsilon$, $g_1|Y_1 = \mathrm{id}$, $g_1 f_1 = f_2$ on U, and $g_1|V$ is a C-embedding. By [TV_3, Theorem 4.4] there is a C-homeomorphism $h: R^n \to R^n$ such that $d(h, \mathrm{id}) < \varepsilon$, $h|Y = \mathrm{id}$, and $h = g_1$ near $f_1[I^k(3/4) \times I^m(1/2)]$. Then $hf_1 = f_2$ on $I^k \times I^m(1/2)$. □

4.10. Lemma. *Let C, n, k, m, ε be as in 4.9 with $m \geqq 1$ and let $f: I_+^{n-1} \to R^n$ be an embedding. Then there is $\delta > 0$ with the following property: If $f_1, f_2: I_+^{n-1} \to R^n$ are locally C-flat embeddings such that $d(f_i, f) < \delta$ and $f_1 = f_2$ on $(I^k \setminus I^k(1/2)) \times I_+^m$, there is an ambient ε-isotopy H of R^n rel $R^n \setminus N(fI_+^{n-1}(1/2), \varepsilon)$ such that H_t is a C-homeomorphism for each $t \in I$ and $H_1 f_1 = f_2$ on $I^k \times I_+^m(1/2)$.*

Proof. We reduce the lemma to 4.9. We begin with some preliminary constructions. Let $N = N(fI_+^{n-1}(1/2), \varepsilon)$. Let $\sigma: R^1 \to I^1$ and $\tau: R^1 \to I$ be the unique nondecreasing retractions; then we define a retraction $r: R^n \to I_+^{n-1}$ by $r(x) = (\sigma(x_1), \ldots, \sigma(x_{n-2}), \tau(x_{n-1}))$. For $s > 0$, let $B_s = I^{n-2}(1+s) \times [-s, 1+s] \times I^1(s) \subset R^n$ and define homeomorphisms $\alpha_s, \beta_s: R^1 \to R^1$ by

$$\alpha_s(t) = \begin{cases} st + s - 1, & t \leqq -1 \\ t, & -1 \leqq t \leqq 1 \\ st + 1 - s, & t \geqq 1 \end{cases} \qquad \beta_s(t) = \begin{cases} st, & t \leqq 0 \\ \alpha_s(t), & t \geqq 0. \end{cases}$$

Then $\psi_s: R^n \to R^n$, $\psi_s(x) = (\alpha_s(x_1), \ldots, \alpha_s(x_{n-2}), \beta_s(x_{n-1}), sx_n)$, is a PL homeomorphism such that $\psi_s|I_+^{n-1} = \mathrm{id}$, $\psi_s B_1 = B_s$, and $r\psi_s = r$. We choose $a \in (0, 1)$ such that $|fr(x) - fr(y)| < \varepsilon/15$ for all $x, y \in R^n$ with $|x - y| < a$. Consider $b \in (0, a/2)$ and the subset

$$A = I^k \times I^{m-1}(1/2 + b) \times [b, 1/2] \cup (I^k \setminus J^k(1/2 + b)) \times I^{m-1}(1/2 + b) \times [0, 1/2]$$

of I_+^{n-1}. It is obvious that if b is sufficiently small, then $fI_+^{n-1}(1/2 + b) \subset N$ and there is an ambient a-isotopy Γ of R^n rel $R^n \setminus N_0$, where $N_0 = N(I^{n-2}(1/2), a)$, such that Γ_t is PL and $\Gamma_t I_+^{n-1} \subset I_+^{n-1}$ for each $t \in I$ and that $\Gamma_1[I^k \times I_+^m(1/2)] \subset A$. We fix such a number b.

Obviously, there is a PL homeomorphism $\varphi: I^{n-1} \to I_+^{n-1}$ such that

$$\varphi[(I^k \setminus I^k(1/2)) \times I^m] = (I^k \setminus I^k(1/2)) \times I_+^m,$$

$$\varphi I^{n-1}(1/2) = I^k(1/2) \times I^{m-1}(1/2 + b) \times [b, 1/2] \subset I_+^{n-1}(1/2 + b),$$

$$\varphi[I^k \times I^m(1/2)] = A.$$

Thus by 4.9 there is $\delta_0 > 0$ with the following property: If $f_1, f_2: I_+^{n-1} \to R^n$ are locally C-flat embeddings such that $d(f_i, f) < \delta_0$ and $f_1 = f_2$ on $(I^k \setminus I^k(1/2)) \times I_+^m$, there is an ambient $(\varepsilon/3)$-isotopy H^0 of R^n rel $R^n \setminus N$ such that H_t^0 is a C-homeomorphism for each $t \in I$ and that $H_1^0 f_1 = f_2$ on A, whence $H_1^0 f_1 \Gamma_1 = f_2 \Gamma_1$ on $I^k \times I_+^m(1/2)$.

We show that for every locally C-flat embedding $g: I_+^{n-1} \to R^n$ with $d(g, f) < \varepsilon/15$ there is an ambient $(\varepsilon/3)$-isotopy G of R^n rel $R^n \setminus N$ such that G_t is a C-homeomorphism for each $t \in I$ and $G_1 g = g\Gamma_1|I_+^{n-1}$. By [Ru, Corollary 3.4.1]

if $C = \mathrm{TOP}$ or by [GV, Theorem 4.3] if $C \neq \mathrm{TOP}$, there is a C-homeomorphism $g': R^n \to R^n$ extending g. Since $d(r|B_s, \mathrm{id}) \to 0$ as $s \to 0$, there is $s > 0$ such that $d(g'|B_s, gr|B_s) < \varepsilon/15$. Then $\bar{g} = g'\psi_s: R^n \to R^n$ is a C-homeomorphism extending g with $d(\bar{g}|B_1, gr|B_1) = d(g'\psi_s|B_1, gr\psi_s|B_1) = d(g'|B_s, gr|B_s) < \varepsilon/15$. We set $G_t = \bar{g}\Gamma_t\bar{g}^{-1}$ for $t \in I$. Then $G_t|R^n \setminus \bar{g}N_0 = \mathrm{id}$ and $G_t g = g\Gamma_t|I_+^{n-1}$. If $x \in R^n$, $y \in I^{n-2}(1/2)$, and $|x - y| < a$,

$$|\bar{g}(x) - f(y)| \leqq |\bar{g}(x) - gr(x)| + |gr(x) - fr(x)| + |fr(x) - fr(y)|$$
$$< 3 \cdot \varepsilon/15 = \varepsilon/5,$$

whence $\bar{g}N_0 \subset N$. If $x \in N_0 \subset B_1$,

$$|\bar{g}\Gamma_t(x) - \bar{g}(x)| \leqq 2d(\bar{g}|B_1, gr|B_1) + 2d(g, f) + |fr\Gamma_t(x) - fr(x)|$$
$$< 5 \cdot \varepsilon/15 = \varepsilon/3.$$

Hence G is the desired isotopy.

We show that $\delta = \min(\delta_0, \varepsilon/15)$ satisfies the lemma. Let f_1, f_2 be as in the lemma. By what we have proved, there are ambient $(\varepsilon/3)$-isotopies H^0, H^1, H^2 of R^n rel $R^n \setminus N$ such that H_t^i is a C-homeomorphism for all $t \in I$ and $i = 0, 1, 2$, that $H_1^0 f_1 \Gamma_1 = f_2 \Gamma_1$ on $I^k \times I_+^m(1/2)$, and that $H_1^i f_i = f_i \Gamma_1|I_+^{n-1}$, $i = 1, 2$. Then $H = (H^2)^{-1} * H^0 * H^1$ is the desired isotopy. $\square$

The rest of the proof of 4.1 will now be standard. The next lemma is the second stage of this proof.

4.11. Lemma. *Let $C \in \{\mathrm{TOP}, \mathrm{LQC}, \mathrm{LIP}\}$ and $n \neq 4$, let G be an open subset of R_+^{n-1}, let A, B, U be open in G such that $\bar{A} \cap G \subset U$, $\bar{B} \subset G$, and $\bar{B}$ is compact, let $f: G \to R^n$ be an embedding, and let $\varepsilon > 0$. Then there is $\delta > O$ with the following property: If $f_1, f_2: G \to R^n$ are locally C-flat embeddings such that $d(f_i, f) < \delta$ and $f_1|U = f_2|U$, there is an ambient ε-isotopy φ of R^n rel $R^n \setminus N(fB, \varepsilon)$ such that φ_t is a C-homeomorphism for each $t \in I$ and $\varphi_1 f_1 = f_2$ on $A \cup B$.*

Proof. Let $A_1 = \bar{A} \cap G$ and $N = N(fB, \varepsilon)$. Choose a compact PL $(n-1)$-manifold $P \subset f^{-1}N$ with $\bar{B} \subset \mathrm{int}_G P$. Choose a triangulation (K, K_0) of $(P, \partial P)$. Let K' be the first barycentric subdivision of K and K'' the second one. For $\sigma \in K$, let $\hat{\sigma}$ be the barycenter of σ, $H_\sigma = \cup\{\tau | \hat{\sigma} \in \tau \in K''\}$, and $i(H_\sigma) = \dim \sigma$. Write $K = \{\sigma_1, \ldots, \sigma_r\}$ with $\dim \sigma_j \leqq \dim \sigma_{j+1}$. Then $\mathscr{H} = \{H_{\sigma_1}, \ldots, H_{\sigma_r}\}$ is a handle decomposition of P as in [H, pp. 224—227], [RS, p. 82] or [Ru, pp. 31—33]. In fact, if $\sigma \in K$, $P_\sigma = \cup\{H_\tau | \tau \in K, \dim \tau < \dim \sigma\}$, $\tilde{H}_\sigma = \mathrm{cl}\,((\cup\{\tau | \hat{\sigma} \in \tau \in K'\}) \setminus P_\sigma)$, $k = i(H_\sigma)$, and $m = n - 1 - k$ (thus $m \geqq 1$ whenever $\sigma \in K_0$), it is easy to show that $(\tilde{H}_\sigma, H_\sigma, \tilde{H}_\sigma \cap P_\sigma, \tilde{H}_\sigma \cap \partial P)$ is PL homeomorphic either to $(I^k \times 2I^m, I^k \times I^m, \partial I^k \times 2I^m, \emptyset)$ whenever $\sigma \in K \setminus K_0$ or to $(I^k \times 2I_+^m, I^k \times I_+^m, \partial I^k \times 2I_+^m, I^k \times 2I^{m-1})$ whenever $\sigma \in K_0$. Setting $H_\sigma \leqq H_\tau$ if $H_\sigma \cap H_\tau \neq \emptyset$ and $i(H_\sigma) \leqq i(H_\tau)$ (i.e., if σ is a face of τ) we obtain a partial order for $\mathscr{H}$. Let $\mathscr{H}_0$ and $\mathscr{H}_1$, respectively, be the sets of the handles $H \in \mathscr{H}$ such that there is $H' \in \mathscr{H}$ with $H \leqq H'$ and with either

$H' \cap A_1 \neq \emptyset$ or $H' \cap \bar{B} \neq \emptyset$, respectively. Then $H \leqq H' \in \mathcal{H}_i$ implies $H \in \mathcal{H}_i$ for $i = 0, 1$. We may assume that $\cup \mathcal{H}_0 \subset U$ and $\cup \mathcal{H}_1 \subset \text{int}_G P$. Observe that $(\text{int}_G P) \cap \partial P = (\text{int}_G P) \cap R^{n-2}$. Write $\mathcal{H}_1 \setminus \mathcal{H}_0 = \{H_1, \ldots, H_s\}$ with $i(H_j) \leqq i(H_{j+1})$. Let $\mathcal{H}_j' = (\mathcal{H}_0 \cap \mathcal{H}_1) \cup \{H_1, \ldots, H_j\}$ and $P_j = \cup \mathcal{H}_j'$ for $0 \leqq j \leqq s$. Then $P_s \supset B$. For each $H \in \mathcal{H}$ with $H \cap H_j \neq \emptyset$ we have $i(H) < i(H_j)$ if and only if $H \in \mathcal{H}_{j-1}'$.

We set $U_0 = U \supset A_1 \cup P_0$. Inductively, for each $j \in \{1, \ldots, s\}$ we first choose either a PL embedding $\mu_j \colon I^{n-1} \to f^{-1}N \setminus A_1$ whenever $H_j \cap R^{n-2} = \emptyset$ or a PL embedding $\mu_j \colon I_+^{n-1} \to f^{-1}N \setminus A_1$ whenever $H_j \cap R^{n-2} \neq \emptyset$ and then define an open neighborhood U_j of $A_1 \cup P_j$ in G such that, setting $k = i(H_j)$, $m = n - 1 - k$, we have either

$$\mu_j^{-1} R^{n-2} = \emptyset,$$

$$\mu_j[I^k \times I^m(1/3)] = H_j,$$

$$\mu_j I^{n-1} \cap P_{j-1} = \mu_j[\partial I^k \times I^m],$$

$$\mu_j[(I^k \setminus I^k(1/2)) \times I^m] \subset U_{j-1},$$

$$U_j = (U_{j-1} \setminus \mu_j[I^k(2/3) \times I^m]) \cup \mu_j[I^k \times J^m(1/2)]$$

whenever $H_j \cap R^{n-2} = \emptyset$ or

$$\mu_j^{-1} R^{n-2} = I^{n-2},$$

$$\mu_j[I^k \times I_+^m(1/3)] = H_j,$$

$$\mu_j I_+^{n-1} \cap P_{j-1} = \mu_j[\partial I^k \times I_+^m],$$

$$\mu_j[(I^k \setminus I^k(1/2)) \times I_+^m] \subset U_{j-1},$$

$$U_j = (U_{j-1} \setminus \mu_j[I^k(2/3) \times I_+^m]) \cup \mu_j[I^k \times J_+^m(1/2)]$$

whenever $H_j \cap R^{n-2} \neq \emptyset$. In both cases, the image of μ_j is locally PL flat in G.

Choose numbers $0 < \delta_j \leqq \varepsilon_j \leqq \varepsilon/s$ for $1 \leqq j \leqq s$ such that $\varepsilon_j \leqq \delta_{j+1}/2$ for $j < s$, that either

$$N_j = N(f \mu_j I^{n-1}(1/2), \varepsilon_j) \subset N \setminus N(f[G \setminus \mu_j[I^k(2/3) \times I^m]], \delta_j)$$

if $H_j \cap R^{n-2} = \emptyset$ or

$$N_j = N(f \mu_j I_+^{n-1}(1/2), \varepsilon_j) \subset N \setminus N(f[G \setminus \mu_j[I^k(2/3) \times I_+^m]], \delta_j)$$

if $H_j \cap R^{n-2} \neq \emptyset$, where k, m are as above, and that δ_j is the number given either by 4.9 if $H_j \cap R^{n-2} = \emptyset$ or by 4.10 if $H_j \cap R^{n-2} \neq \emptyset$ with the substitution $(C, k, m) \mapsto (C, k, m)$, $f \mapsto f \mu_j$, $\varepsilon \mapsto \varepsilon_j$. Then $\delta = \delta_1$ satisfies the lemma. In fact, let f_1, f_2 be as in the lemma. Then it follows inductively that there exist ambient ε_j-isotopies φ^j, $1 \leqq j \leqq s$, of R^n rel $R^n \setminus N_j$ such that φ_t^j is a C-homeomorphism for all t and $\varphi_1^j \ldots \varphi_1^1 f_1 = f_2$ on U_j. Hence $\varphi = \varphi^s * \ldots * \varphi^1$ is the desired isotopy. $\square$

4.12. *Proof of 4.1 for* $n \geqq 2$. We may assume that B is closed and $U \subset B$. Choose an open set $Q_0 \subset Q$ such that f defines a closed embedding $f' \colon B \to Q_0$, $x \mapsto f(x)$. We may assume that $N \subset \text{int } Q_0$ and that N is closed either in Q_0 if

$C = \text{TOP}$ or in Q if $C \neq \text{TOP}$. Then there is $\varepsilon' \in C_+(Q_0)$ such that $\varepsilon'|N \leq \varepsilon$ and $\varepsilon'(x) \leq d(x, Q \setminus Q_0)/2$ for each $x \in Q_0$. There is a locally finite family $\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}'$ of open subsets of M in M, where $\mathscr{V}_0 = \{U\}$, such that $A \subset \cup \mathscr{V}$ and that for each $V \in \mathscr{V}'$, V is C-homeomorphic to an open set in R_+^{n-1}, $\overline{V}$ is compact in B, and $f\overline{V}$ is contained in an open subset $Q_V \subset N$ of Q C-homeomorphic to R^n. By [Mu, Lemma 2.7] we may assume that $\mathscr{V}'$ has a partition $\mathscr{V}' = \mathscr{V}_1 \cup \ldots \cup \mathscr{V}_n$ with each family $\mathscr{V}_j$ being disjoint. We choose for each $V \in \mathscr{V}$ open sets $V = V^{-1} \supset V^0 \supset \ldots \supset V^n$ such that $\overline{V}^j \subset V^{j-1}$ and that $A \subset \cup \{V^n | V \in \mathscr{V}\}$. Define open sets

$$D_j = \cup \{V^{j-i} | 0 \leq i \leq j, V \in \mathscr{V}_i\} \subset B \quad (0 \leq j \leq n),$$

$$D_j' = \cup \{V^{j-i+1} | 0 \leq i \leq j, V \in \mathscr{V}_i\} \subset B \quad (0 \leq j < n),$$

in M. Then $D_0 = U^0$, $D_n \supset A$, and for each $1 \leq j \leq n$ we have $\overline{D}_{j-1}' \subset D_{j-1}$ and

$$D_j = D_{j-1}' \cup (\cup \{V^0 | V \in \mathscr{V}_j\}).$$

Consider $j \in \{1, \ldots, n\}$ and a function $\varepsilon_j \in C_+(Q_0)$, to be specified below. For each $V \in \mathscr{V}_j$ we choose $\varepsilon_V > 0$ such that $N_V = \text{cl } N(fV^0, \varepsilon_V)$ is compact in Q_V, $\varepsilon_V \leq \min \varepsilon_j N_V$, $N_V \cap f[B \setminus V] = \emptyset$, and that $\{N_V | V \in \mathscr{V}_j\}$ is a locally finite disjoint family in Q_0. It is easy to see that 4.11 also holds if the embedding $f : G \to R^n$ is replaced by the embedding $f|V : V \to Q_V$ for $V \in \mathscr{V}_j$. Let $\delta_V > 0$ with $\delta_V \leq d(fV, Q \setminus Q_V)$ be the number given by this version of 4.11 with the substitution $A \mapsto D_{j-1}' \cap V$, $B \mapsto V^0$, $U \mapsto D_{j-1} \cap V$, $\varepsilon \mapsto \varepsilon_V$. There is $\delta_j \in C_+(B)$ such that $\max \delta_j \overline{V} \leq \delta_V$ and $\delta_j(x) \leq d(f(x), N_V)$ for all $V \in \mathscr{V}_j$ and $x \in B \setminus V$. We fix the functions $\varepsilon_1, \delta_1, \ldots, \varepsilon_n, \delta_n$ so small that

$$U(\text{id}_{Q_0}, \varepsilon_n) \circ \ldots \circ U(\text{id}_{Q_0}, \varepsilon_1) \subset U(\text{id}_{Q_0}, \varepsilon'),$$

$$U(\text{id}_{Q_0}, \varepsilon_j) \circ U(f', \delta_j) \subset U(f', \delta_{j+1}) \quad \text{if} \quad 1 \leq j < n.$$

There is $\delta \in C_+(M)$ such that $\delta(x) \leq \min(\delta_1(x), d(f(x), Q \setminus Q_0))$ for each $x \in B$ and $N(A, \delta) \subset D_n$.

We show that this δ satisfies the theorem. Let f_1, f_2 be as in 4.1. Then f_i defines an embedding $f_i' : B \to Q_0$, $x \mapsto f_i(x)$. We construct inductively ambient ε_j-isotopies H^j, $1 \leq j \leq n$, of Q_0 rel $Q_0 \setminus N$ such that H_t^j is a C-homeomorphism for all t and j and that $H_1^j \ldots H_1^1 f_1' = f_2'$ on D_j for each j. The desired isotopy H is then the union of $H^n * \ldots * H^1$ and of the trivial ambient isotopy of $Q \setminus Q_0$. Let $1 \leq j \leq n$. The embedding $g_{j-1} = H_1^{j-1} \ldots H_1^1 f_1' : B \to Q_0$ $(g_0 = f_1')$ is locally C-flat on $\text{int}_M B$, $g_{j-1}, f_2' \in U(f', \delta_j)$, and $g_{j-1} = f_2'$ on D_{j-1}. Then for each $V \in \mathscr{V}_j$ there is, by the definition of δ_V, an ambient ε_V-isotopy H^V of Q_0 rel $Q_0 \setminus N_V$ such that H_t^V is a C-homeomorphism for each t and $H_1^V g_{j-1} = f_2'$ on $D_j \cap V$. We set $H_t^j(x) = H_t^V(x)$ for each $t \in I$ whenever there is $V \in \mathscr{V}_j$ with $x \in N_V$ and $H_t^j(x) = x$ otherwise. Then H^j is an ambient ε_j-isotopy of Q_0 rel $Q_0 \setminus N$ such that H_t^j is a C-homeomorphism for each t. We finally show that if $x \in D_j$, then $H_1^j g_{j-1}(x) = f_2'(x)$. If $x \in V \in \mathscr{V}_j$, then for each $W \in \mathscr{V}_j$, $W \neq V$, we have $x \notin W$,

whence $d\big(g_{j-1}(x), f(x)\big) < \delta_j(x) \leqq d(f(x), N_W)$, and thus $g_{j-1}(x) \notin N_W$. Hence $H_1^j g_{j-1}(x) = H_1^V g_{j-1}(x) = f_2'(x)$. If $x \notin \cup \mathscr{V}_j$, then $g_{j-1}(x) \notin \cup \{N_V | V \in \mathscr{V}_j\}$ and $x \in D_{j-1}'$, whence $H_1^j g_{j-1}(x) = g_{j-1}(x) = f_2'(x)$. $\quad\square$

4.13. Lemma. *Let* $Q = R_+^1 \times R^1$, *let* $f : [0, 2] \to Q$ *be an embedding with* $f^{-1}\partial Q = \{0\}$, *and let* $\varepsilon > 0$. *Then there is* $\delta > 0$ *with the following property: If* $f_1, f_2 : [0, 2] \to Q$ *are embeddings such that* $f_i^{-1}\partial Q = \{0\}$ *and* $d(f_i, f) < \delta$, *there is an* ε-*push* h *of* (Q, fI) *with* $hf_1 = f_2$ *on* I.

Proof. Since f can be extended to a selfhomeomorphism of Q, it suffices to prove the modification of 4.7, with the additional assertion omitted, where $(R^2, [-1, 2])$ is replaced by $(Q, [0, 2])$ and where $f_i^{-1}\partial Q = \{0\}$ (it is not difficult to construct the isotopy). The proof is otherwise the same as earlier, but we now define A_{i0k} to be the segment in ∂Q whose endpoints are $f_i(0)$ and $(0, (-1)^k \eta)$. $\quad\square$

4.14. Lemma. *The modification of* 4.10 *is true where* $n = 2$ *or* 3, R^n *is replaced by* R_+^n, $f_i^{-1}R^{n-1} = f^{-1}R^{n-1} = I^{n-2}$, *and, whenever* $C \neq \mathrm{TOP}$, H_t *is not claimed to be a* C-*homeomorphism for* $t \neq 1$.

Proof. For $C = \mathrm{TOP}$, $n = 2$, this is proved in 4.13. The case $C = \mathrm{TOP}$, $n = 3$ follows from 4.5. For $C \neq \mathrm{TOP}$, the lemma can be proved by the aid of the TOP case, 3.1(b), and [TV$_3$, Theorem 4.8] as the case $C \neq \mathrm{TOP}$ of 4.9 (construct the isotopy gradually). $\quad\square$

4.15. Lemma. *The modification of* 4.11 *is true where* $n = 2$ *or* 3, R^n *is replaced by* R_+^n, $f_i^{-1}R^{n-1} = f^{-1}R^{n-1} = G \cap R^{n-2}$, *and, whenever* $C \neq \mathrm{TOP}$, φ_t *is not claimed to be a* C-*homeomorphism for* $t \neq 1$.

Proof. The proof is a slight modification of that of 4.11; Lemmas 4.9 and 4.14 are applied. $\quad\square$

4.16. Lemma. *The modification of* 4.9 *is true where* $C = \mathrm{TOP}$, $n = 3$, $k = 0$, $(R^n, I^{n-1}, I^{n-1}(1/2))$ *is replaced by* $(R_+^3, I \times I, [0, 1/2] \times [0, 1/2])$, *and* $f_i^{-1}R^2 = f^{-1}R^2 = I$.

Proof. This follows from 4.5. $\quad\square$

4.17. Lemma. *The modification of* 4.11 *is true where* $C = \mathrm{TOP}$, $n = 3$, (R^n, R_+^{n-1}) *is replaced by* $(R_+^3, R_+^1 \times R_+^1)$, *and* $f_i^{-1}R^2 = f^{-1}R^2 = G \cap R_+^1$.

Proof. The proof is a slight modification of that of 4.11; Lemmas 4.9, 4.10, 4.14, and 4.16 are applied. $\quad\square$

4.18. Proof of 4.2 for $n \geqq 2$. The proof is an obvious modification of 4.12; Lemmas 4.11, 4.15, and, whenever $C = \mathrm{TOP}$, $n = 3$, also 4.17 are applied. $\quad\square$

5. Locally C-flat approximation

In this section we first prove the following three theorems. The first two of them deal with the relative locally C-flat approximation of embeddings or of immersions, respectively, and the third one with the extension of locally C-flat approximations. We then give two applications, 5.10 and 5.11, on "LQC or LIP taming" of locally flat embeddings. For a discussion on 5.1—5.3, see 5.4.

5.1. Theorem. *Let* $C \in \{\text{TOP}, \text{LQC}, \text{LIP}\}$ *and* $n \neq 4$, *let* M *be an* $(n-1)$- *and* Q *an n-dimensional C-manifold with* $\partial Q = \emptyset$, *let* $A \subset M$ *be closed, let* $f : M \to Q$ *be an embedding which is locally C-flat near* A, *and let* $\varepsilon \in C_+(M)$. *Then there is a locally C-flat embedding* $f^* : M \to Q$ *in* $U(f, \varepsilon)$ *such that* $f^* = f$ *near* A.

5.2. Theorem. *Let* C, n, M, Q, A *be as in* 5.1, *let* $B \subset M$ *be closed and* $B' \subset M$ *open with* $B \subset B'$, *let* $f : M \to Q$ *be an immersion which is locally C-flat near* A, *and let* $\varepsilon \in C_+(B')$. *Then there is an immersion* $f^* : M \to Q$ *such that* $f^* = f$ *near* A *and on* $M \setminus B'$, *that* $f^* | B' \in U(f | B', \varepsilon)$, *and that* f^* *is locally C-flat near* $A \cup B$.

5.3. Theorem. *Let* $C, n, M, Q, A, \varepsilon$ *be as in* 5.1, *let* $U \subset M$ *be an open neighborhood of* A, *and let* $f : M \to Q$ *be an embedding. Then there is* $\delta \in C_+(M)$ *with the following property: If* $f_0 : U \to Q$ *is a locally C-flat embedding in* $U(f | U, \delta | U)$, *there is a locally C-flat embedding* $f^* : M \to Q$ *in* $U(f, \varepsilon)$ *such that* $f^* = f_0$ *near* A.

5.4. Remarks. 1. Theorem 5.1 implies immediately the following version of 5.2, which generalizes 5.1: Suppose that f in 5.2 is an embedding (and $B = \emptyset$); then f^* can be chosen to be an embedding such that $f^* | B'$ is locally C-flat. The sets B and B' of 5.1 and 5.2 could similarly be included in the analogous result [LT, Theorem 4.4]; see 5.9.

2. Several special cases of 5.1 and 5.2 are known. In particular, for $C = \text{TOP}$, $n \geq 5$, and $A = \emptyset$, Theorem 5.1 has been proved by Ancel and Cannon [AC, p. 61]. (In [AC] this result is only stated in the case $\partial M = \emptyset$, but its general case [D, Theorem 5B.1] is obtained as follows. Using a collar of ∂M in M construct a closed locally flat embedding $h : M \to \text{int } M$ sufficiently close to id. Choose a locally flat embedding $g : \text{int } M \to Q$ sufficiently close to $f | \text{int } M$. Then $f^* = gh$ is the desired embedding.) Theorems 5.1 and 5.2 hold with $f^* = f$ in the case $n = 1$ trivially and in the case $C = \text{TOP}$, $n = 2$ by the Schoenflies theorem. For $C \in \{\text{LQS}, \text{LIP}\}$ and $n = 2$, 5.1 and 5.2 follow from the relative C-approximation result [LT, Theorem 4.4] (cf. 5.4.1) and the C-flatness of C-embeddings $I^1 \to R^2$ ([Ri, Theorem 1], [T₂, Theorem B]); however, we will give a new proof for this special case. Theorem 5.1 for $C = \text{TOP}$, $n = 3$ follows easily from Bing's results [B₁, Theorem 9] and [B₂, Theorems 7 and 8]. In the case $C \in \{\text{LQC}, \text{LIP}\}$, $n = 3$, and $A = \emptyset$, we may assume that M and Q are PL manifolds; hence 5.1 now follows from the fact that f can even be

approximated by PL embeddings, by the case $C =$ TOP and $[B_1$, Theorem 6 and its proof].

3. To prove 5.1 and 5.3 for $C =$ TOP, $n \geqq 5$, we apply, together with 4.1, the Ancel—Cannon theorem in its general form, mentioned above. However, a slight reorganization of this section shows that the special case $(M, Q) = (R^{n-1}, R^n)$ of this result would suffice. In fact, the special case $(M, Q) = (I^{n-1}, R^n)$ is enough in a more laborious proof.

5.5. Lemma. *Theorem 5.3 holds if* $C =$ TOP.

Proof. We may assume that f is closed. Then there are $\eta \in C_+(Q)$ and $\delta_0 \in C_+(M)$ with $U(\mathrm{id}_Q, \eta) \circ U(f, \delta_0) \subset U(f, \varepsilon)$. Choose a closed neighborhood B of A with $B \subset U$. Let $\delta_1 \in C_+(U)$ be the function given by 4.1 with the substitution $C \mapsto$ TOP, $n \mapsto n$, $M \mapsto U$, $Q \mapsto Q$, $A \mapsto A$, $B \mapsto B$, $U \mapsto \emptyset$, $f \mapsto f|U$, $N \mapsto Q$, $\varepsilon \mapsto \eta$. There is $\delta \in C_+(M)$ such that $\delta \leqq \delta_0$ and $\delta|B \leqq \delta_1|B$. We show that this δ satisfies 5.3. Let f_0 be as in 5.3. By the special case $C =$ TOP, $A = \emptyset$ of 5.1, which is known to be true (see 5.4.2), there is a locally flat embedding $f_1 : M \to Q$ in $U(f, \delta)$. Then by 4.1 there is a homeomorphism $h : Q \to Q$ in $U(\mathrm{id}, \eta)$ such that $hf_1 = f_0$ near A. Hence $f^* = hf_1$ is the desired embedding.   □

5.6. Lemma. *Theorem 5.1 holds if* $C =$ TOP.

Proof. This follows from 5.5.   □

5.7. Lemma. *For* $C \neq$ TOP, *5.1 holds if* $M = R^{n-1}$ *or* R_+^{n-1} *and if there is an embedding* $F : R^n \to Q$ *extending* f.

Proof. Choose an open neighborhood U of A in R^n such that $f|U \cap M$ is locally C-flat. By 3.1 if $M = R^{n-1}$ or by 3.3 if $M = R_+^{n-1}$, there is a C-embedding $F_1 : U \to Q$ extending $f|U \cap M$ such that F_1 can be extended by $F|R^n \setminus U$ to an embedding $F_2 : R^n \to Q$. Then $F_2|M = f$. By $[TV_3$, Theorem 4.4] there is a C-embedding $F^* : R^n \to Q$ such that $F^*|M \in U(f, \varepsilon)$ and $F^* = F_2$ near A. Then $f^* = F^*|M$ is the desired locally C-flat embedding.   □

5.8. Lemma. *Let* $C \in \{$TOP, LQC, LIP$\}$ *and* $n \neq 4$, *let* (X, Y) *be either* $(2J^{n-1}, J^{n-1})$ *or* $(2J_+^{n-1}, J_+^{n-1})$, *let* $f : X \to R^n$ *be an embedding which is locally* C-*flat near a closed subset* A *of* X, *let* $B \subset Y$ *be compact, and let* $\varepsilon > 0$. *Then there is an embedding* $f^* : X \to R^n$ *such that* $f^* = f$ *on* $X \setminus Y$ *and near* $A, d(f^*, f) < \varepsilon$, *and* f^* *is locally* C-*flat near* $A \cup B$.

Proof. Choose a closed neighborhood A_1 of A in X such that f is locally C-flat near A_1. By 5.6 there is a locally flat embedding $g : Y \to R^n$ such that $g = f$ on $A_1 \cap Y$ and that g is so close to $f|Y$ that $f_1 = g \cup (f|X \setminus Y)$ is an embedding with $d(f_1, f) < \varepsilon$. Then $f_1 = f$ on A_1. Hence $f^* = f_1$ satisfies the lemma for $C =$ TOP.

Consider $C \in \{LQC, LIP\}$. We may replace f by f_1 and, hence, assume that $f|Y$ is locally flat. Choose $r \in (0, 1)$ with $B \subset rY$. Then $f|r\overline{Y}$ is locally flat. We may well replace Y by rY in the lemma. Thus we may assume that $f|\overline{Y}$ is locally flat. Then by [Ru, Corollary 3.4.1] there is a homeomorphism $F: R^n \to R^n$ extending $f|\overline{Y}$. Thus, if we repeat the above construction of f_1, we may assume by 5.7 that g is locally C-flat. Then again $f^* = f_1$ satisfies the lemma. $\square$

5.9. *Proof of* 5.1, 5.2, *and* 5.3. For $C = TOP$, the proof is only needed for 5.2 by 5.5 and 5.6. The joint proof of 5.1 and of the special case $B = M$ of 5.2 is almost the same as that of [LT, Theorem 4.4]; it is only based on 5.8 in place of the analogous [LT, Lemma 4.2]. To obtain the general case of 5.2, we modify this proof as follows: In the notation of [LT], we choose the family $\mathscr{V}$ in such a way that $\cup \hat{\mathscr{V}} \subset B'$, where $\hat{\mathscr{V}} = \{V \in \mathscr{V} | V \cap B \neq \emptyset\}$. In the construction of the sets C^i and the immersions f_i we then replace $\mathscr{V}$ by $\hat{\mathscr{V}}$ and $\mathscr{V}_i$ by $\mathscr{V}_i \cap \hat{\mathscr{V}}$ for $1 \leq i \leq k$.

Theorem 5.3 for $C \neq TOP$ now follows from 4.1 and 5.1 as in 5.5 for $C = TOP$. $\square$

5.10. **Theorem.** *Let* $C \neq TOP$, n, M, Q, A *be as in* 5.1, *let* $f: M \to Q$ *be a locally flat embedding which is locally C-flat near* A, *and let* $\varepsilon \in C_+(Q)$. *Then there is an ε-push h of* $(Q, f[M \setminus A])$ *such that* $hf = f$ *near* A *and* hf *is locally C-flat.*

Proof. Choose open sets $U, V \subset M$ such that $A \subset V \subset \overline{V} \subset U$ and $f|U$ is locally C-flat. By 5.1 for every $\delta \in C_+(M)$ there is a locally C-flat embedding $g: M \to Q$ in $U(f, \delta)$ with $g|V = f|V$. By 4.1 there is an ε-push h of $(Q, f[M \setminus A])$ such that $hf = g$, provided that δ is small enough. Then h is the desired homeomorphism. $\square$

5.11. **Theorem.** *Let* $C \in \{LQC, LIP\}$ *and* $n \neq 4, 5, 6$, *let* M *be an* $(n-1)$- *and* Q *an n-manifold with* $\partial Q = \emptyset$, *and let* $f: M \to Q$ *be a locally flat embedding. Then* M *and* Q *can be given C-structures such that* f *becomes locally C-flat.*

Moreover, if M *(or* ∂M, *respectively) is already a C-manifold, this can be done without changing the C-structure of* M *(of* ∂M, *respectively) for all* $n \neq 4$ *($n \neq 4, 5$, respectively).*

Proof. We may assume that M and Q are C-manifolds by [TV_3, Theorems 4.6 and 4.8] and, whenever $\partial M \neq \emptyset$ and the parenthetic assertion is considered, by 8.2. Then 5.11 follows from 5.10. $\square$

An application of 5.11 is given in 8.3.2.

6. Locally LIP flat approximation of quasisymmetric
or bilipschitz embeddings

In this section we prove the following two theorems concerning, respectively, quasisymmetric embeddings and bilipschitz embeddings.

6.1. Theorem. *Let $n\neq 4$ and let $\eta: R^1_+ \to R^1_+$ be a homeomorphism. Then there is a homeomorphism $\eta^*: R^1_+ \to R^1_+$ with the following property: Let either $Y \subset X \subset R^{n-1}$ with Y open in R^{n-1} or $Y \subset X \subset R^{n-1}_+$ with Y open in R^{n-1}_+, let $f: X \to R^n$ be an η-quasisymmetric embedding, and let $\varepsilon \in C_+(Y)$. Then there is an η^*-quasisymmetric embedding $f^*: X \to R^n$ such that*

(1) $f^*|X \setminus Y = f|X \setminus Y$,

(2) $f^*|Y \in U(f|Y, \varepsilon)$,

(3) $f^*|Y$ *is locally* LIP *flat.*

6.2. Theorem. *Let $n\neq 4$ and let $L \geq 1$. Then there is $L^* \geq 1$ with the following property: Let X, Y, ε be as in 6.1 and let $f: X \to R^n$ be an L-bilipschitz embedding. Then there is an L^*-bilipschitz embedding $f^*: X \to R^n$ satisfying the conditions* (1), (2), *and* (3) *of* 6.1.

6.3. Remarks. 1. Theorems 6.1 and 6.2 are trivial for $n=1$. They are new for $n \geq 5$ only because of their analogues mentioned in the Introduction and proved in [LT, Corollary 2.20 and Theorem 3.2]. In the proofs for $n \geq 2$ we closely follow [LT]; we use Carleson's finiteness idea [Ca] (also used in [V₂], [T₁], [T₂], [Kk], [TV₂], [TV₃]) but replace the PL methods of [LT] by 5.3 for LIP. As in [LT], in 6.1 and 6.2 we could suppose about (X, Y) more generally that $X \subset R^{n-1}$ and that Y is only an open subset of X which is the union of a subfamily, locally finite in Y, of the cube family $\mathscr{L}$ of 6.4 and which is a manifold.

2. One obtains easily an analogue of 6.1 and 6.2 for weakly quasisymmetric embeddings, defined in [TV₁]; cf. [LT, Theorem 2.22].

6.4. *Cubical decompositions.* Let $n \geq 2$, let Y be an open subset of R^{n-1} or of R^{n-1}_+, let $\varphi: Y \to Z$ be a continuous map into a metric space, and let $\varepsilon \in C_+(Y)$. We define a decomposition $\mathscr{K}$ of Y into closed $(n-1)$-cubes as follows. Let $\mathscr{L}$ be the family of all closed $(n-1)$-cubes in R^{n-1} with side length 2^{-k} and with vertices in $2^{-k}Z^{n-1}$, $k \in \mathbb{N}$. For each $Q \in \mathscr{L}$, we let z_Q be its center and $2\lambda_Q$ its side length. Define $\alpha_Q: R^{n-1} \to R^{n-1}$ by $\alpha_Q(x) = z_Q + \lambda_Q x$, and let $Q(t) = \alpha_Q J^{n-1}(t)$, $\overline{Q}(t) = \overline{Q(t)}$ for $t > 0$; then $Q = \overline{Q}(1)$. Let $\mathscr{K}^0$ be the family of all maximal cubes Q of $\mathscr{L}$ contained in Y and satisfying either the conditions $\overline{Q}(3) \subset Y$ and $d(\varphi \overline{Q}(3)) <$ min $\varepsilon \overline{Q}(3)$ whenever $Q \cap \partial Y = \emptyset$ or the conditions $\overline{Q}(3) \cap R^{n-1}_+ \subset Y$ and $d(\varphi[\overline{Q}(3) \cap R^{n-1}_+]) < $ min $\varepsilon[\overline{Q}(3) \cap R^{n-1}_+]$ whenever $Q \cap \partial Y \neq \emptyset$. Dividing each cube of $\mathscr{K}^0$ into $N = 2^{n-1}$ cubes by bisecting the sides we obtain the family $\mathscr{K}$. We express $\mathscr{K}$ as a disjoint union $\mathscr{K} = \mathscr{K}_1 \cup \ldots \cup \mathscr{K}_N$ as follows: Divide I^{n-1} into cubes

$Q_1, \ldots, Q_N$ by bisecting the sides. Then set $\mathcal{K}_i = \{\alpha_Q Q_i \mid Q \in \mathcal{K}^0\}$. The following properties of $\mathcal{K}$ are easily verified:

(1) $Y = \cup \mathcal{K}$.
(2) If $Q, R \in \mathcal{K}$, $Q \cap R \neq \emptyset$, and $Q \neq R$, then $\operatorname{int} Q \cap \operatorname{int} R = \emptyset$ and $\lambda_Q / \lambda_R \in \{1/2, 1, 2\}$.
(3) If $Q \in \mathcal{K}$ and $Q \cap \partial Y = \emptyset$, then $\bar{Q}(3) \subset Y$. If $Q \in \mathcal{K}$ and $Q \cap \partial Y \neq \emptyset$, then $\bar{Q}(3) \cap R_+^{n-1} \subset Y$.
(4) If $Q, R \in \mathcal{K}$ and $Q \cap R = \emptyset$, then $Q(3/2) \cap R(3/2) = \emptyset$.
(5) If $x \in Q \in \mathcal{K}$, then $d(\varphi Q) < \varepsilon(x)$.
(6) If $Q, R \in \mathcal{K}_i$ and $Q \neq R$, then $Q \cap R = \emptyset$.

We call $\mathcal{K}$ the *cubical* (φ, ε)-*decomposition* of Y or also a *cubical decomposition* of Y if we do not specify φ and ε.

6.5. From now on to the end of the proof of Lemma 6.13, we suppose that n, η, X, Y, f are as in 6.1 with $n \geq 2$, that $\mathcal{K}$ is a cubical decomposition of Y, and that $\sigma > 0$. We may assume card $X \geq 2$; then $\eta(1) \geq 1$.

For each $Q \in \mathcal{K}$ we set $\varrho_Q = |f(z_Q) - f(z_Q + \lambda_Q e_1)|$ and $Q^*(t) = Q(t) \cap Y$ $(= Q(t) \cap X)$ for $t \in (0, 3]$; then $\alpha_Q^{-1} Q^*(t) = J^{n-1}(t)$ if $Q \cap \partial Y = \emptyset$ and $\alpha_Q^{-1} Q^*(t) = J^{n-1}(t) \cap (R_+^{n-1} - e_{n-1})$ if $Q \cap \partial Y \neq \emptyset$.

The following lemma can be proved in the same way as [LT, Lemma 2.6].

6.6. Lemma. *There are constants $c_1 \geq 1$ and $c_2 \geq 1$ depending only on n and η with the following properties:*
(1) *If $Q, R \in \mathcal{K}$, $Q \cap R \neq \emptyset$, and $Q \neq R$, then $\varrho_Q / c_1 \leq |f(z_Q) - f(z_R)| \leq c_1 \varrho_Q$.*
(2) *If $Q, R \in \mathcal{K}$ and $Q \cap R \neq \emptyset$, then $\varrho_Q / \varrho_R \leq c_1^2$.*
(3) *If $Q \in \mathcal{K}$, $x \in Q^*(5/4)$, and $y \in X \setminus Q^*(4/3)$, then $|f(x) - f(y)| \geq \varrho_Q / c_2$.*   □

6.7. *Constructions.* Let $\varkappa_1, \varkappa_2$ be the least natural numbers such that $2^{\varkappa_1} \geq 2\sqrt{n}$ and $2^{\varkappa_2} \geq 2c_1^2$. For each $Q \in \mathcal{K}$ let μ_Q be the greatest integer such that $s_Q = 2^{\mu_Q} \leq \varrho_Q$. By 6.6(2) we have $s_Q / s_R \leq 2^{\varkappa_2}$ if $Q, R \in \mathcal{K}$, $Q \cap R \neq \emptyset$. We choose $b_Q \in 2^{\mu_Q - \varkappa_1} \mathbf{Z}^n$ such that $|b_Q - f(z_Q)| \leq \sqrt{n} \, 2^{\mu_Q - \varkappa_1 - 1} (\leq s_Q / 4)$. Define $\beta_Q : R^n \to R^n$ by $\beta_Q(x) = (x - b_Q) / s_Q$.

6.8. Lemma. *There are a finite set $\mathcal{A}$ depending only on n and a finite set $\mathcal{B}$ depending only on n and η such that if $Q, R \in \mathcal{K}$, $Q \cap R \neq \emptyset$, then $\alpha_R^{-1} \alpha_Q \in \mathcal{A}$ and $\beta_Q \beta_R^{-1} \in \mathcal{B}$.*

Proof. The existence of $\mathcal{A}$ follows from 6.4(2). The existence of $\mathcal{B}$ follows from an obvious analogue of [LT, Lemma 2.8], which can be proved by the aid of 6.6(1) and 6.6(2) exactly as [Kk, Lemma 2.4].   □

6.9. *Constructions.* We set for $0 \leqq i \leqq N$

$$\mathcal{K}_i^* = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_i,$$
$$U_i = \cup \{Q^*(1+2^{-i-1}) \mid Q \in \mathcal{K}_i^*\},$$
$$V_i = \cup \{Q^*(1+2^{-i-2}) \mid Q \in \mathcal{K}_i^*\}.$$

Then U_i, V_i are open in Y, $V_{i-1} \subset U_i$, and $U_N = Y$. If $i \geqq 1$ and $Q \in \mathcal{K}_i$, we set

$$U_Q = \alpha_Q^{-1}[Q^*(3/2) \cap U_{i-1}],$$
$$V_Q = \alpha_Q^{-1}[Q^*(4/3) \cap V_{i-1}].$$

Then $\overline{V}_Q \subset U_Q$. There is a finite set $\mathscr{S}$ depending only on n such that $(U_Q, V_Q) \in \mathscr{S}$.

Let $\mathscr{T} = \{I^{n-1}(2), I^{n-1}(2) \cap (R_+^{n-1} - e_{n-1})\}$. For $T \in \mathscr{T}$ let $\mathscr{F}_\eta(T)$ be the set of all η-quasisymmetric embeddings $g: T \to R^n$ with $|g(0)| \leqq 1/4$ and $3/4 \leqq |g(e_1)| \leqq 3$. In the topology of uniform convergence $\mathscr{F}_\eta(T)$ is compact by [TV$_1$, Theorems 3.4 and 3.7 and Remark 3.6]. For each $Q \in \mathcal{K}$ we set $T_Q = \alpha_Q^{-1}[\overline{Q}(2) \cap Y]$ and $f_Q = \beta_Q f \alpha_Q | T_Q$. Clearly $T_Q \in \mathscr{T}$ and $f_Q \in \mathscr{F}_\eta(T_Q)$.

6.10. Lemma. *For every* $\varepsilon' > 0$ *there is* $\delta^*(\varepsilon') \in (0, \varepsilon']$ *depending only on* $n, \eta,$ *and* ε' *which has the following property: If* $\mathcal{K}$ *is as above,* $Q \in \mathcal{K}$, $g \in \mathscr{F}_\eta(T_Q)$, *and if* $h: U_Q \to R^n$ *is a locally* LIP *flat embedding with* $d(h, g|U_Q) < \delta^*(\varepsilon')$, *there is a locally* LIP *flat embedding* $h': T_Q \to R^n$ *such that* $d(h', g) < \varepsilon'$ *and* $h'|V_Q = h|V_Q$.

Proof. Consider $T \in \mathscr{T}$, an arbitrary embedding $g: T \to R^n$, and $\varepsilon' > 0$. Given $\mathcal{K}$ and $Q \in \mathcal{K}$ with $T_Q = T$, by 5.3 there is $\delta > 0$ such that for every locally LIP flat embedding $h: U_Q \to R^n$ with $d(h, g|U_Q) < \delta$ there is h' as in the lemma. Since $\mathscr{S}$ is finite, there is the greatest $\delta = \delta(T, g, \varepsilon') \in (0, \varepsilon']$ having this property for all $\mathcal{K}$ and $Q \in \mathcal{K}$ with $T_Q = T$.

We set $\delta^*(\varepsilon') = \inf \{\delta(T, g, \varepsilon') \mid T \in \mathscr{T}, g \in \mathscr{F}_\eta(T)\}$. Then as in [V$_2$, Lemma 2.6] we get $\delta^*(\varepsilon') > 0$ by the compactness of the spaces $\mathscr{F}_\eta(T)$. Thus $\delta^*(\varepsilon')$ is the desired number. $\square$

6.11. *Construction.* We choose numbers $\delta_N \geqq \delta_{N-1} \geqq \ldots \geqq \delta_0 > 0$ such that

$$\delta_N < \min (\sigma/2, 1/4c_1^2 c_2, 1/4c_1^2 \eta(2) \eta(2\sqrt{n-1})),$$
$$\delta_i < \delta^*(\delta_{i+1})/(2^{\varkappa_2+1}+1).$$

The numbers δ_i depend only on $n, \eta,$ and σ.

We next prove by induction on i that the following lemma is true for every $i \in \{0, 1, \ldots, N\}$:

6.12$_i$. Lemma. *There are a finite set* $\mathcal{H}_i$, *depending only on* (n, η, σ, i), *of locally* LIP *flat embeddings* $J^{n-1}(1+2^{-i-1}) \cap T \to R^n$, $T \in \mathscr{T}$, *and a locally* LIP *flat embedding* $\varphi_i: U_i \to R^n$, *such that for each* $Q \in \mathcal{K}_i^*$

(1) $\beta_Q \varphi_i \alpha_Q | \alpha_Q^{-1} Q^*(1+2^{-i-1}) \in \mathcal{H}_i$,

(2) $d(\varphi_i | Q^*(1+2^{-i-1}), f | Q^*(1+2^{-i-1})) \leqq 2\delta_i s_Q$.

Proof. Since $U_0=\emptyset$, 6.12_0 is trivially true. Assume that $1\leq i\leq N$ and that 6.12_{i-1} is true. For $Q\in\mathscr{K}_i$ we define a locally LIP flat embedding $h_Q=\beta_Q\varphi_{i-1}\alpha_Q|U_Q$. If $x\in U_Q$, there is $R\in\mathscr{K}_{i-1}^*$ such that $\alpha_Q(x)\in R^*(1+2^{-i})$, whence $R\cap Q\neq\emptyset$. Then $h_Q(x)=\beta_Q\beta_R^{-1}\psi\alpha_R^{-1}\alpha_Q(x)$, where $\psi=\beta_R\varphi_{i-1}\alpha_R|\alpha_R^{-1}R^*(1+2^{-i})\in\mathscr{H}_{i-1}$. The sets $U_Q\cap\alpha_Q^{-1}R^*(1+2^{-i})$ belong to a finite set depending only on n. Thus by 6.8 the maps h_Q belong to a finite set $\mathscr{C}_i$ depending only on (n,η,σ,i).

For each $T\in\mathscr{T}$, by the compactness of $\mathscr{F}_\eta(T)$ there is a finite set $\mathscr{F}'(T)\subset\mathscr{F}_\eta(T)$ depending only on (n,η,σ,i) such that $d(g,\mathscr{F}'(T))<\delta_{i-1}$ if $g\in\mathscr{F}_\eta(T)$. Consider $Q\in\mathscr{K}_i$. Choose $g\in\mathscr{F}'(T_Q)$ with $d(f_Q,g)<\delta_{i-1}$. If $x\in U_Q$ and R is as above,

$$|h_Q(x)-f_Q(x)| = |\varphi_{i-1}(\alpha_Q(x))-f(\alpha_Q(x))|/s_Q$$
$$\leq 2\delta_{i-1}s_R/s_Q \leq 2^{\varkappa_2+1}\delta_{i-1},$$

whence $d(h_Q,g|U_Q)<\delta^*(\delta_i)$. Hence there is a locally LIP flat embedding $h'_Q\colon T_Q\to R^n$ such that $d(h'_Q,g)<\delta_i$ and $h'_Q=h_Q$ on V_Q. We can choose the maps h'_Q in such a way that the maps $h'_Q|\alpha_Q^{-1}Q^*(1+2^{-i-1})$ belong to a finite set $\mathscr{D}_i$, depending only on (n,η,σ,i), of locally LIP flat embeddings $J^{n-1}(1+2^{-i-1})\cap T\to R^n$, $T\in\mathscr{T}$.

We set $\varphi_i|V_{i-1}=\varphi_{i-1}|V_{i-1}$ and $\varphi_i(x)=\beta_Q^{-1}h'_Q\alpha_Q^{-1}(x)$ for all $Q\in\mathscr{K}_i$ and $x\in Q^*(1+2^{-i-1})$. Then $\varphi_i\colon U_i\to R^n$ is a well-defined locally LIP flat immersion, and it satisfies (1) with

$$\mathscr{H}_i = \mathscr{D}_i\cup\{h|J^{n-1}(1+2^{-i-1})\cap T\,|\,h\colon J^{n-1}(1+2^{-i})\cap T\to R^n,\,T\in\mathscr{T},\,h\in\mathscr{H}_{i-1}\}.$$

If $Q\in\mathscr{K}_i$ and $x\in Q^*(1+2^{-i-1})$,

$$|\varphi_i(x)-f(x)| = s_Q|h'_Q(\alpha_Q^{-1}(x))-f_Q(\alpha_Q^{-1}(x))| \leq 2\delta_i s_Q,$$

which implies (2). We finally prove that φ_i is an embedding. Since φ_i is the embedding $\beta_Q^{-1}h'_Q\alpha_Q^{-1}$ on $Q^*(4/3)\cap U_i$ for each $Q\in\mathscr{K}_i$, it suffices to show that if $Q,R\in\mathscr{K}_i^*$, $x\in Q^*(1+2^{-i-1})$, $y\in R^*(1+2^{-i-1})$, $s_Q\geq s_R$, $\Delta=|\varphi_i(x)-\varphi_i(y)|$, and if $x\notin R^*(4/3)$ or $y\notin Q^*(4/3)$, then $\Delta\geq\lambda s_Q$, where $\lambda=1/c_1^2c_2-4\delta_i>0$. We have $\Delta\geq|f(x)-f(y)|-4\delta_i s_Q$. Thus, if $y\notin Q^*(4/3)$, the claim follows from 6.6(3), whereas if $y\in Q^*(4/3)$, in which case $Q\cap R\neq\emptyset$ and $x\notin R^*(4/3)$, the claim follows from 6.6(3) and 6.6(2). $\quad\square$

6.13. Lemma. *Let* $2\leq n\neq 4$, *let* $\eta\colon R_+^1\to R_+^1$ *be a homeomorphism, and let* $\sigma>0$. *Then there are a homeomorphism* $\eta^*\colon R_+^1\to R_+^1$ *and a finite set* $\mathscr{D}$ *of locally LIP flat embeddings* $T\to R^n$, $T\in\mathscr{T}$, *with the following property: Let either* $Y\subset X\subset R^{n-1}$ *with* Y *open in* R^{n-1} *or* $Y\subset X\subset R_+^{n-1}$ *with* Y *open in* R_+^{n-1}, *let* $f\colon X\to R^n$ *be an* η-*quasisymmetric embedding, and let* $\mathscr{K}$ *be a cubical decomposition of* Y. *Then there is an* η^*-*quasisymmetric embedding* $f^*\colon X\to R^n$ *such that*

(1) $f^*|X\setminus Y=f|X\setminus Y$,

(2) $f^*|Y$ *is locally* LIP *flat*,

(3) $d(f^*|Q,f|Q)\leq\sigma\varrho_Q$ *for each* $Q\in\mathscr{K}$,

(4) $\beta_Q f^*\alpha_Q|T_Q\in\mathscr{D}$ *for each* $Q\in\mathscr{K}$.

Proof. Let $\varphi_N \colon Y \to R^n$ be the locally LIP flat embedding of 6.12_N. We define $f^* = \varphi_N \cup (f | X \setminus Y)$. Then (1), (2), and (3) are satisfied. One can find $\mathscr{D}$ and prove (4) in the same way as one obtained $\mathscr{C}_i$ and the relation $h_Q \in \mathscr{C}_i$ in the proof of 6.12_i. The proof of [LT, Theorem 2.16] now shows that f^* is an η^*-quasisymmetric embedding with η^* depending only on (n, η, σ). $\square$

6.14. *Proof of 6.1 for* $n \geqq 2$. Apply 6.13 with $\sigma = 1$ and with $\mathscr{K}$ being the cubical (f, ε)-decomposition of Y. $\square$

6.15. *Proof of 6.2 for* $n \geqq 2$. Every L-bilipschitz embedding is η_L-quasisymmetric, $\eta_L(t) = L^2 t$. Theorem 6.2 follows from an application of 6.13 with the substitution $\eta \mapsto \eta_L$, $\sigma \mapsto 1/3L^2$. The details are almost the same as those of the proof of [LT, Theorem 3.2]. $\square$

7. Relative LQS or LIP approximation

In this section we prove the following two theorems dealing with relative LQS or LIP approximation of embeddings or of immersions, respectively. Their essence is the special case $A \neq \emptyset$, $U = B = B' = M$.

7.1. **Theorem.** *Let* $C \in \{\text{LQS}, \text{LIP}\}$ *and* $n \neq 4$, *let* M *be an* $(n-1)$- *and* Q *an* n-*dimensional* C-*manifold with* $\partial Q = \emptyset$, *let* $A \subset M$ *be closed and* $U, B \subset M$ *open with* $A \subset U$, *let* $f \colon M \to Q$ *be an embedding which is a* C-*embedding near* A, *and let* $\varepsilon \in C_+(B)$. *Then there is an embedding* $f^* \colon M \to Q$ *such that* $f^* = f$ *near* A *and on* $M \setminus B$, *that* $f^* | B \in U(f | B, \varepsilon)$, *that* f^* *is a* C-*embedding near* $A \cup B$, *and that* f^* *is locally* C-*flat near* $B \setminus U$.

7.2. **Theorem.** *Let* C, n, M, Q, A, U *be as in 7.1, let* $B \subset M$ *be closed and* $B' \subset M$ *open with* $B \subset B'$, *let* $f \colon M \to Q$ *be an immersion which is a* C-*immersion near* A, *and let* $\varepsilon \in C_+(B')$. *Then there is an immersion* $f^* \colon M \to Q$ *such that* $f^* = f$ *near* A *and on* $M \setminus B'$, *that* $f^* | B' \in U(f | B', \varepsilon)$, *that* f^* *is a* C-*immersion near* $A \cup B$, *and that* f^* *is locally* C-*flat near* $B \setminus U$.

We begin with an analogue of [LT, Lemma 4.2] and of 5.8.

7.3. **Lemma.** *Let* $C \in \{\text{LQS}, \text{LIP}\}$ *and* $n \neq 4$, *let* $B \subset A$ *be open subsets of* R_+^{n-1} *such that* $\bar{B} \subset A$ *is compact, let* $f \colon A \to R^n$ *be an embedding which is a* C-*embedding near a closed subset* E *of* A, *and let* $\varepsilon > 0$. *Then there is an embedding* $f^* \colon A \to R^n$ *such that* $f^* = f$ *on* $A \setminus B$, $d(f^*, f) < \varepsilon$, *and* f^* *is a* C-*embedding near* $E \cup B$.

Proof. We may assume that E is compact. Define $v \in C_+(B)$ by $v(x) = \min(\varepsilon, d(f(x), f[A \setminus B])/2)$. Choose an open neighbourhood X of E in A such that $\bar{X} \subset A$ is compact and $f | \bar{X}$ is a C-embedding. Then $f | X$ is quasisymmetric

if $C = LQS$ or bilipschitz if $C = LIP$. Let $Y = X \cap B$, and define $\delta \in C_+(Y)$ by $\delta(x) = (1/2) \min\left(v(x), d(f(x), f[B \setminus Y])\right)$. By 6.1 if $C = LQS$ or by 6.2 if $C = LIP$, there is a C-embedding $g: X \to R^n$ such that $g = f$ on $X \setminus Y$, $g|Y \in U(f|Y, \delta)$, and $g|Y$ is locally LIP flat. Extending $g|Y$ by $f|B \setminus Y$ we obtain an embedding $g_1: B \to R^n$.

Choose an open neighborhood V of E in A with $\overline{V} \subset X$. Since g_1 is locally LIP flat on $Y \supset \overline{V} \cap B$, by 5.1 there is a locally LIP flat embedding $f_1: B \to R^n$ in $U(g_1, v/2)$ such that $f_1 = g_1$ on $V \cap B$. Then $f_1 \in U(f|B, v)$. Thus f_1 and $f|A \setminus B$ define an embedding $f^*: A \to R^n$, which is the desired one since $f^* = g$ on V. $\square$

7.4. Lemma. *Theorem 7.1 holds if $U = B = M$, and Theorem 7.2 holds if $U = M$.*

Proof. This follows from 7.3 in the same way as 5.1 and 5.2 followed from 5.8 in 5.9. $\square$

7.5. *Proof of 7.2.* Choose an open $V \subset M$ with $B \setminus U \subset V \subset \overline{V} \subset B' \setminus A$. By 5.2 there is an immersion $g: M \to Q$ such that $g = f$ on $M \setminus V$, $g|V \in U(f|V, (\varepsilon/2)|V)$ and g is locally C-flat near $B \setminus U$. Then $g = f$ near A, whence g is a C-immersion near $A \cup (B \setminus U)$. Thus by 7.4 there is an immersion $f^*: M \to Q$ such that $f^* = g$ near $A \cup (B \setminus U)$ and on $M \setminus B'$, $f^*|B' \in U(g|B', \varepsilon/2)$, and f^* is a C-immersion near $A \cup B$. Then f^* is the desired map. $\square$

7.6. Lemma. *Theorem 7.1 holds if $B = M$.*

Proof. This reduces to 7.4 by 5.1; the proof is similar to 7.5. $\square$

7.7. *Proof of 7.1.* We may assume that $\varepsilon(x) \leqq d(f(x), f[M \setminus B])/2$ for each $x \in B$. Choose an open $V \subset M$ such that $A \subset V \subset \overline{V} \subset U$ and that f is a C-embedding near $\overline{V}$. By 7.6 there is a C-embedding $g: B \to Q$ in $U(f|B, \varepsilon)$ such that $g = f$ on $V \cap B$ and g is locally C-flat near $B \setminus U$. Then $f^* = g \cup (f|M \setminus B)$ is the desired embedding since $f^* = f$ on V. $\square$

8. Extension of LQC or LIP structures

In this section $C \in \{LQC, LIP\}$, M is an n-manifold, and $\mathscr{A}$ is a C-structure on ∂M. We prove the following two theorems.

8.1. Theorem. *Suppose that M is of the form $M = \partial M \times R^1_+$. Then $\mathscr{A}$ can be extended to a C-structure on M.*

8.2. Theorem. *Suppose $n \neq 4$. Then $\mathscr{A}$ can be extended to a C-structure on M.*

Moreover, if $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are two C-structures on* M *both extending* $\mathscr{A}$, *then for every* $\varepsilon \in C_+(M)$ *there is an ambient* ε-*isotopy* H *of* M *rel* ∂M *such that* $H_1 : (M, \mathscr{A}_1) \to (M, \mathscr{A}_2)$ *is a C-homeomorphism.*

8.3. Remarks. 1. By [LT, Remark 4.7] every LQC manifold $(N, \mathscr{B})$ has a metric ϱ such that id: $(N, \mathscr{B}) \to (N, \varrho)$ is an LQS homeomorphism. In 8.1 for $C = \text{LQC}$, $n = 2$, [T_3, Example 4] shows that if $(\partial M, \mathscr{A})$ is given such a suitable metric, a product metric is not necessarily a suitable one for the extended LQC structure on M.

2. One gets the existence part of 8.2 immediately from 5.11 applying it to the inclusion of $(\partial M, \mathscr{A})$ into the double of M. However, we give below a more elementary proof.

8.4. *Proof of* 8.1 *and* 8.2. The uniqueness part of 8.2 follows immediately from [TV_3, Theorem 4.8]. Since ∂M has a collar in M, it is easy to see by [TV_3, Theorem 4.6 and Corollary 4.5] that the existence part of 8.2 follows from 8.1. Theorem 8.1 is trivial for LIP. For $n \neq 5$, the LQC case of 8.1 follows from the LIP case, because the LQC structure $\mathscr{A}$ contains a LIP structure by [TV_3, Theorem 4.6 and Corollary 4.5]. Thus it suffices to prove 8.1 for $C = \text{LQC}$, $n = 5$. Since $(\partial M, \mathscr{A})$ has an open cover by n sets each C-embeddable into R^{n-1} (cf. [Mu, Lemma 2.7]), the following lemma completes the proof. $\square$

8.5. Lemma. *Let* M *be as in* 8.1 *with* $n \neq 4$, *write* $X' = X \times R_+^1$ *whenever* $X \subset \partial M \equiv \partial M \times \{0\}$, *let* U, V *be open subsets of* ∂M, *let* $\mathscr{A}_1$ *be a C-structure on* U' *with* $\mathscr{A}_1|U = \mathscr{A}|U$, *and let* $f : (V, \mathscr{A}|V) \to R^{n-1}$ *be a C-embedding. Then there is a C-structure* $\mathscr{A}_2$ *on* $(U \cup V)'$ *such that* $\mathscr{A}_2|U \cup V = \mathscr{A}|U \cup V$ *and* $\mathscr{A}_2|U' = \mathscr{A}_1$.

Proof. Let $\mathscr{A}_3$ be the unique C-structure on V' which makes the open embedding $f \times \text{id} : V' \to R_+^n$ a C-embedding. Then $\mathscr{A}_3|V = \mathscr{A}|V$. Let $W = U \cap V$. By the uniqueness part of 8.2 there is a C-homeomorphism $h_0 : (W', \mathscr{A}_1|W') \to (W', \mathscr{A}_3|W')$ which can be extended by the identity of $V' \setminus W'$ to a homeomorphism $h : V' \to V'$ such that $h|V = \text{id}$. Let $\mathscr{A}_4$ be the unique C-structure on V' which makes $h : (V', \mathscr{A}_4) \to (V', \mathscr{A}_3)$ a C-homeomorphism. Then $\mathscr{A}_4|V = \mathscr{A}|V$ and $\mathscr{A}_4|W' = \mathscr{A}_1|W'$. Hence the C-structure $\mathscr{A}_2$ on $(U \cup V)'$ induced by $\mathscr{A}_1 \cup \mathscr{A}_4$ is the desired C-structure. $\square$

9. Extension of LQC homeomorphisms

In this section we prove that for every n, every LQC homeomorphism $f : R^n \to R^n$ can be extended to an LQC homeomorphism $F : R_+^{n+1} \to R_+^{n+1}$; cf. [$\text{TV}_3$, Theorem 4.10]. The proof makes use of the fundamental Theorem 3.2 of Tukia and Väisälä [TV_2] on similar problems for quasiconformal embeddings. For $n \neq 3$ the result is simple: [TV_3, Theorem 4.8] produces F as an LQC approxima-

tion of $f \times \mathrm{id}$. (In the case $n=3$, it seems that [Ca] does not suffice for us.) In 9.4 we give a short proof for a slightly improved version of [G_2, Theorem 2].

9.1. Lemma. *Let $f: R^n \to R^n$ ($n \geq 1$) be an LQC homeomorphism and let $A \subset R_+^{n+1}$ be compact. Then there is $r_0 > 0$ such that for every $r \geq r_0$ there is an embedding $g: R^n \cup B_+^{n+1}(r) \to R_+^{n+1}$ such that $g|R^n = f$, $g|B_+^{n+1}(r)$ is quasiconformal, and $gB_+^{n+1}(r_0) \supset A$.*

Proof. We may assume that f is sense-preserving. Choose consecutively $a, \varrho, r_0 > 0$ such that $A \subset B^n(a) \times [0, a]$, $f^{-1}B^n(a) \subset B^n(\varrho)$, $d(fB^n(\varrho)) \geq 2a$, and $B^n(\varrho) \times [0, 2\varrho] \subset B_+^{n+1}(r_0)$. We extend f to a homeomorphism $F_f: R_+^{n+1} \to R_+^{n+1}$ by setting $F_f(x, t) = (f(x), \tau(x, t))$ for $x \in R^n$, $t \geq 0$, where

$$\tau(x, t) = \max \{|f(y) - f(x)|: y \in R^n, |y - x| \leq t\}$$

(cf. [TV_2, (2.4)]). For each $x \in B^n(\varrho)$ we have $\tau(x, 2\varrho) \geq a$ since $B^n(x, 2\varrho) \supset B^n(\varrho)$. Hence $F_f B_+^{n+1}(r_0) \supset A$. Consider $r \geq r_0$. Let $b = 9 + 5\sqrt{n}$, $r' = (1+b)r$, $U = B^n(r')$, and $U^* = \{(x, t) \in R_+^{n+1}: |x| + bt < r'\}$. Then $U^* \cap R^n = U$ and $\bar{B}_+^{n+1}(r) \subset U^*$. Since $f|U$ is quasiconformal, by [TV_2, Theorem 3.2] there is a quasiconformal embedding $F: U^* \to R_+^{n+1}$ such that $F|U = f|U$ and $FB_+^{n+1}(r_0) \supset A$. Then $g = f \cup (F|B_+^{n+1}(r))$ is the desired embedding. □

9.2. Theorem. *For every $n \geq 0$, every LQC homeomorphism $f: R^n \to R^n$ can be extended to an LQC homeomorphism $F: R_+^{n+1} \to R_+^{n+1}$.*

Proof. We may assume $n \geq 1$. We construct inductively numbers $s_j > r_j > j$ and embeddings $g_j: R^n \cup B_+^{n+1}(s_j) \to R_+^{n+1}$, $j \in \mathbf{N}$, such that $r_j < r_{j+1}$, $g_j|R^n = f$, $g_j|B_+^{n+1}(s_j)$ is quasiconformal, $g_j B_+^{n+1}(r_j) \supset \bar{B}_+^{n+1}(j)$, and $g_{j+1} = g_j$ on $\bar{B}_+^{n+1}(r_j)$. We obtain s_1, r_1, g_1 from 9.1.

Suppose that we have constructed s_j, r_j, g_j. Let $H = R_+^{n+1} \setminus B_+^{n+1}(r_j)$, $H' = R_+^{n+1} \setminus g_j B_+^{n+1}(r_j)$. Clearly there is a quasiconformal homeomorphism $\alpha: R^{n+1} \to R^{n+1}$ with $\alpha H = R_+^{n+1}$. We next construct a quasiconformal homeomorphism $\beta: R^{n+1} \to R^{n+1}$ with $\beta H' = R_+^{n+1}$. Extend $g_j|B_+^{n+1}(s_j)$ by reflection to a quasiconformal embedding $\varphi: B^{n+1}(s_j) \to R^{n+1}$. By [GV, Relative Schoenflies theorem 2.4] there is a quasiconformal homeomorphism $\psi: R^{n+1} \to R^{n+1}$ such that $\psi = \varphi$ on $\bar{B}^{n+1}(r_j)$ and $\psi R^n = R^n$. Then $\psi H = H'$. Thus $\beta = \alpha \psi^{-1}$ is the desired map. Using the equivalent definition of LQC embeddings in a Euclidean space as LQS embeddings [V_3, Corollary 2.6], we see that the homeomorphism $h_0 = \beta g_j \alpha^{-1}: R^n \to R^n$ is LQC. By 9.1 there is $r_0 > 0$ such that for every $r \geq r_0$ there is an embedding $h: R^n \cup B_+^{n+1}(r) \to R_+^{n+1}$ such that $h|R^n = h_0$, $h|B_+^{n+1}(r)$ is quasiconformal, and $hB_+^{n+1}(r_0) \supset \beta[\bar{B}_+^{n+1}(j+1) \cap H']$. Choose $r_{j+1} > \max(r_j, j+1)$ such that $\alpha[B_+^{n+1}(r_{j+1}) \cap H] \supset B_+^{n+1}(r_0)$. Set $s_{j+1} = 2r_{j+1}$, and fix r such that $\alpha[B_+^{n+1}(s_{j+1}) \cap H] \subset B_+^{n+1}(r)$. Define an embedding $h_1: (R^n \cup B_+^{n+1}(s_{j+1})) \cap H \to R_+^{n+1}$ by $h_1(x) = \beta^{-1} h \alpha(x)$. Then $h_1 = g_j$ on ∂H, $h_1|B_+^{n+1}(s_{j+1}) \cap H$ is quasiconformal, and $h_1[B_+^{n+1}(r_{j+1}) \cap H] \supset$

$\bar{B}_+^{n+1}(j+1) \cap H'$. Thus, extending h_1 by $g_j|B_+^{n+1}(r_j)$ we obtain the desired embedding g_{j+1}.

Setting $F = g_j$ on $B_+^{n+1}(r_j)$ for each $j \in \mathbf{N}$ we obtain an LQC homeomorphism $F : R_+^{n+1} \to R_+^{n+1}$ which extends f. $\square$

9.3. Corollary. *For all* $0 \leqq n \leqq p$, *every* LQC *homeomorphism* $f : R^n \to R^n$ *can be extended to an* LQC *homeomorphism* $F : R^p \to R^p$. $\square$

9.4. Theorem. *Let* $C \in \{\text{LQC, LIP}\}$, $n \neq 5$, *and let* M_1, M_2 *be homeomorphic compact locally* C-*flat* $(n-1)$-*submanifolds of* S^n *without boundary. Then there are open neighborhoods* U_i *of* M_i *in* S^n *and a* C-*homeomorphism* $f : (U_1, M_1) \to (U_2, M_2)$.

Moreover, if $f_0 : M_1 \to M_2$ *is a* C-*homeomorphism, this also holds for* $n = 5$, *and we may assume* $f|M_1 = f_0$.

Proof. We may assume that M_1, M_2 are connected and $n \geqq 2$. By [Gr, Theorem 27.10], $S^n \setminus M_i$ consists of two components; their closures M_{i1}, M_{i2} are C-manifolds with $\partial M_{ij} = M_i$. By [TV$_3$, Corollary 4.5] there is a C-homeomorphism $f_0 : M_1 \to M_2$ whenever $n \neq 5$. By [TV$_3$, Theorem 4.10 and Remark 4.11] there are open neighborhoods U_{ij} of M_i in M_{ij} and C-homeomorphisms $f_j : U_{1j} \to U_{2j}$, $i, j = 1, 2$, such that $f_j|M_1 = f_0$. Then $U_i = U_{i1} \cup U_{i2}$ and $f = f_1 \cup f_2$ satisfy the theorem. $\square$

References

[AC]   ANCEL, F. D., and J. W. CANNON: The locally flat approximation of cell-like embedding relations. - Ann. of Math. (2) 109, 1979, 61—86.

[B$_1$]   BING, R. H.: Locally tame sets are tame. - Ibid. 59, 1954, 145—158.

[B$_2$]   BING, R. H.: Approximating surfaces with polyhedral ones. - Ibid. 65, 1957, 456—483.

[Ca]   CARLESON, L.: The extension problem for quasiconformal mappings. - Contributions to analysis, edited by L. V. Ahlfors et al., Academic Press, New York—London, 1974, 39—47.

[Cr]   CRAGGS, R.: Small ambient isotopies of a 3-manifold which transform one embedding of a polyhedron into another. - Fund. Math. 68, 1970, 225—256.

[D]   DAVERMAN, R. J.: Embeddings of $(n-1)$-spheres in Euclidean n-space. - Bull. Amer. Math. Soc. 84, 1978, 377—405.

[E]   EDWARDS, R. D.: The equivalence of close piecewise linear embeddings. - General Topology Appl. 5, 1975, 147—180.

[G$_1$]   GAULD, D.: Lipschitz and quasiconformal tubular neighbourhoods of spheres in codimension two. - Ann. Acad. Sci. Fenn. Ser. A I Math. 5, 1980, 175—178.

[G$_2$]   GAULD, D.: Lipschitz and quasiconformal flattening of manifolds in codimension one. - Univ. of Auckland, Dep. Math., Report Series 172, 1981, 1—8.

[GV]   GAULD, D. B., and J. VÄISÄLÄ: Lipschitz and quasiconformal flattening of spheres and cells. - Ann. Acad. Sci. Fenn. Ser. A I Math. 4, 1978/1979, 371—382.

[Gr]   GREENBERG, M. J.: Lectures on algebraic topology. - W. A. Benjamin, Inc., New York—
        Amsterdam, 1967.

[H]   HUDSON, J. F. P.: Piecewise linear topology. - W. A. Benjamin, Inc., New York—Amsterdam,
        1969.

[Kk]   KIIKKA, M.: Piecewise linear approximation of quasiconformal and Lipschitz homeo-
        morphisms. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 30, 1980, 1—24.

[Ks]   KISTER, J. M.: Isotopies in 3-manifolds. - Trans. Amer. Math. Soc. 97, 1960, 213—224.

[LV$_1$] LEHTO, O., and K. I. VIRTANEN: On the existence of quasiconformal mappings with prescribed
        complex dilatation. - Ann. Acad. Sci. Fenn. Ser. A I Math. 274, 1960, 1—24.

[LV$_2$] LEHTO, O., and K. I. VIRTANEN: Quasiconformal mappings in the plane. - Die Grundlehren
        der mathematischen Wissenschaften in Einzeldarstellungen 126, Springer-Verlag,
        Berlin—Heidelberg—New York, 1973.

[LT]   LUUKKAINEN, J., and P. TUKIA: Quasisymmetric and Lipschitz approximation of embeddings.
        - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 343—367.

[LuV]   LUUKKAINEN, J., and J. VÄISÄLÄ: Elements of Lipschitz topology. - Ibid. 3, 1977, 85—122.

[M]   MATSUMOTO, Y.: Wild embeddings of piecewise linear manifolds in codimension two. -
        Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), edited by
        J. C. Cantrell, Academic Press, New York—San Francisco—London, 1979, 393—428.

[Ma]   MATVEEV, S. V.: Approximation of embeddings of manifolds by locally flat embeddings in
        codimension greater than 2. - Moscow Univ. Math. Bull. 27 (1972), 1973, 112—116.

[Mi]   MILLER, R. T.: Approximating codimension 3 embeddings. - Ann. of Math. (2) 95, 1972,
        406—416.

[Mu]   MUNKRES, J. R.: Elementary differential topology. - Annals of Mathematics Studies 54,
        Princeton University Press, Princeton, N. J., 1963.

[PS]   PRICE, T. M., and C. L. SEEBECK III: Somewhere locally flat codimension one manifolds
        with 1-ULC complements are locally flat. - Trans. Amer. Math. Soc. 193, 1974,
        111—122.

[Ri]   RICKMAN, S.: Characterization of quasiconformal arcs. - Ann. Acad. Sci. Fenn. Ser. A I
        Math. 395, 1966, 1—30.

[RS]   ROURKE, C. P., and B. J. SANDERSON: Introduction to piecewise-linear topology. - Ergebnisse
        der Mathematik und ihrer Grenzgebiete 69, Springer-Verlag, Berlin—Heidelberg—
        New York, 1972.

[Ru]   RUSHING, T. B.: Topological embeddings. - Pure and Applied Mathematics 52, Academic
        Press, New York—London, 1973.

[S]   SULLIVAN, D.: Hyperbolic geometry and homeomorphisms. - Geometric topology (Proc.
        Georgia Topology Conf., Athens, Ga., 1977), edited by J. C. Cantrell, Academic
        Press, New York—San Francisco—London, 1979, 543—555.

[T$_1$]   TUKIA, P.: Lipschitz approximation of homeomorphisms. - Ann. Acad. Sci. Fenn. Ser.
        A I Math. 4, 1978/1979, 137—144.

[T$_2$]   TUKIA, P.: The planar Schönflies theorem for Lipschitz maps. - Ibid. 5, 1980, 49—72.

[T$_3$]   TUKIA, P.: A quasiconformal group not isomorphic to a Möbius group. - Ibid. 6, 1981,
        149—160.

[TV$_1$] TUKIA, P., and J. VÄISÄLÄ: Quasisymmetric embeddings of metric spaces. - Ibid. 5, 1980,
        97—114.

[TV$_2$] TUKIA, P., and J. VÄISÄLÄ: Quasiconformal extension from dimension n to $n+1$. - Ann.
        of Math. (2) 115, 1982, 331—348.

[TV$_3$] TUKIA, P., and J. VÄISÄLÄ: Lipschitz and quasiconformal approximation and extension. -
        Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 303—342.

[V$_1$]   VÄISÄLÄ, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Mathe-
        matics 229, Springer-Verlag, Berlin—Heidelberg—New York, 1971.

[V₂]   VÄISÄLÄ, J.: Piecewise linear approximation of lipeomorphisms. - Ann. Acad. Sci. Fenn. Ser. A I Math. 3, 1977, 377—383.
[V₃]   VÄISÄLÄ, J.: Quasi-symmetric embeddings in Euclidean spaces. - Trans. Amer. Math. Soc. 264, 1981, 191—204.
[Ve]   VENEMA, G. A.: Approximating codimension two embeddings of cells. - Abstracts Amer. Math. Soc. 2, 1981, 590.

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