DIFFEOMORPHIC APPROXIMATION OF QUASICONFORMAL AND QUASISYMMETRIC HOMEOMORPHISMS

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Introduction

Piecewise linear approximations for quasiconformal, bilipschitz and quasi-symmetric embeddings have been constructed by Carleson [C], Väisälä [V], Kiikka [K] and by Luukkainen and Tukia [LT]. We apply a result of Munkres [M, Theorem 4.1] to smooth these approximations into $C^\infty$-embeddings in the homeomorphic case. By [LV, V. 4.3] every finite $K$-quasiconformal mapping in dimension 2 is locally the limit of a sequence of regular $K$-quasiconformal $C^\infty$-mappings.

I wish to thank Pekka Tukia for suggesting this problem and Jouni Luukkainen for a careful reading of the manuscript and for making several corrections.

1. Preliminaries

We will use the same notation and definitions as in [K] and [LT]. If $Q$ is a closed $n$-cube of $R^n$ with side length $2\lambda_Q$ and center $z_Q$, define $\varphi_Q(x) = z_Q + \lambda_Q x$. In open sets of $R^n$ we have canonical decompositions $\mathcal{K}$ into cubes as in [K]. Let $I^n$ denote the closed cube $[-1, 1]^n$ in $R^n$ and $J^n$ the open cube $]-1, 1[^n$. If $Q \in \mathcal{K}$, then $Q = \varphi_Q I^n$. For $Q \in \mathcal{K}$ define

$$P_Q = P_Q(\mathcal{K}) = \{ \varphi_Q^{-1} Q' | Q' \in \mathcal{K}, Q' \cap Q \neq \emptyset, Q' \neq Q \}.$$ Denote by $\mathcal{H}^n$ the set of all canonical decompositions of open sets of $R^n$. Set

$$\mathcal{P}^n = \{ P_Q(\mathcal{K}) | Q \in \mathcal{K} \in \mathcal{K}^n \}.$$ Then $\mathcal{P}^n$ is finite.

Let $T$ be a triangulation of an $n$-dimensional PL manifold $X$ in $R^n$. If a $k$-dimensional open simplex $\sigma$ of $T$ has vertices $a_0, \ldots, a_k$, we write $\sigma = (a_0, \ldots, a_k)$. If $0 \leq k \leq n-1$, let $\{ \sigma_1, \ldots, \sigma_p \}$ be the set of those $n$-simplices of $T$ which have $\sigma$ as a face. Let

$$\sigma_i = (a_0, \ldots, a_k, a_{k+1}^i, \ldots, a_n^i), \quad i \in \{1, \ldots, p\}.$$
If \( x \in \sigma_i \), denote by \( \lambda_j^i(x), j \in \{0, \ldots, n\} \), the barycentric coordinates of \( x \) in \( \sigma_i \). If \( s \in [0, \infty[ \), set
\[
V_s(\sigma) = \bigcup_{l=1}^p \{ x \in \sigma_i | \lambda_j^l(x) \equiv s \lambda_j^l(x) \text{ for all } l \leq k, j > k \}.
\]
Then \( V_s(\sigma) \) is a closed neighborhood of \( \sigma \) in \( X \).

For \( \sigma \in T \) set \( \text{St}(\sigma, T) = \{ \tau \in T \mid \sigma \text{ is a face of } \tau \} \).

Denote by \( T^k \) the set of \( k \)-simplices of \( T \). If \( A \subset X \), set \( T | A = \{ \sigma \in T \mid \sigma \subset A \} \) and \( T^k | A = \{ \sigma \in T^k \mid \sigma \subset A \} \).

## 2. Results

The following theorem is analogous to [K, Theorem 2.1].

**Theorem 1.** Let \( n \geq 2 \) or \( n = 3 \) and let \( K \geq 1 \). Then there exists \( K \geq 1 \) with the following property: Let \( G \) and \( G' \) be domains in \( \mathbb{R}^n \), let \( f: G \to G' \) be a \( K \)-quasiconformal homeomorphism and let \( \varepsilon: G \to [0, \infty[ \) be continuous. Then there exists a \( K \)-quasiconformal \( C^\infty \)-diffeomorphism \( \tilde{f}: G \to G' \) such that \( |\tilde{f}(x) - f(x)| < \varepsilon(x) \) for every \( x \in G \). Theorem 2 is analogous to [K, Theorem 3.1] and [LT, Corollary 3.3], and Theorems 3 and 4 to [LT, Corollary 2.21] and [LT, Theorem 2.16], respectively.

**Theorem 2.** Let \( n \geq 2 \) or \( n = 3 \) and let \( L \geq 1 \). Then there exists \( L \geq 1 \) with the following property: Let \( G \) and \( G' \) be open sets in \( \mathbb{R}^n \), let \( f: G \to G' \) be an \( L \)-bilipschitz homeomorphism and let \( \varepsilon: G \to [0, \infty[ \) be continuous. Then there exists an \( L \)-bilipschitz \( C^\infty \)-diffeomorphism \( \tilde{f}: G \to G' \) such that \( |\tilde{f}(x) - f(x)| < \varepsilon(x) \) for every \( x \in G \).

**Theorem 3.** Let \( n \geq 2 \) or \( n = 3 \) and let \( \eta: \mathbb{R}_+^n \to \mathbb{R}_+^n \) be a homeomorphism. Then there exists a homeomorphism \( \tilde{\eta}: \mathbb{R}_+^n \to \mathbb{R}_+^n \) with the following property: Let \( G \) and \( G' \) be open sets in \( \mathbb{R}^n \), let \( f: G \to G' \) be an \( \eta \)-quasisymmetric homeomorphism and let \( \varepsilon: G \to [0, \infty[ \) be continuous. Then there exists an \( \eta \)-quasisymmetric \( C^\infty \)-diffeomorphism \( \tilde{f}: G \to G' \) such that \( |\tilde{f}(x) - f(x)| < \varepsilon(x) \) for every \( x \in G \).

Let \( \varrho_G \) and \( \beta_G \) be as in [K] and [LT].

**Theorem 4.** Let \( n \) and \( \eta \) be as in Theorem 3 and let \( \varepsilon > 0 \). Then there exist a homeomorphism \( \tilde{\eta}: \mathbb{R}_+^n \to \mathbb{R}_+^n \) and a finite set \( \tilde{D} \) of \( C^\infty \)-embeddings \( \tilde{g}: 2I^n \to \mathbb{R}^n \) with the following property: Let \( G \) be open in \( \mathbb{R}^n \), \( f: G \to \mathbb{R}^n \) an \( \eta \)-quasisymmetric embedding and let \( \mathcal{X} \) be a canonical decomposition of \( G \). Then there exists an
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\[ \hat{f} \text{-quasisymmetric } C^\infty \text{-embedding } \hat{f}: G \rightarrow R^n \text{ such that} \]

\[ d(\hat{f}|Q, f|Q) \equiv \varepsilon_Q \]
and
\[ \beta_Q \hat{f}_{\mid 2I^n} \in \hat{D} \]

for every \( Q \in \mathcal{K} \).

3. Proofs

Proof of Theorem 1. Let \( n=2 \) or \( n=3 \) and \( K \geq 1 \). Let \( \delta_M = \delta_{M(0)} > 0 \) be the constant of [K, p. 12]. The proof of [K, Lemma 2.2] shows that there exists a finite set \( D=D(K) \) of PL embeddings of \( 2I^n \) into \( R^n \) with the following property: Let \( G \) be a domain in \( R^n, f: G \rightarrow R^n \) a \( K-QC \) embedding and \( \mathcal{K} \in \mathcal{K}^n \) with \( \cup \mathcal{K} = G \). Then there exists a PL embedding \( \hat{f}^*: G \rightarrow R^n \) such that \( \beta_Q \hat{f}^* \in \hat{D} \) and \( |f^*(x) - f(x)| < \varepsilon_Q \) if \( x \in Q \in \mathcal{K} \). In fact, to see this, consider the maps \( \gamma_Q \) in [K, p. 12].

We choose a triangulation \( \tau \) of \( I^n \) such that \( g|\sigma \) is affine and \( d(g|\sigma) < \delta_M \) if \( g \in D \) and \( \sigma \in \tau \) and such that \( g|\alpha_R \sigma \) is affine if \( g \in D, R \in \mathcal{P}, \sigma \in \tau \) and \( \alpha_R \sigma \subset 2I^n \). We may assume that \( \tau \) is a triangulation of \( Q \) and that \( \tau \) is a full subcomplex (see [RS, p. 31]) of \( \tau \) for each of the \( 4^n \) subcubes \( Q \in \{ z+[0,1/2] \} \) \( z \in (1/2)Z^n \) of \( I^n \). Furthermore, if \( n=3 \), we may assume for all 2-dimensional faces \( S \) of these cubes \( Q \) that \( \tau \) is a full subcomplex of \( \tau |S \).

For each \( P \in \mathcal{P} \) we next define a subdivision \( \tau_P \) of \( \tau \) such that, if \( G \) is open in \( R^n \) and if \( \mathcal{K} \) is a canonical decomposition of \( G \), then

\[ T = \{ z_Q \mid Q \in \mathcal{K}, \sigma \in \tau_{P_Q} \} \]

is a triangulation of \( G \).

We first suppose that \( P \in \mathcal{P} \) and set

\[ I_P = \cup \{ \partial I^3 \cap \partial R_1 \cap \partial R_2 \mid R_1, R_2 \in P \text{ and } R_1 \neq R_2 \}. \]

Let \( \gamma_1 \) be the triangulation of \( \partial I^3 \) satisfying

\[ \gamma_1^0 = (\tau^0 \cap I_P) \cup (\bigcup_{R \in P} (\alpha_R \tau^0 \cap \partial I^3)) \]

and such that \( \gamma_1 \) is a subdivision of \( \tau \) for every \( R \in P \). Let \( \gamma_2 \) be the subdivision of \( \tau \) having

\[ \gamma_2^0 = (\tau^0 \cap I^3) \cup (\bigcup_{R \in P} (\alpha_R \tau^0 \cap I_P)). \]

Then \( \gamma_1 I_P = \gamma_2 I_P \). Set \( \gamma_3 = \{ \sigma_1 \cap \sigma_2 \mid \sigma_1, \sigma_2 \in \gamma_1 \} \). Then \( \gamma_3 \) is a cell complex on \( \partial I^3 \). By replacing every (open) 2-cell \( \sigma \in \gamma_3 \) which is not a simplex by the simplices obtained by joining the barycenter of \( \sigma \) with the cells of \( \gamma_3 \) in \( \partial \sigma \), we get a triangulation \( \gamma_4 \) of \( \partial I^3 \). Then \( \gamma_4 \) is a subdivision of \( \tau \).

Let \( \tau_P \) be the subdivision of \( \tau \) having \( \tau_P^0 = \tau^0 \cup \gamma_4^0 \) and \( \tau_P \cap I^3_3 = \gamma_4 \).

To see that \( \{ \tau_P \mid P \in \mathcal{P} \} \) is the desired family, suppose that above \( P = P_Q \),
where $Q \in \mathcal{X} \in \mathcal{X}^n$, and denote $\gamma_i(Q) = \gamma_i$ if $1 \leq i \leq 4$. Then, for each $Q' \in \mathcal{X}$ with $Q' \cap Q \neq 0, Q' \neq Q$, we have $\alpha_Q \gamma_i(Q) | Q \cap Q' = \alpha_{Q'} \gamma_i(Q') | Q \cap Q'$, whence $\alpha_Q \gamma_i(Q) | Q \cap Q' = \alpha_{Q'} \gamma_i(Q') | Q \cap Q'$ for $i = 3$ and thus also for $i = 4$. Therefore $\alpha_Q \tau_{P_Q} | Q \cap Q' = \alpha_{Q'} \tau_{P_Q} | Q \cap Q'$.

If $P \in \mathcal{P}$, the construction of $\tau_P$ is similar but easier.

For every $\mathcal{X} \in \mathcal{X}^n$ and $Q \in \mathcal{X}$ we define a triangulation $T_{P_Q} = T(\mathcal{X}, Q)$ of $2I^n$ by setting

$$T_{P_Q} = \alpha_Q^{-1}T|2I^n,$$

where $T$ is defined by (3).

One can show that $T_{P_Q}$ depends only on the set $P_Q \in \mathcal{P}$. Instead of this fact we could have used below the easier fact that the set $\{T(\mathcal{X}, Q) | Q \in \mathcal{X} \in \mathcal{X}^n\}$ is finite.

We are going to smooth the PL maps $g \in D$ in some neighborhoods of $I^n$. The groups $I^i$ of Milnor and Thom, cf. [M, Chapter 1], are zero for $i \in \{1, 2, 3\}$; see [M, Proof of Theorem 6.3].

For $P \in \mathcal{P}^n$ and $0 \leq i \leq n - 1$ set

$$U^i_P = \cup \{V^i(\sigma) : \sigma \in T^i_P|2J^i\}.$$

Then $U^i_P$ is a closed neighborhood of $I^n$ in $2I^n$.

We first suppose that $n = 2$. Let $g \in D$ and $P \in \mathcal{P}^2$. Then $g|\bar{\sigma}$ is affine if $\sigma \in T_P$. Let $\sigma \in T^1_P|2J^2$. We apply [M, Theorem 4.1] with $\mathcal{V} = \text{int} V_1(\sigma), \mathcal{W} = \mathcal{V} \setminus \sigma, \mathcal{V}' = \text{int} V_2(\sigma)$ to obtain a homeomorphism

$$\varphi = \varphi_{g, \sigma} : V_1(\sigma) \to gV_1(\sigma)$$

such that

$$\varphi = g \quad \text{in} \quad \sigma \cup (V_1(\sigma) \setminus V_2(\sigma)),
\quad d(\varphi, g|V_1(\sigma)) < d(g|\sigma)/2,$$

and such that $\varphi|\text{int} V_1(\sigma)$ is a $C^1$-embedding and $\varphi$ is smooth on $\text{int} V_1(\sigma)$ near the vertices of $\sigma$; see [M, Definition 2.2]. This is possible because $g|V_1(\sigma)$ is smooth on $\text{int} V_1(\sigma) \setminus \sigma$ near $\sigma$ and near the vertices of $\sigma$ (cf. [M, Proof of Theorem 2.8]), and because $\gamma(g|V_1(\sigma)) \in I^1 = 0$ (cf. [M, Definition 3.4]).

Define a homeomorphism

$$\tilde{g}^1_P = \cup \{\varphi_{g, \sigma} : \sigma \in T^1_P|2J^2\} : U^1_P \to gU^1_P.$$

Let $v \in T^0_P|2J^2$. Because $\tilde{g}^1_P|V_1(v)$ is smooth on $\text{int} V_1(v) \setminus v$ near $v$ and $\gamma(\tilde{g}^1_P|V_1(v)) \in I^2 = 0$, we may apply [M, Theorem 4.1] to get a homeomorphism

$$\varphi = \varphi_{g, v} : V_1(v) \to \tilde{g}^1_P V_1(v)$$

such that

$$\varphi = \tilde{g}^1_P \quad \text{in} \quad v \cup (V_1(v) \setminus V_2(v)),
\quad d(\varphi, \tilde{g}^1_P|V_1(v)) < \min\{d(g|\sigma^2)|v|v|v\}$$

and $\varphi|\text{int} V_1(v)$ is a $C^1$-embedding. We get a homeomorphism

$$\tilde{g}^0_P = \cup \{\varphi_{g, v} : v \in T^0_P|2J^2\} : U^0_P \to \tilde{g}^1_P U^0_P.$$
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Then \( \tilde{g}_P^0 \mid \text{int } U_P^0 \) is a \( C^1 \)-embedding. One may replace above \( C^1 \) by \( C^\infty \); see [M, Chapter 9].

We construct the maps \( \tilde{g}_P^0 \) and \( \bar{g}_P^0 \) for every \( g \in D \) and \( P \in \mathcal{D}_2 \). It is possible to do this in such a way that the following condition is satisfied:

Let \( i \in \{0, 1\}, j \in \{1, 2\}, P_i \in \mathcal{D}_2, \sigma_j \in T_P^j \mid \mathcal{F}^j \) and let \( g, h \in D \). If \( \psi_1 \) and \( \psi_2 \) are affine bijections of \( R^2, \psi_1 \text{ St}(\sigma_2, T_{P_j})=\text{St}(\sigma_1, T_{P_i}) \) and \( h=\psi_2 g \psi_1 \) in \( V_1(\sigma_2) \), we have

\[
R_{P_j} = \psi_2 \tilde{g}_P^1 \psi_1
\]

in \( V_1(\sigma_2) \). Here \( V_1(\sigma_2) \) is taken in \( T_{P_j} \).

Let \( \tilde{g}_P^0 \colon \text{int } U_P^0 \to R^2 \) be the \( C^\infty \)-embedding defined by \( \tilde{g}_P^0 \).

If \( x \in I^2 \), we have

\[
|\tilde{g}_P^0(x) - g(x)| \leq |\tilde{g}_P^0(x) - \tilde{g}_P^1(x)| + |\tilde{g}_P^1(x) - g(x)| < \delta_M
\]

because \( d(gos^2)<\delta_M \) for all \( s^2 \in T_P^2 \mid I^2 \).

If \( n=3 \), we may proceed in the same way, because \( I^1=0 \) for \( i \in \{1, 2, 3\} \).

Let \( G \) and \( G' \) be domains in \( R^n, f \colon G \to G' \) a \( K-\mathcal{Q}C \) homeomorphism, \( \varepsilon \colon G \to 0, \varepsilon \) continuous and \( \mathcal{K} \) a canonical \( (f, \varepsilon/2) \)-decomposition of \( G \). Let \( f^+ \colon G \to R^n \) be the PL embedding given in the first paragraph of the proof. We define \( f^+ \colon G \to R^n \) setting

\[
\tilde{f}^+:G \to R^n \text{ setting}
\]

\[
\tilde{f}^+|Q = \beta_Q^{-1} \tilde{g}_P^0 \bar{z}_Q^{-1}|Q
\]

for \( Q \in \mathcal{K} \) whenever \( \beta_Q f^+ \bar{z}_Q \mid 2I^n=g \mid ( \rho D ) \).

We show that \( \tilde{f}^+ \) is well-defined and that

\[
\tilde{f}^+|Q \in \mathcal{K} \text{ whenever } \beta_Q f^+ \bar{z}_Q \mid 2I^n=g \mid ( \rho D ) \text{ and } \beta_Q \bar{z}_Q^{-1}
\]

for each \( Q \in \mathcal{K} \). Let \( R \in \mathcal{K}, R \neq Q, R \cap Q=0 \) and \( x \in R \cap U_Q \). There is \( v \in T^0 \mid R \cap \bar{z}_Q(2I^n) \) such that \( x \in V_1(v) \), where \( V_1(v) \) is taken in the triangulation \( T \) of \( G \).

Let \( v_Q=\bar{z}_Q^{-1} v, v_R=\bar{z}_R^{-1} v, g=\beta_Q f^+ \bar{z}_Q \mid 2I^n \) and \( h=\beta_R f^+ \bar{z}_R \mid 2I^n \). Then \( \text{St}(v_Q, T_{P_Q})=\bar{z}_Q^{-1} \text{St}(v_R, T_{P_R}) \) and \( h=\beta_R \beta_Q^{-1} g \bar{z}_Q^{-1} \bar{z}_R \) in \( V_1(v_R) \). Because \( \psi_1=\bar{z}_Q^{-1} \bar{z}_R \) and \( \psi_2=\beta_R \beta_Q^{-1} \) are affine bijections of \( R^n \), it follows from (4) that

\[
\beta_R^{-1} \tilde{h}_{P_R} \bar{z}_R^{-1}(x) = \beta_R^{-1} (\beta_R \beta_Q^{-1} \tilde{g}_P^0 \bar{z}_Q^{-1} \bar{z}_R) \bar{z}_R^{-1}(x) = \beta_Q^{-1} \tilde{g}_P^0 \bar{z}_Q^{-1}(x).
\]

Hence \( \tilde{f}^+ \) is well-defined and (6) holds.

By the construction, \( \tilde{f}^+ \) is a \( C^\infty \)-embedding. The maps \( \tilde{g}_P^0 \mid J^n \) are quasiconformal. Set

\[
\bar{G} = \max \{K(\tilde{g}_P^0 \mid J^n) \mid g \in D, P \in \mathcal{D}\}
\]

Then \( \tilde{f}^+ \) is a \( \bar{G}-\mathcal{Q}C \) embedding.

For \( Q \in \mathcal{K} \) we have

\[
d(\tilde{f}^+|Q, f^+|Q) \leq d(\tilde{f}^+|Q, f^+|Q) + d(f^+|Q, f|Q) \leq \delta_M s_Q + \delta_Q.
\]

Hence \( |\tilde{f}(x) - f(x)| \leq 2\delta_Q < \varepsilon(x) \) if \( x \in Q \in \mathcal{K} \). We may assume that

\[
\varepsilon(x) \equiv \min \{d(f(x), \partial G), (1+|f(x)|)^{-1}\}
\]

for every \( x \in G \). Therefore \( \tilde{f} G = G' \); see [K, p. 8]. Theorem 1 is proved. \( \Box \)
Proof of Theorem 4. Let \( D \) be the set of PL embeddings given by [LT, Theorem 2.16] and let \( \delta_M = \delta_M(n) > 0 \) be the constant of [LT, 2.13], both with \( q = n \) and with \( \varepsilon \) replaced by \( \varepsilon/2 \). Let \( D_1 = \{ g \in D \mid g : 2I^n \to R^n \} \). For each \( g \in D_1 \) and \( P \in \mathcal{P}^n \), we obtain a \( C^\infty \)-embedding \( \bar{g}_P : int U^0_P \to R^n \) with \( d(\bar{g}_P \mid I^n, g \mid I^n) < \delta_M \) in the same way as in proving Theorem 1.

Let \( G \) be open in \( R^n, f : G \to R^n \) an \( \eta \)-quasisymmetric embedding and \( \mathcal{K} \in \mathcal{K}^n \) with \( \cup \mathcal{K} = G \). Define \( \bar{f} \mid Q \) by (5) for each \( Q \in \mathcal{K} \). Then \( \bar{f} : G \to R^n \) is a \( C^\infty \)-embedding and (1) holds, because \( d(\bar{f} \mid Q, f \mid Q) \equiv \delta_M s_Q + \varepsilon s_Q/2 \equiv \varepsilon s_Q \) for every \( Q \in \mathcal{K} \); see (7).

One can find \( \bar{D} \), prove (2), and then, since every \( \bar{g} \in \bar{D} \) is quasisymmetric, construct \( \bar{\eta} \) with \( \bar{f} \) being \( \bar{\eta} \)-quasisymmetric as \( D \) and \( \eta^* \) were obtained in the proof of [LT, Theorem 2.16].

Proof of Theorem 3. Let \( \mathcal{K} \) be a canonical \((f, \varepsilon)\)-decomposition of \( G \). Apply Theorem 4 with \( \varepsilon = 1 \). It follows from (1) that \( |\bar{f}(x) - f(x)| \equiv \varepsilon s_Q \) if \( x \in Q \in \mathcal{K} \). We may assume (8). Hence \( \bar{f}G = G^* \).

Proof of Theorem 2. Theorem 2 can be proved similarly to Theorem 1; cf. the proof of [K, Theorem 3.1]. Also Theorem 2 follows easily from Theorem 4; cf. the proof of [LT, Theorem 3.2].

References


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Received 7 March 1983