DOMAINS WITH LIPSCHITZ MAPPING FUNCTIONS

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1. Introduction

Suppose that Γ is a closed Jordan curve in the ω-plane, with interior Ω and exterior Ω^*. Let f and f^* respectively be conformal mappings of $D=\{|\zeta|<1\}$ and $D^*=\{|\zeta|>1\}$ onto Ω and Ω^*. Both f and f^* may be extended to homeomorphisms between the closures of their domains and images, and we denote these extensions by f and f^* also. It is a much studied phenomenon that various geometric properties of Γ correspond to properties of the mapping functions or their inverses. In particular, there are many results in which properties of Γ imply properties of the mapping functions. There are fewer where one assumes properties of the functions to infer properties of Γ, and there are still fewer equivalences between behavior of Γ and boundary behavior of f or f^*. In some cases, a condition on f alone (for example) is equivalent to a condition on Γ. For instance, absolute continuity of $f(e^{i\theta})$ is equivalent to the rectifiability of Γ (and $f^*(e^{i\theta})$ is then absolutely continuous as well). Sometimes a "two sided condition", one involving both f and f^*, is needed. An example of this is Ahlfors' characterization of quasiconformal curves in terms of $f^{*-1} \circ f$ [1, Chapter 4].

In this paper we are interested in consequences of Lipschitz continuity on ∂D of the mapping functions. We first recall a few definitions.

For ω_1 and ω_2 on Γ, let $\Gamma(\omega_1, \omega_2)$ be the arc of smaller diameter between ω_1 and ω_2. Then Γ is a c-quasiconformal curve (or a quasiconformal curve) if there exist positive c and δ such that for $|\omega_1-\omega_2|<\delta$ and $\omega_0\in\Gamma(\omega_1, \omega_2)$,

$$(1.1) \qquad \frac{|\omega_1-\omega_0|+|\omega_0-\omega_2|}{|\omega_1-\omega_2|} < c.$$

The curve Γ is an asymptotically conformal curve if

$$(1.2) \qquad \lim_{|\omega_1-\omega_2|\to 0} \sup_{\omega_0 \in \Gamma(\omega_1, \omega_2)} \frac{|\omega_1-\omega_0|+|\omega_0-\omega_2|}{|\omega_1-\omega_2|} = 1.$$

Suppose now that Γ is rectifiable. For $\omega_1, \omega_2\in\Gamma$, let Δs be the length of the shorter arc between ω_1 and ω_2. Then Γ is a bounded arclength-chordlength

curve, or a quasismooth curve, if there exists $c > 0$ such that

(1.3)
$$\frac{\Delta s}{|\omega_1 - \omega_2|} \leqq c$$

for all $\omega_1, \omega_2 \in \Gamma$. Γ is asymptotically smooth if

(1.4)
$$\lim_{|\omega_1 - \omega_2| \to 0} \frac{\Delta s}{|\omega_1 - \omega_2|} = 1.$$

The properties (1.2) and (1.4) have been studied by Pommerenke, who showed them to be equivalent to certain properties of f ([10], see also [2], [8]). In [7], Hölder continuity of f and f^* (as well as f^{-1} and f^{*-1}) was inferred from (1.1) with Hölder exponents depending on c (see [9] also). The question then arose as to whether Hölder continuity of all of the mapping functions implied (1.1). This was settled by Becker and Pommerenke [3] who constructed a curve Γ for which all four functions are Hölder continuous but which is not a quasiconformal curve.

In this paper we consider the hypothesis that f and f^* are Lipschitz continuous and ask what can be said about Γ.

We begin by remarking that this implies Lipschitz continuity of f^{-1} and f^{*-1} on Γ [6]. From the Lipschitz continuity of f and f^{-1} it is easy to see that (1.3) holds; Γ is quasismooth. Our main result is Theorem 3. If Γ is a quasiconformal curve and f and f^* are locally Lipschitz at $\zeta_0 \in \partial D$ with $f(\zeta_0) = f^*(\zeta_0) = \omega_0$, then a local version of (1.2) holds. This "local asymptotic conformality" has been studied by Rodin and Warschawski [12] and has several consequences, both for the mapping functions and for Γ. For example, the "Visser—Ostrowski limit" for f exists and equals 1 at ζ_0 (Corollary 1). As another example, near ω_0, Γ may be sandwiched between two asymptotically smooth arcs containing ω_0 (Corollary 2).

If f and f^* are uniformly Lipschitz continuous on ∂D then Γ is locally asymptotically conformal at every point of Γ, so that one might expect (1.2) to hold, since the hypothesis is a uniform one. However, in Section 3 we construct a curve Γ for which f and f^* are Lipschitz but for which (1.2) fails. In fact, for any $\alpha > 1$, Γ may be constructed so that the limit in (1.2) is at least α. In other words, the c in (1.1) cannot be taken to be smaller than α, no matter how small δ is.

In both sections we work with mappings between strip domains which correspond to f and f^*, and the principal tools are the Ahlfors inequalities. In Section 2 we show that the Lipschitz continuity of f and f^* at ξ_0 implies that the strips corresponding to Ω and Ω^* have width nearly π in a certain sense (Theorem 1). Then we show that the strips both have the "Ahlfors distortion property" which means that the real parts of the strip mappings are given asymptotically by the "Ahlfors integral" (Theorem 2). The geometric consequences of this property [11], together with our Theorem 1 then yield our main result.

None of the three results follows from the hypothesis that f, f^{-1} and f^{*-1} are all Lipschitz continuous, as is seen from Theorem 2 in [6].

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2. Consequences of Lipschitz continuity

In all that follows we assume that f and f^* are conformal mappings of $D = \{|\zeta| < 1\}$ and $D^* = \{|\zeta| > 1\}$ onto $\Omega = \text{Int } \Gamma$ and $\Omega^* = \text{Ext } \Gamma$ respectively. Γ is a closed Jordan curve and the functions f and f^* are extended continuously to the boundaries of their domains. Without loss of generality in what follows, we assume that for a certain $\zeta_0 \in \partial D$, $f(\zeta_0) = \omega_0 = f^*(\zeta_0)$, $f(i\zeta_0) = \omega_0' = f^*(i\zeta_0)$ and $f(-\zeta_0) = \omega_0'' = f^*(-\zeta_0)$. We then transform Ω and Ω^* into strip domains via

$$w(\omega) = \log \frac{(\omega_0'' - \omega)}{(\omega_0 - \omega)}$$

where the branch of the logarithm in both Ω and Ω^* is chosen so that for $w_0' = w(\omega_0')$, we have $\text{Im } w_0' \in (-\pi, \pi]$.

The domain Ω corresponds to a strip Σ_1 which is bounded by two Jordan arcs C_1 and C_2, each with $-\infty$ and $+\infty$ as endpoints. Let C_1 be the arc containing w_0'. Then Ω^* corresponds to a strip Σ_2 bounded by C_1 and $C_2' = \{w + 2\pi i, w \in C_2\}$. Letting $z = \log((\zeta_0 + \zeta)/(\zeta_0 - \zeta))$ for suitable branches of the logarithm, D corresponds to $S_1 = \{z = x + iy; |y| < \pi/2\}$ and D^* corresponds to $S_2 = \{z = x + iy : \pi/2 < y < 3\pi/2\}$. The mappings f and f^* correspond to mappings $w_1(z)$ and $w_2(z)$ between S_1 and Σ_1, and S_2 and Σ_2 respectively. We denote their inverses by $z_1(w)$ and $z_2(w)$. The mappings extend continuously to the closures of their domains and we use the same names for the extensions. Note that $w_1(-\infty) = w_2(-\infty) = -\infty$, $w_1(+\infty) = w_2(+\infty) = +\infty$, and $w_1(\pi i/2) = w_2(\pi i/2) = w_0'$.

Next, we let Λ_u be the vertical line $\{w : \text{Re } w = u\}$. For each $i = 1, 2$, $\Sigma_i - \Lambda_u$ contains a component Σ_i' in $\{\text{Re } w < u\}$ which has $-\infty$ as a boundary point. Let $\sigma_i(u)$ be the maximal subinterval of $\Lambda_u \cap \bar{\Sigma}_i$ which is a common boundary arc in $\bar{\Sigma}_i$ of $\bar{\Sigma}_i'$ and the component of $\bar{\Sigma}_i - \bar{\Sigma}_i'$ with $+\infty$ as a boundary point. We let $\theta_i(u)$ be the length of $\sigma_i(u)$ and we note that $\theta_1(u) + \theta_2(u) \leq 2\pi$.

We recall the Ahlfors distortion theorem [5]: fix i and define

$$\underline{x}_i(u) = \min_{w \in \sigma_i(u)} x_i(w) \quad \text{and} \quad \bar{x}_i(u) = \max_{w \in \sigma_i(u)} x_i(w),$$

where $x_i(w) = \text{Re } z_i(w)$. For $u' < u$, if

$$\int_{u'}^{u} \frac{dt}{\theta_i(t)} > 1$$

then

(2.1) $$x_i(u) - \bar{x}_i(u') \geq \pi \int_{u'}^{u} \frac{dt}{\theta_i(t)} - 2\pi.$$

The following lemma characterizes the Lipschitz continuity of f, f^{-1}, f^* and f^{*-1} in terms of the strip mappings (see [6]). We state the result for f; a similar statement holds for f^*. In all that follows $M, M_1, M_2, \ldots, K, K_1, K_2, \ldots$ denote constants. The same symbols may be used for different constants in the statements of different results.

Lemma 1. *Suppose that* $|f(\zeta) - f(\zeta_0)| \leq K_1 |\zeta - \zeta_0|$ *for* $\zeta_0 \in \partial D$ *and all* $\zeta \in \bar{D}$. *Then there exist* M_1 *and* M_2 *such that for* $w = u + iv \in \bar{\Sigma}_1$ *with* $u > 0$, *we have*

(2.2) $$x_1(w) - u \leq M_1 \quad and$$

(2.3) $$\int_{0}^{u} \frac{\pi - \theta_1(t)}{\theta_1(t)} \, dt \leq M_2.$$

If $|f^{-1}(\omega) - f^{-1}(\omega_0)| \leq K_2 |\omega - \omega_0|$ *for* $\omega_0 \in \Gamma$ *and all* $\omega \in \bar{\Omega}$ *then there exists* M_3 *for which*

(2.4) $$M_3 \leq x_1(w) - u.$$

The lemma is stated locally, but if f and f^{-1} are uniformly Lipschitz continuous in $\bar{D}$ and $\bar{\Omega}$ respectively, then M_1, M_2 and M_3 are independent of ζ_0.

We need a sharpening of the inequality between the harmonic and arithmetic means.

Lemma 2. *Suppose that* $\theta_1, \theta_2, \ldots, \theta_n$ *are positive real numbers for which* $\sum_{i=1}^{n} \theta_i \leq 2\pi$, *and that* $\bar{\theta} = (1/n) \sum_{i=1}^{n} \theta_i$. *Then*

(2.5) $$\sum_{i=1}^{n} \frac{1}{\theta_i} \geq \frac{1}{\bar{\theta}} \left(n + \frac{1}{\pi^2} \left(\sum_{i=1}^{n} \left(\theta_i - \frac{2\pi}{n} \right)^2 \right) - \frac{n}{\pi^2} \left(\bar{\theta} - \frac{2\pi}{n} \right)^2 \right).$$

Proof. We recall a well known identity of Lagrange. For $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ real,

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 = \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) - \frac{1}{2} \sum_{j,k=1}^{n} (a_j b_k - a_k b_j)^2.$$

Letting $a_i = \sqrt{\theta_i}$ and $b_i = 1/\sqrt{\theta_i}$, and noting that $\theta_j \theta_k \leq \pi^2$, we have

$$\sum_{i=1}^{n} \frac{1}{\theta_i} = \frac{1}{\sum_{i=1}^{n} \theta_i} \left(n^2 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{(\theta_j - \theta_k)^2}{\theta_j \theta_k} \right)$$

$$\geq \frac{1}{\bar{\theta}} \left(n + \frac{1}{2n} \sum_{j,k=1}^{n} \frac{((\theta_j - 2\pi/n) - (\theta_k - 2\pi/n))^2}{\pi^2} \right)$$

$$= \frac{1}{\bar{\theta}} \left(n + \frac{1}{\pi^2} \sum_{i=1}^{n} \left(\theta_i - \frac{2\pi}{n} \right)^2 - \frac{1}{n\pi^2} \sum_{j,k=1}^{n} \left(\theta_j - \frac{2\pi}{n} \right) \left(\theta_k - \frac{2\pi}{n} \right) \right).$$

The lemma follows by summing the last term.

Theorem 1. *Let* Γ *be a closed Jordan curve and let* f *and* f^* *be conformal mappings of* D *and* D^* *onto* $\Omega = \operatorname{Int} \Gamma$ *and* $\Omega^* = \operatorname{Ext} \Gamma$ *respectively. Suppose that* f *and* f^* *are Lipschitz continuous at* $\zeta_0 \in \partial D$, *with* $f(\zeta_0) = f^*(\zeta_0) = \omega_0$. *Then there exists* M *such that for* $\theta_1(u)$ *and* $\theta_2(u)$ *as above,*

$$(2.6) \qquad \sum_{i=1}^{2} \int_0^{\infty} (\theta_i(u) - \pi)^2 \, du < M.$$

Proof. By (2.3) applied to f and f^*, there exists M_1 such that for $\bar{\theta}(t) = (\theta_1(t) + \theta_2(t))/2$ and for all $u > 0$,

$$(2.7) \qquad M_1 \geqq \int_0^u \frac{\pi}{\theta_1(t)} + \frac{\pi}{\theta_2(t)} - \frac{2\bar{\theta}(t)}{\bar{\theta}(t)} \, dt$$

$$\geqq \int_0^u \frac{2(\pi - \bar{\theta}(t))}{\bar{\theta}(t)} \, dt + \frac{1}{\pi} \int_0^u \frac{1}{\bar{\theta}(t)} \left(\sum_{i=1}^{2} (\theta_i(t) - \pi)^2 \right) dt - \frac{2}{\pi} \int_0^u \frac{(\bar{\theta}(t) - \pi)^2}{\bar{\theta}(t)} \, dt,$$

by Lemma 2. From the proof of Lemma 2 we see that the difference of the last two integrals is non-negative so that

$$\int_0^{\infty} \frac{(\pi - \bar{\theta}(t))}{\bar{\theta}(t)} \, dt \leqq \frac{M_1}{2},$$

from which it follows that

$$\int_0^{\infty} \frac{(\pi - \bar{\theta}(t))^2}{\bar{\theta}(t)} \, dt \leqq \frac{M_1 \pi}{2}.$$

Applying this to (2.7) we obtain (2.6), with $M = 2M_1 \pi^2$.

Theorem 2. *Under the conditions of Theorem 1, each* Σ_i *has the Ahlfors distortion property: For* $w = u + iv$ *and* $w' = u' + iv' \in \Sigma_i$

$$(2.8) \qquad \left| x_i(w) - x_i(w') - \int_{u'}^{u} \frac{\pi \, dt}{\theta_i(t)} \right| = o(1)$$

where $o(1) \to 0$ *as* $u > u' \to \infty$, *uniformly in* v *and* v'.

Proof. We work with Σ_1, the proof for Σ_2 being identical. It suffices to show that for some M,

$$x_1(w) - \int_0^u \frac{\pi}{\theta_1(t)} \, dt \leqq M,$$

since by a result of Eke [4, Theorem 2], this implies that for some β,

$$x_1(w) - \int_0^u \frac{\pi}{\theta_1(t)} \, dt \to \beta \quad \text{as} \quad u \to \infty,$$

from which (2.8) follows.

Now, by (2.2) applied to x_1 and x_2, and by the distortion theorem, we have

$$x_1(w) - u \leqq u - x_2(w) + M_3 \leqq \int_0^u \frac{\theta_2(t) - \pi}{\theta_2(t)} \, dt + M.$$

From the inequality $(\theta_2 - \pi)/\theta_2 \leqq (\pi - \theta_1)/\theta_1$, it follows that

$$x_1(w) \leqq \int_0^u \frac{\pi - \theta_1(t)}{\theta_1(t)} \, dt + M_4$$

so that

$$x_1(w) \leqq \int_0^u \frac{\pi}{\theta_1(t)} \, dt + M_4,$$

and Theorem 2 is proved.

The inequality (2.8) has many geometric consequences, which are studied extensively in [11]. The first result which we will use is the following. Fix i and let $H_t = \{w \in \Sigma_i : y_i(w) = t\}$. Here $|t| < \pi/2$ for Σ_1 and $\pi/2 < t < 3\pi/2$ for Σ_2. Then for u_0 sufficiently large the level line $H_t \cap \{w = u + iv : u > u_0\}$ is the graph of a function $v = \varphi_t(u)$ for which $\varphi_t'(u) \to 0$ as $u \to \infty$, [11, Theorem 7]. We will use this result and Theorem 1 to show that Γ is locally asymptotically conformal at ζ_0, assuming that Γ is a quasiconformal curve. To this end we need to know that the functions $w_i(z)$ are uniformly continuous for large x. This will follow from the next two lemmas.

Lemma 3. *Suppose that Γ is a bounded quasiconformal curve. Let Σ be the image of $\Omega = \mathrm{Int}\, \Gamma$ under*

$$w = \log\left((\omega - \omega_0'')/(\omega - \omega_0)\right),$$

for ω_0 and ω_0'' on Γ as above and suppose that Σ is bounded by the curves C_1 and C_2, each with $-\infty$ and $+\infty$ as endpoints. Then there exist $\delta_0 > 0$ and $K > 1$ such that for $w_1, w_2, w_3 \in C_1$ (or all on C_2) with w_2 between w_1 and w_3 and $|w_1 - w_3| < \delta_0$, we have

$$\frac{|w_1 - w_2|}{|w_1 - w_3|} < K.$$

Proof. Let $\zeta = (\omega - \omega_0)/(\omega - \omega_0'')$ and let C be the image of Γ under this mapping. Then C is also a quasiconformal curve so that there exists $\varkappa > 1$ such that for $\zeta_1, \zeta_2, \zeta_3 \in C$ with ζ_2 on the finite arc $\widehat{\zeta_1 \zeta_3}$, we have ([1], Chapter 4),

(2.9) $|\zeta_1 - \zeta_2|/|\zeta_1 - \zeta_3| < \varkappa.$

Now let w_1, w_2, w_3 be on C_1 with $w_j = \log(1/\zeta_j)$. Then

$$\frac{|w_1 - w_2|}{|w_1 - w_3|} = \frac{|\log(\zeta_2/\zeta_1)|}{|\log(\zeta_3/\zeta_1)|} \leqq \frac{3|\zeta_2/\zeta_1 - 1|}{|\zeta_3/\zeta_1 - 1|} \leqq \frac{3|\zeta_2 - \zeta_1|}{|\zeta_3 - \zeta_1|} < 3\varkappa$$

as long as

(2.10)
$$\left|\frac{\zeta_2}{\zeta_1}-1\right| < \frac{1}{3} \quad \text{and} \quad \left|\frac{\zeta_3}{\zeta_1}-1\right| < \frac{1}{3}.$$

To guarantee (2.10) we note that by (2.9) it suffices to have

(2.11)
$$\left|\frac{\zeta_3}{\zeta_1}-1\right| < \frac{1}{3\varkappa}.$$

On account of the inequality

$$\left|\frac{\zeta_3}{\zeta_1}-1\right| = |e^{w_1-w_3}-1| \leqq \frac{|w_1-w_3|}{1-|w_1-w_3|} \leqq 2|w_1-w_3|$$

for $|w_1-w_3|<1/2$, it follows that (2.11) is true for

$$|w_1-w_3| \leqq \frac{1}{6\varkappa} = \delta_0.$$

Lemma 4. *Let Γ be a bounded quasiconformal curve and let f map D conformally onto $\Omega = \operatorname{Int}\Gamma$. For $\zeta_0 \in \partial D$ and $\omega_0 = f(\zeta_0) \in \Gamma$, suppose that f and f^{-1} are Lipschitz continuous at ζ_0 and ω_0 respectively. Let S and Σ be the corresponding strip domains with $w(z)$ mapping S onto Σ as above. Then there exists a constant M such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $z_0 = x_0 + iy_0 \in \partial S$ with $x_0 \geqq M$ and $z'' \in \bar{S}$ with $|z''-z_0| \leqq \delta$, we have $|w(z_0)-w(z'')| < \varepsilon$.*

Proof. We begin by citing a slight modification of Lemma 1 in [13]: For $z_0 \in \partial S$ and r, $0 < r < 1$, let k_r be the semicircle $\{|z-z_0|=r\} \cap S$. Then for every such $r < 1$ there exists a ϱ_1 with $r \leqq \varrho_1 \leqq r^{1/2}$ such that the image of k_{ϱ_1} is a crosscut of Σ with length

$$l_{\varrho_1} \leqq \left(\frac{2\pi A_{z_0,r}}{\log(1/r)}\right)^{1/2}$$

where $A_{z_0,r}$ is the area of $T_r = w(\{|z-z_0|<r, z \in S\})$.

We must now show that $A_{z_0,r}$ is uniformly bounded for all sufficiently large x_0.

As in our Lemma 1, the local Lipschitz continuity of f and f^{-1} implies the existence of constants M, M_1, and M_3 such that for $x > M-1$ and $z = x+iy \in \bar{S}$, we have

$$M_3 \leqq x - u(z) \leqq M_1.$$

Thus for $x_0 > M$ and $x_0 + r \geqq x \geqq x_0 - r$ we have

$$x_0 + r - M_3 \geqq u(z) \geqq x_0 - r - M_1$$

and T_r is contained in a vertical strip of width $2r + (M_1 - M_3)$. Since the length of $\Sigma \cap l$ does not exceed 2π for any vertical line l, it follows that for $x_0 > M$,

$$A_{z_0,r} \leqq 2\pi(2r + M_1 - M_3)$$

so that

$$l_{\varrho_1} \leq \left(\frac{4\pi^2 (2r + M_1 - M_3)}{\log (1/r)} \right)^{1/2}.$$

Now, let $T_{\varrho_1} = w(\{|z - z_0| < \varrho_1, z \in S\})$ so that $T_r \subset T_{\varrho_1}$. Let w_1 and w_3 be the endpoints of the image of $k\varrho_1$ on C_1 (or C_2) and let C' be the arc of C_1 (or C_2) with w_1 and w_3 as endpoints. Then, for $|z_0 - z''| = r$ we have

$$|w(z_0) - w(z'')| \leq \text{diam } T_r \leq \text{diam } T_{\varrho_1} \leq \text{diam } C' + l_{\varrho_1}.$$

For δ_0 and K as in Lemma 3, choose $\delta > 0$ so small that $r < \delta$ implies that $l_{\varrho_1} < \max (\delta_0, \varepsilon/(2K+1))$. Since $|w_1 - w_3| < l_{\varrho_1}$ we apply Lemma 3 to see that

$$|w(z_0) - w(z'')| \leq 2K l_{\varrho_1} + l_{\varrho_1} < \varepsilon,$$

which proves Lemma 4. We may now prove

Theorem 3. *Assume the conditions of Theorem 1 and that Γ is a quasiconformal curve. Then Γ is asymptotically conformal at ω_0: As ω_1 and ω_2 approach ω_0 from opposite sides*

$$\lim_{\omega_i \to \omega_0} \frac{|\omega_1 - \omega_0| + |\omega_0 - \omega_2|}{|\omega_1 - \omega_2|} = 1.$$

Proof. We may assume that $\omega_0 = 0$. We work with $w_1(z): S_1 \to \Sigma_1$, which we shall refer to as $w(z)$, with inverse $z(w)$. Then for $w(\omega_j) = w_j = u_j + iv_j \in C_j$, $j = 1, 2$, we have

$$\frac{|\omega_1| + |\omega_2|}{|\omega_1 - \omega_2|} = \frac{1 + \left| \dfrac{1 - e^{w_1}}{1 - e^{w_2}} \right|}{\left| 1 - \dfrac{1 - e^{w_1}}{1 - e^{w_2}} \right|} = \frac{1 + e^{u_1 - u_2}}{|1 + e^{u_1 - u_2 + i(v_1 - v_2 - \pi)}|} + o(1)$$

where $o(1) \to 0$ as $u_1, u_2 \to \infty$. We may assume that $u_1 \geq u_2$ so that it suffices to show that

$$\lim_{u_2 \to \infty} \frac{1 + e^{u_1 - u_2}}{|1 + e^{u_1 - u_2 + i(v_1 - v_2 - \pi)}|} = 1.$$

To this end, given $\varepsilon > 0$ we first choose $a \geq 0$ such that $u_1 - u_2 \geq a$ implies that

(2.12)
$$\frac{1 + e^{u_1 - u_2}}{|1 + e^{u_1 - u_2 + i(v_1 - v_2 - \pi)}|} < 1 + \varepsilon,$$

for any v_1, v_2.

Next, choose b such that $0 \leq u_1 - u_2 \leq a$ and $|v_1 - v_2 - \pi| \leq b$ yields (2.12). Finally we show that there exists a real c such that $u_2 \geq c$ and $u_1 - u_2 \leq a$ implies that $|v_1 - v_2 - \pi| \leq b$. By Lemma 4 there exist M and $\delta > 0$, such that for $z = x + iy \in \partial S_1$ with $x \geq M$, and for $z'' \in \bar{S}_1$ with $|z - z''| \leq \delta$, we have

$$|w(z) - w(z'')| < \frac{b}{14}.$$

Next choose $u_0 > M$ so large that for $u \geqq u_0$, the level curves $H_{\pi/2 - \delta}$ and $H_{-\pi/2 + \delta}$ are given by the graphs of functions $\varphi_{\pi/2 - \delta}(u)$ and $\varphi_{-\pi/2 + \delta}(u)$, with $|\varphi'_{\pi/2 - \delta}(u)|$ and $|\varphi'_{-\pi/2 + \delta}(u)|$ both less than or equal to $K_1 = \max(b/(7a), 1)$.

Suppose that $w = u + iv \in C_1$ and $w' = u + iv' \in H_{\pi/2 - \delta}$. Then

$$(2.13) \qquad\qquad |v - v'| \leqq \frac{b}{7}.$$

The inequality (2.13) also holds for $w = u + iv \in C_2$ and $w' = u + iv' \in H_{-\pi/2 + \delta}$.

To establish (2.13), suppose that $z(w) = x + i\pi/2 = z$, $z(w') = x' + i(\pi/2 - \delta) = z'$, $z'' = x + i(\pi/2 - \delta)$ and $w'' = w(z'')$. Then

$$|v - v'| \leqq |v - v''| + |v'' - v'| \leqq |w - w''| + K_1 |u'' - u| \leqq 2|w - w''| \leqq \frac{b}{7}.$$

Now, by Theorem 1 we may choose U_0 such that $u \geqq U_0$ implies that

$$(2.14) \qquad\qquad \int_{U_0}^{\infty} (\theta_1(t) - \pi)^2 \, dt \leqq \left(\frac{b}{7}\right)^2 a.$$

We then let $c = \max(u_0, U_0)$ and suppose that $u_1 \geqq u_2 \geqq c$, $w_1 = u_1 + iv_1 \in C_1$, $w_2 = u_2 + iv_2 \in C_2$ and $u_1 - u_2 \leqq a$. We then choose $u_1 + iv'_1 \in H_{\pi/2 - \delta}$ and $u_2 + iv'_2 \in H_{-\pi/2 + \delta}$. By (2.14) there exists $u_3 \in (u_2, u_2 + a)$ such that $|\theta_1(u_3) - \pi| \leqq b/7$. Taking $u_3 + iv_3 \in C_1$ and $u_3 + iv_4 \in C_2$ to be endpoints of $\sigma(u_3)$, with $u_3 + iv'_3 \in H_{\pi/2 - \delta}$ and $u_3 + iv'_4 \in H_{-\pi/2 + \delta}$, we finally have

$$|v_1 - (v_2 + \pi)| \leqq |v_1 - v'_1| + |v'_1 - v'_3| + |v'_3 - v_3|$$
$$+ |v_3 - (v_4 + \pi)| + |v_4 - v'_4| + |v'_4 - v'_2| + |v'_2 - v_2|$$
$$\leqq \frac{b}{7} + K_1 a + \frac{b}{7} + \frac{b}{7} + \frac{b}{7} + K_1 a + \frac{b}{7} \leqq b,$$

which proves Theorem 3.

The following are consequences of Theorem 3 which follow from results in [12]. The first is an immediate application of our result and Theorem 7 of [12].

Corollary 1. *Assume the hypotheses of Theorem 3. Then*

$$\lim_{\substack{\zeta \to \zeta_0 \\ \zeta \in D}} \frac{f'(\zeta)(\zeta - \zeta_0)}{f(\zeta) - f(\zeta_0)} = 1$$

for nontangential approach. The corresponding result holds for f^*.

Corollary 2. *Assume the hypotheses of Theorem 3. Then $\Omega \cup \{\omega_0\}$ and $\Omega^* \cup \{\omega_0\}$ each contain an open asymptotically smooth arc containing ω_0.*

As we shall see, Γ need not itself be asymptotically smooth. The foregoing means, however, that near ω_0, Γ is trapped between asymptotically smooth arcs.

Proof. Recall the definition of an *L*-strip. R is called an *L-strip* of inclination 0 if

$$R = \{w = u+iv: a<u<\infty, \varphi_-(u)<v<\varphi_+(u)\}$$

where $a \geqq -\infty$ and φ_+, φ_- are continuous functions such that for $u'>u$,

$$\lim_{u\to+\infty} \frac{\varphi_\pm(u')-\varphi_\pm(u)}{u'-u} = 0.$$

By a construction of Rodin and Warschawski [12, p. 135], the conclusion of Theorem 3 implies that each of Σ_1 and Σ_2 contains an *L*-strip of inclination 0. In fact, if R_1 is the *L*-strip contained in Σ_1 with boundary curves given by the graphs of φ_{1+} and φ_{1-}, each of φ_{1+} and φ_{1-} may be taken to be smooth, with

$$\varphi_{1+}'(u) \to 0 \quad \text{and} \quad \varphi_{1-}'(u) \to 0$$

and

$$\varphi_{1+}(u)-\varphi_{1-}(u) \to \pi \quad \text{as} \quad u \to \infty.$$

The image of the graphs of φ_{1+} and φ_{1-} in the ω plane is an asymptotically smooth arc in $\Omega \cup \{\omega_0\}$, containing ω_0. Similarly, the boundary of the *L*-strip contained in Σ_2 corresponds to the asymptotically smooth arc in $\Omega^* \cup \{\omega_0\}$ which contains ω_0.

If the functions f and f^* are uniformly Lipschitz continuous on Γ, then the curve Γ is locally asymptotically conformal at every point. One might then ask whether Γ is then asymptotically conformal, or in light of Corollary 2, even asymptotically smooth. In the next section we shall see that Γ may fail to be asymptotically conformal.

3. Construction of an example

We again use mappings between strip domains, and we shall use the same notations as in Section 2. We shall construct a strip domain Σ_1 in the $w=u+iv$ plane, bounded by Jordan arcs C_1 and C_2, each with $-\infty$ and $+\infty$ as endpoints, i.e. C_j may be given parametrically by $w=w_j(t)$, $-\infty<t<+\infty$, with $\operatorname{Re} w_j(t) \to -\infty$ as $t \to -\infty$ and $\operatorname{Re} w_j(t) \to +\infty$ as $t \to +\infty$. As above Σ_2 will be the "complementary" domain bounded by C_1 and $C_2'=\{w+2\pi i: w\in C_2\}$. We consider the conformal mappings $w_1(z)$ and $w_2(z)$ from S_1 and S_2 onto Σ_1 and Σ_2 respectively with $w_j(-\infty)=-\infty$, $w_j(+\infty)=+\infty$, and $w_j(\pi i/2)=\pi i/2\in\partial\Sigma_j$, for $j=1,2$. The inverses of these functions will again be denoted by $z_j(w)$. Similar notation will denote inverses of defined functions. We define

$$\omega(w) = \frac{e^w-1}{e^w+1}, \quad \zeta(z) = \frac{e^z-1}{e^z+1}$$

in the appropriate domains, so that Σ_1 and Σ_2 correspond to the interior Ω and

exterior Ω^* of a closed Jordan curve Γ, while S_1 and S_2 correspond to D and D^*. Then $f(\zeta)=\omega(w_1(z(\zeta)))$ is a conformal mapping of D onto Ω, and $f^*(\zeta)=\omega(w_2(z(\zeta)))$ maps D^* conformally onto Ω^*.

We shall construct the curves C_1 and C_2 so that Γ is not an asymptotically conformal curve, but so that f and f^* are Lipschitz continuous. We begin by constructing "building blocks".

Choose and fix $\alpha\in(0,\pi/2)$. For each integer $k\geqq3$, consider the following circles in the $\omega=s+it$ plane:

$$T_1: (t-\pi)^2+(s-k)^2 = \left(\frac{\pi}{2}+\alpha\right)^2,$$

$$T_2: t^2+(s-u_2)^2 = \frac{\pi^2}{4}, \quad T_3: t^2+(s-u_3)^2 = \frac{\pi^2}{4},$$

where $u_2=k-(\alpha^2+2\alpha\pi)^{1/2}$ and $u_3=k+(\alpha^2+2\alpha\pi)^{1/2}$, so that T_1 is tangent to T_2 and T_3. Let $L=\{s+it: s\geqq0, t=\pi/2\}$. We trace a curve Γ_k as follows. Starting at $\pi i/2$, move to the right, first on L to T_2, then on T_2 to T_1, T_1 to T_3, T_3 to L and then on L to $+\infty$. Let Γ_k' be the reflection of Γ_k across the s axis and take Ω_k to be the domain bounded by $\Gamma_k\cup\Gamma_k'\cup\{ti: |t|\leqq\pi/2\}$.

Now let $w(\omega)=-ie^{-\omega}+w_k$ for $w_k=2k+\pi i/2$, and let D_k be the image of Ω_k under this mapping. Let $\Sigma=\{w=u+iv: |v|<\pi/2\}$. Delete from Σ the half disks $\{w: |w-w_k|<1, v<\pi/2\}$ and replace them with the D_k. We christen the resulting domain Σ_1 and we let $\Sigma_2=\{w=u+iv: -\pi/2<v<3\pi/2\}-\bar\Sigma_1$. The upper boundary of Σ_1 is then a curve C_1, with a sequence of double bumps going out to $+\infty$, decreasing in size. Under the mapping $\omega=(e^w+1)/(e^w-1)$, Σ_1 is mapped onto a domain Ω which is nearly a unit disk, with a sequence of double bumps converging to $\omega=1$.

In order to show that $\partial\Omega$ is not asymptotically conformal, it suffices to consider the image of $\partial\Sigma_1$ under $W(w)=e^{-w}$, since

$$\omega = \frac{1+e^{-w}}{1-e^{-w}} = \frac{1+W}{1-W} = h(W)$$

and $h(W)$ is conformal at $W=0$.

For each $k(\geqq3)$, let $w_k'=-ie^{-k-i(\pi/2-\alpha)}+w_k$ and $w_k''=-ie^{-k+i(\pi/2-\alpha)}+w_k$. Consider the ratio

(3.1)
$$\frac{|W(w_k')-W(w_k)|+|W(w_k)-W(w_k'')|}{|W(w_k')-W(w_k'')|}$$

$$= \frac{|e^{-w_k'}-e^{-w_k}|+|e^{-w_k}-e^{-w_k''}|}{|e^{-w_k'}-e^{-w_k''}|} = \frac{|e^{w_k-w_k'}-1|+|e^{w_k-w_k''}-1|}{|e^{w_k-w_k'}-e^{w_k-w_k''}|}.$$

Then $w_k-w_k'=e^{-k}(\sin(\pi/2-\alpha)+i\cos(\pi/2-\alpha))=u+iv$ and $w_k-w_k''=-u+iv$.

Furthermore $v=cu$ for $c=1/\tan(\pi/2-\alpha)$. Thus the quantity in (3.1) is

$$\frac{|e^{-u+icu}-1|+|e^{u+icu}-1|}{(e^u-e^{-u})} = (1+c^2)^{1/2}+o(1)$$

as $k\to\infty$, so that $u\to 0$. Thus we see that

$$\lim_{|\omega'-\omega''|\to 0} \sup_{\omega\in\Gamma(\omega',\omega'')} \frac{|\omega'-\omega|+|\omega-\omega''|}{|\omega'-\omega''|} \geqq (1+c^2)^{1/2} = \frac{1}{\sin(\pi/2-\alpha)},$$

which may be made as large as desired by taking α near $\pi/2$.

We must now show that f and f^* are uniformly Lipschitz continuous on ∂D. We shall consider only f, as the proof for f^* is essentially the same. The Lipschitz continuity of f follows from the following result.

Lemma 5. *With the notations above, suppose that there exist positive constants* M_1, K_1 *and* K_2 *such that for almost every* $z\in\partial S_1$ *with* $\operatorname{Re} z=x>M_1$ *we have*

$$(3.2) \qquad\qquad \left|\frac{dw_1}{dz}(z)\right| < K_1$$

and for $u_1(z)=\operatorname{Re} w_1(z)$ *we have*

$$(3.3) \qquad\qquad x-u_1(z) < K_2.$$

Then there exist positive K_3 *and* δ *such that for almost every* $\zeta\in\partial D$ *with* $|\zeta-1|<\delta$,

$$\left|\frac{df}{d\zeta}(\zeta)\right| < K_3.$$

Proof. We have, for $\zeta\neq\pm 1$, $z=z(\zeta)$ and $w=w_1(z)$,

$$\left|\frac{df}{d\zeta}\right| = \left|\frac{d\omega}{dw}\right|\left|\frac{dw_1}{dz}\right|\left|\frac{dz}{d\zeta}\right| = \left|\frac{2e^w}{(e^w+1)^2}\right|\left|\frac{dw_1}{dz}\right|\left|\frac{(e^z+1)^2}{2e^z}\right|$$

$$= \left|\frac{dw_1}{dz}\right||e^{z-w}|\left|\frac{1+e^{-z}}{1+e^{-w}}\right|^2 = \left|\frac{dw_1}{dz}\right|e^{x-u_1(z)}\big(1+o(1)\big) \quad\text{as}\quad x\to\infty. \qquad\text{QED.}$$

We first show that (3.3) holds. Recall the Ahlfors upper inequality proved in [5, Theorem 3]: For $u>0$ and $x_1(w)=\operatorname{Re} z_1(w)$,

$$(3.4) \qquad\qquad \bar{x}_1(u)-\underline{x}_1(0) \leqq \pi\int_0^u \frac{dt}{\theta_1(t)}+M_2$$

where the constant M_2 depends only on α in our case. From this we see that

$$(3.5) \qquad\qquad x_1(w)-u \leqq \int_0^u \frac{\pi-\theta_1(t)}{\theta_1(t)}\,dt+M_3$$

for $M_3=M_2+\underline{x}_1(0)$. Since $\theta(t)=\pi$ off of the intervals

$$[2k-e^{-k+(\alpha^2+2\alpha\pi)^{1/2}}, 2k+e^{-k+(\alpha^2+2\alpha\pi)^{1/2}}]$$

and
$$0 \leqq \pi - \theta_1(t) \leqq e^{-k+(\alpha^2+2\alpha\pi)^{1/2}}$$
on the corresponding interval, the integral in (3.5) converges as $u \to \infty$, and (3.3) follows.

We next note that the domain Σ_1 is such that the mapping $z_1(w) = x_1(w) + iy_1(w)$ from Σ_1 onto S_1 is "semiconformal" [14]:

For any r, $0 < r < \pi/2$, there exists u_r such that
$$S_1 \supset S_r = \left\{ u + iv: \ u \geqq u_r, \ -\frac{\pi}{2} + r < v < \frac{\pi}{2} - r \right\}$$
and
(3.6) $$\lim_{\substack{w \to +\infty \\ w \in S_r}} y_1(w) - v = 0.$$

This will be used in showing that (3.2) holds for w_1. The proof of (3.2) is somewhat delicate, and we must consider properties of the Ω_k.

For fixed $k \geqq 3$, let $z_k = z_1(w_k)$ and define $\zeta(z) = \mathrm{Log}\left(1/(z-z_k)\right) - (\pi/2)i$ for $z \in S_1$, so that $|\mathrm{Im}\,\zeta(z) < \pi/2|$. Let $\omega(w) = \mathrm{Log}\left(1/(w-w_k)\right) - (\pi/2)i$ for $w \in D_k$. Then $\zeta_k(\omega) = \zeta(z_1(w(\omega)))$ maps Ω_k conformally onto a strip domain S_k which is bounded by the lines
$$\left\{ \zeta = \xi + i\frac{\pi}{2}: \ \xi \geqq \log\left|\frac{1}{z_1(w_k-1)-z_k}\right| \right\}$$
and
$$\left\{ \zeta = \xi - i\frac{\pi}{2}: \ \xi \geqq \log\left|\frac{1}{z_1(w_k+1)-z_k}\right| \right\}.$$

and by a curve γ_k connecting the left endpoints of the lines. Observe that for $\zeta \in \gamma_k$, $|\mathrm{Im}\,\zeta| \leqq \pi/2$.

Lemma 6. *Let* $\zeta_k(\omega) = \xi_k(\omega) + i\eta_k(\omega)$ *map* Ω_k *onto* S_k *as above. Then there exists* M_1 *such that for all* $k \geqq 3$,
(3.7) $$\xi_k(0) \leqq M_1.$$

Furthermore, there exists M_2 *depending only on* α *such that for* $\omega = s > M_2$
(3.8) $$|\eta_k(s)| < (\pi/2 + \alpha)/2 < \pi/2, \quad \text{for all} \quad k.$$

Proof. Consider $\zeta_k' = \zeta(z_1(w_k - i)) \in \gamma_k$. By (3.6), $\mathrm{Im}\left(z_1(w_k-i)\right) \to \pi/2 - 1$ as $k \to \infty$, so that $|z_1(w_k-i) - z_k| > 1/2$ for k sufficiently large. From this it follows that $\xi_k(0) \leqq \mathrm{Re}\,\zeta_k' < \log 2$ for large k and (3.7) then follows.

Next, let $h_k(\omega) = \eta_k(\omega) - t$. Then h_k is harmonic in Ω_k and for $\omega = it \in \partial\Omega_k$, $|t| < \pi/2$, we have $|h_k(\omega)| < \pi$. On the rest of $\partial\Omega_k$, $|h_k(\omega)| \leqq \alpha$. Thus, for any $\varepsilon > 0$, the two constants theorem and a standard comparison argument show that there exists $K > 0$ for which $\mathrm{Re}\,\omega > K$ implies that $|h_k(\omega)| \leqq \alpha + \varepsilon$. We let $\omega = s$ and $\varepsilon = (\pi/2 - \alpha)/2$ to obtain (3.8).

Next we state a result which will be applied to the mapping $w_1(z)$ and to the $\omega_k(\zeta)$. We state it for w_1. The proof is essentially that of Lemma 3 in [6].

Lemma 7. *Let* $w_1(z)$ *map* S_1 *conformally onto* Σ_1 *as above. Let* $w = w_1(z) \in \partial \Sigma_1$ *and let* $r > 0$ *and* $M > 0$ *be such that:*

(3.9) *There exist circles* T *and* T' *with radius* r *and tangent to* $\partial \Sigma_1$ *at* w, *with* $T \subset \bar{\Sigma}_1$ *and* $T' \subset \overline{(-\Sigma_1)}$.

(3.10) *The center of* T *is* $w_0 = w_1(z_0)$, *and* $|\mathrm{Im}\, z_0| \leq M < \pi/2$.

Then there exists $K > 0$, *depending only on* r *and* M, *such that*

$$\left| \frac{dw_1}{dz}(z) \right| < K.$$

Lemmas 6 and 7 will be used to prove

Lemma 8. *There exist positive constants* δ *and* K', *depending only on* α, *such that for* $w \in C_1$ *with* $0 < |w - w_k| < \delta$ *for any* $k \geq 3$, *we have, for* $z = z_1(w)$

$$\left| \frac{dw_1}{dz}(z) \right| < K'.$$

Proof. We shall establish that there exist constants M, K_1 and K_2 such that for $\zeta \in \partial S_k$ with $\xi = \mathrm{Re}\, \zeta > M$, we have

(3.11) $$\left| \frac{d\omega_k}{d\zeta}(\zeta) \right| < K_1, \quad \text{and for} \quad s_k(\zeta) = \mathrm{Re}\, \omega_k(\zeta)$$

we have

(3.12) $$\xi - s_k(\xi) < K_2.$$

We begin by applying Ahlfors' upper inequality to the mapping $\zeta_k(\omega)$ to see that there exists M_2, depending only on α such that

$$\bar{\xi}_k(s) - \underline{\xi}_k(0) \leq \int_0^s \frac{\pi}{\theta(\tau)}\, d\tau + M_2.$$

By (3.7) and the construction of Ω_k,

$$\xi_k(\omega) - s \leq M_1 + M_2 + \int_0^s \frac{\pi - \theta(\tau)}{\theta(\tau)}\, d\tau \leq K_2$$

so that (3.12) holds.

Next we may suppose that $\omega = \omega_k(\zeta) \in \Gamma$ with $\mathrm{Re}\, \omega > \pi/2$. Then there exist circles T and T' which are tangent to Γ_k at ω and whose interiors are respectively interior and exterior to Ω_k. We take the center of T to be at s_0 on the s axis, and the radius of both circles to be $r \geq (\pi/2) - \alpha$. By Lemma 6 there exists $M > 0$ such that for $s_0 > M$, $|\eta_k(s_0)| \leq ((\pi/2) + \alpha)/2 < \pi/2$. Then apply the version of Lemma 7 for the mappings $\omega_k(\zeta)$ between half strips to establish (3.11), where K_1 depends only on r and M, which in turn depend only on α. Lemma 8 is then proved by the argument in the proof of Lemma 5.

We may now show that (3.2) holds. Let $\delta < 1$ be as in Lemma 8 and let $N (\geqq 3)$ be such that for $k \geqq N$, $\exp \left(-k + (\alpha^2 + 2\alpha\pi)^{1/2} \right) < \delta/2$. Let $A = \{ w \in C_1 : 0 < |w - w_k| < \delta,\ k \geqq N \}$ and let $B = \{ w \in \partial S_1 : \operatorname{Re} w \geqq 2N,\ w \in A \cup \{ w_k \} \}$. Then any point of B is on one of the lines $v = \pi/2$ or $v = -\pi/2$, and for any such point, the condition (3.9) holds with $r = \delta/2$. Because of (3.6), there exists $M_1 > 2N$ such that (3.10) holds for $w \in B$ with $\operatorname{Re} w \geqq M_1$. Thus (3.2) holds for almost every $w \in \partial S_1$ for which $\operatorname{Re} w \geqq M_1$ and with $K_1 = \max (K, K')$. Thus Lemma 5 may be applied to $w_1(z)$ and it finally follows that f is Lipschitz continuous on ∂D.

References

[1] AHLFORS, L. V.: Lectures on quasiconformal mappings. - D. Van Nostrand Company, Inc., Princeton, New Jersey—Toronto—New York—London, 1966.

[2] BECKER, J., and CH. POMMERENKE: Über die quasiconforme Fortsetzung schlichter Funktionen. - Math. Z. 161, 1978, 69—80.

[3] BECKER, J., and CH. POMMERENKE: Hölder continuity of conformal mappings and non-quasiconformal Jordan curves. - Comment. Math. Helv., 57, 1982, 221—225.

[4] EKE, B. G.: Remarks on Ahlfors' distortion theorem. - J. Analyse Math. 19, 1967, 97—134.

[5] JENKINS, J. A., and K. OIKAWA: On results of Ahlfors and Hayman. - Illinois J. Math. 15, 1971, 664—671.

[6] LESLEY, F. D.: On interior and exterior conformal mappings of the disk. - J. London Math. Soc. (2) 20, 1979, 67—78.

[7] LESLEY, F. D.: Hölder continuity of conformal mappings at the boundary via the strip method. - Indiana Univ. Math. J. 31, 1982, 341—354.

[8] LESLEY, F. D., and S. E. WARSCHAWSKI: Boundary behavior of the Riemann mapping function of asymptotically conformal curves. - Math. Z. 179, 1982, 299—323.

[9] NÄKKI, R., and B. PALKA: Quasiconformal circles and Lipschitz classes. - Comment. Math. Helv. 55, 1980, 485—498.

[10] POMMERENKE, CH.: On univalent functions, Bloch functions and VMOA. - Math. Ann. 236, 1978, 199—208.

[11] RODIN, B., and S. E. WARSCHAWSKI: Extremal length and univalent functions III. Consequences of the Ahlfors distortion property. - Bull. Inst. Math. Acad. Sinica 6, 1978, 583—597.

[12] RODIN, B., and S. E. WARSCHAWSKI: On the derivative of the Riemann mapping function near a boundary point and the Visser—Ostrowski problem. - Math. Ann. 248, 1980, 125—137.

[13] WARSCHAWSKI, S. E.: On the degree of variation in conformal mapping of variable regions. - Trans. Amer. Math. Soc. 69, 1950, 335—356.

[14] WARSCHAWSKI, S. E.: On the boundary behavior of conformal maps. - Nagoya Math. J. 30, 1967, 83—101.

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