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LOCALLY UNIVALENT FUNCTIONS IN THE UNIT DISK

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1. Introduction

Let f be meromorphic and locally univalent in the unit disk D and denote by

$$\{f, z\} = (f''/f')' - \tfrac{1}{2}(f''/f')^2 = 2q(z) \tag{1}$$

the Schwarzian derivative of f.

A well known theorem of Nehari [7] states that f is univalent in each non-euclidean disk of radius $k^{-1/2}$ $(k \geqq 1)$ if

$$|q(z)| \leqq \frac{k}{(1-|z|^2)^2}. \tag{2}$$

Hille [5] has shown that this theorem is sharp.

Later on, Ahlfors and Weill [1] proved that f extends continuously (with respect to the spherical metric) to the boundary, provided $k<1$; in fact, they showed that $f(e^{i\theta})$ is Hölder-continuous of order $1-k$. This result was improved by Duren and Lehto [3] who replaced the constant k in (2) by a nondecreasing function $\lambda(|z|)$, $0<\lambda<1$, and proved that f has a continuous extension to ∂D if $\lambda(r)$ tends "slowly" to 1:

$$\frac{1}{1-\lambda(r)} = 0\left(\log\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1-. \tag{3}$$

Recently, Gehring and Pommerenke [4] showed that this remains true even under the original Nehari condition $\lambda(r) \equiv k = 1$.

It is well known that f admits the representation

$$f = \frac{w_1}{w_2}, \tag{4}$$

where w_1 and w_2 are linearly independent solutions of the linear differential equation

$$w'' + q(z)w = 0. \tag{5}$$

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We are mainly concerned with the boundary behaviour of f. Since

(6) $$f' = \frac{c}{w_2^2},$$

where

$$-c = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix}$$

is the Wronskian of w_1 and w_2, and

(7) $$f^{\#} := \frac{|f'|}{1+|f|^2} = \frac{|c|}{|w_1|^2+|w_2|^2},$$

we have to estimate the solutions of (5) from below as $|z|\to 1$.

2. A monotonicity theorem

Consider

(8) $$u(x) = |w_1(xe^{i\theta})|^2+|w_2(xe^{i\theta})|^2,$$

θ fixed, where w_1 and w_2 are (not necessarily linearly independent) solutions of (5). Then u is nonnegative in $[0, 1)$, and an easy computation gives

(9) $$u' = 2\,\mathrm{Re}\,[e^{i\theta}(w_1'\overline{w}_1+w_2'\overline{w}_2)]$$

$(z = xe^{i\theta})$,

(10) $$u'^2 \leqq 4u(|w_1'|^2+|w_2'|^2)$$

by the Cauchy—Schwarz inequality, and

(11) $$u'' = 2(|w_1'|^2+|w_2'|^2)-2u\,\mathrm{Re}\,[e^{2i\theta}q(xe^{i\theta})].$$

Finally, we get, using the estimate

(12) $$\mathrm{Re}\,[e^{2i\theta}q(xe^{i\theta})] \leqq Q(x)$$

and (10), (11),

(13) $$uu'' \geqq -2Q(x)\,u^2+\tfrac{1}{2}\,u'^2,$$

i.e., u is a "supersolution" of the differential equation

(14) $$y'' = -2Q(x)\,y+\frac{y'^2}{2y}.$$

It seems quite natural to compare u with "subsolutions" of (14), i.e. solutions of

(15) $$vv'' \leqq -2Q(x)v^2+\tfrac{1}{2}\,v'^2.$$

However, the usual monotonicity theorems do not apply immediately, since the right hand side of (14) is not quasimonotonically increasing.

Theorem 1. *Let* Q *be defined in* $(0, 1)$ *and let* u *and* v *be differentiable in* $[0, 1)$ *and twice differentiable in* $(0, 1)$. *Assume further:*

I. u *is a nonnegative solution of* (13) *in* $0<x<1$;
II. v *is a positive solution of* (15) *in* $0<x<1$;
III. $u(0)=v(0)>0$ *and* $u'(0)\geqq v'(0)$ *or*

$u(x) = x^2+\alpha x^3+o(x^3)$ *and* $v(x) = x^2+\beta x^3+o(x^3)$ *as* $x \to 0$, *where* $\alpha \geqq \beta$. *Then*

$$u \geqq v \quad \text{in} \quad [0,1). \tag{16}$$

Remark. a) It is clear that Theorem 1 remains valid if $(0, 1)$ is replaced by an arbitrary interval (of course, III. has to be modified).

b) If we set $U'=1/u$, $V'=1/v$, inequalities (13) and (15) are equivalent to

$$\{U, x\} \leqq \{V, x\}, \tag{17}$$

and Theorem 1 may be interpreted as a monotonicity theorem for the Schwarzian derivative: (17) implies $U'\leqq V'$ if either $U'(0)=V'(0)>0$, $U''(0)\leqq V''(0)$ or else $U'(x)=1/x^2-\alpha/x+o(1/x)$, $V'(x)=1/x^2-\beta/x+o(1/x)$ as $x\to 0$ $(\alpha\geqq\beta)$.

Proof of Theorem 1. Consider $\omega=u/v$ in the largest interval $(0, x_0)$ containing no zeros of u (if any). A short computation gives

$$\omega v(\omega'' v+\omega' v') \geqq \tfrac{1}{2}(\omega' v)^2 \tag{18}$$

and so

$$(\omega' v)' \geqq 0 \quad \text{in} \quad (0, x_0). \tag{19}$$

In both cases III. we have

$$\omega(0) = 1 \quad \text{and} \quad \omega'(0) \geqq 0, \tag{20}$$

implying $\omega' v\geqq 0$ in $(0, x_0)$. Therefore, ω is nondecreasing in $(0, x_0)$ and so $\omega(x)\geqq 1$, or, equivalently,

$$u(x) \geqq v(x) > 0 \quad \text{in} \quad (0, x_0). \tag{21}$$

By definition we must have $x_0=1$, and Theorem 1 is proved.

3. Normal functions

A function f meromorphic in D is called normal if

$$\sup_D (1-|z|^2)f^{\#}(z)<\infty \tag{22}$$

(see Lehto and Virtanen [6]). In our case f is normal if and only if

$$\inf_D \frac{|w_1(z)|^2+|w_2(z)|^2}{1-|z|^2} > 0. \tag{23}$$

It is clear how Theorem 1 leads to normality criteria involving $q(z)$. In the simplest case we have $w_1(0)=w_2'(0)=1$, $w_1'(0)=w_2(0)=0$, and we may apply Theorem 1 to $u(x)=|w_1(xe^{i\theta})|^2+|w_2(xe^{i\theta})|^2$ (we have $u(x)=1+O(x^2)$ near $x=0$). Since $v(x)=1-x^2$ satisfies $vv''=-2/(1-x^2)^2v^2+1/2v'^2$, by Theorem 1 f is normal if

$$\operatorname{Re}[e^{2i\theta}q(xe^{i\theta})] \leqq \frac{1}{(1-x^2)^2}. \tag{24}$$

By Nehari's theorem this is trivial if the real part is replaced by the modulus of q.

Theorem 2. *Let* (12) *be satisfied and suppose that, for some* $x_0\in[0, 1)$, *the differential inequality* (15) *has a positive solution in* $(x_0, 1)$ *such that either*

$$v(x) = 1+o(x-x_0) \tag{25}$$

or else

$$v(x) = (x-x_0)^2+o(x-x_0)^3 \tag{26}$$

as $x\to x_0+$. *Then there exists a positive constant* $C=C(x_0, f)$ *such that*

$$f^{\#}(z) \leqq \frac{C}{v(|z|)} \quad in \quad x_0 < |z| < 1. \tag{27}$$

Corollary. *Suppose*

$$\operatorname{Re}[e^{2i\theta}q(xe^{i\theta})] \leqq \frac{1}{(1-x^2)^2}+\frac{M}{1-x^2}, \tag{28}$$

where M *is a constant. Then* f *is normal in* D.

Proof of Corollary. It is easily seen that

$$v= (x-x_0)^2-\frac{(x-x_0)^4}{(1-x_0)^2} \tag{29}$$

satisfies

$$vv'' = -2Q^*(x)\,v^2+\tfrac{1}{2}\,v'^2 \tag{30}$$

in $x_0\leqq x<1$, where

$$Q^*(x) = \frac{1}{(1-x^2)^2}\left(\frac{(1+x)(1-x_0)}{1+x-2x_0}\right)^2+\frac{2}{1-x^2}\cdot\frac{1+x}{1+x-2x_0}. \tag{31}$$

Since

$$\frac{1+x}{1+x-2x_0} \geqq \frac{1}{1-x_0}, \tag{32}$$

we have

$$Q^*(x) \geqq \frac{1}{(1-x^2)^2}+\frac{2}{1-x_0}\,\frac{1}{1-x^2}, \tag{33}$$

which is greater than the right hand side of (28) in $x_0 \leqq x < 1$ for x_0 sufficiently near to 1. Thus, Theorem 2 gives

$$f^{\#}(z) = O\left(\frac{1}{v(|z|)}\right) = O\left(\frac{1}{1-|z|^2}\right)$$

as $|z| \to 1$.

4. Proof of Theorem 2

Let $\theta, 0 \leqq \theta < 2\pi$, be fixed and set $z_0 = x_0 e^{i\theta}$. If $(\mathring{w}_1, \mathring{w}_2)$ is the canonical fundamental system at z_0 defined by $\mathring{w}_1(z_0) = \mathring{w}_2'(z_0) = 1$, $\mathring{w}_2(z_0) = \mathring{w}_1'(z_0) = 0$, we have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1(z_0) & w_1'(z_0) \\ w_2(z_0) & w_2'(z_0) \end{pmatrix} \begin{pmatrix} \mathring{w}_1 \\ \mathring{w}_2 \end{pmatrix} \tag{34}$$

and therefore

$$|w_1|^2 + |w_2|^2 \geqq |c|^2 K^{-1}(|\mathring{w}_1|^2 + |\mathring{w}_2|^2), \tag{35}$$

where c is the Wronskian of w_1 and w_2 and

$$K = \max_{|z|=x_0} \left(|w_1(z)|^2 + \ldots + |w_2'(z)|^2\right) \tag{36}$$

is independent of θ. For $u(x) = |\mathring{w}_1(xe^{i\theta})|^2 + |\mathring{w}_2(xe^{i\theta})|^2$ or $u(x) = |\mathring{w}_2(xe^{i\theta})|^2$ we have $u(x) = 1 + O(x-x_0)^2$ or $u(x) = (x-x_0)^2 + O(x-x_0)^4$, respectively (note that $\mathring{w}_2''(z_0) = 0$). In both cases Theorem 1 (applied to the interval $(x_0, 1)$) gives $u(x) \geqq v(x)$ in $x_0 \leqq x < 1$. Thus, by (7) and (35)

$$f^{\#}(z) \leqq \frac{K/|c|}{v(|z|)} \quad \text{in} \quad x_0 < |z| < 1. \tag{37}$$

5. Conformal mapping

Since both the Schwarzian of f and the boundary behaviour of f are invariant under Möbius transforms, we are free to choose special solutions of (1).

We will assume that w_1 and w_2 are defined by the initial values

$$w_1(0) = w_2'(0) = 1, \quad w_2(0) = w_1'(0) = 0. \tag{38}$$

Theorem 3. *Let* (12) *be satisfied and assume that the corresponding differential inequality* (15) *has a positive solution* v *in* (0, 1) *such that either*

$$v(x) = x^2 + o(x^3) \tag{39}$$

or

$$v(x) = 1 + o(x) \tag{40}$$

as $x \to 0$, *and*

(41)
$$\int_{1/2}^{1} \frac{dx}{v(x)} < \infty.$$

Then f *or* $1/f$, *respectively, has boundary values everywhere with modulus of continuity*

(42)
$$\Omega(h) \leqq \min_{h \leqq \eta \leqq 1/2} \left[\frac{h}{v(1-\eta)} + 2 \int_{1-\eta}^{1} \frac{dx}{v(x)}\right] \quad (0 < h \leqq 1/2).$$

Corollary 1. *If*

(43)
$$\mathrm{Re}\,[e^{2i\theta} q(xe^{i\theta})] \leqq \lambda^2 < \pi^2$$

holds, then f' *is bounded in* $0 < \delta \leqq |z| < 1$.

Remark. The constant λ cannot be replaced by π, as the example

(44)
$$\{f, z) = 2\pi^2$$

with the special solution

(45)
$$f(z) = -\pi \cot \pi z$$

shows.

Corollary 2. *Suppose*

(46)
$$\mathrm{Re}\,[e^{2i\theta} q(xe^{i\theta})] \leqq \frac{\delta(2-\delta)}{(1-x^2)^2} + \frac{\delta(1+\delta)}{1-x^2},$$

where $\delta \in (0, 1)$ *is a constant. Then* $f(e^{i\theta})$ *is Hölder-continuous of order* $1-\delta$.

Remark. If (2) holds true $(0 < k < 1)$, then by Corollary 2 (with $\delta = 1 - \sqrt{1-k}$) $f(e^{i\theta})$ is Hölder-continuous of order $\sqrt{1-k} > 1-k$. This is the correct order, since one particular solution of

(47)
$$\{f, z\} = \frac{2k}{(1-z^2)^2}$$

is

(48)
$$f(z) = \left(\frac{1-z}{1+z}\right)^{\varkappa}, \quad \varkappa = \sqrt{1-k}.$$

Corollary 3. *Let*

(49)
$$\mathrm{Re}\,[e^{2i\theta} q(xe^{i\theta})] \leqq Q(x)$$

and assume that the linear differential equation

(50)
$$y'' + Q(|x|)y = 0$$

possesses a positive solution in $-1 < x < 1$. *Then* f *has continuous boundary values. Moreover, if*

(51)
$$Q(x) \leqq \frac{1}{(1-x^2)^2}$$

then $f(e^{i\theta})$ *has modulus of continuity*

(52)
$$\Omega(h) = O\left(\frac{1}{\log(1/h)}\right).$$

Remark. Recently, Gehring and Pommerenke [4] showed that under the special Nehari condition

(53)
$$|q(z)| \leqq \frac{1}{(1-|z|^2)^2}$$

f maps the unit disk onto a Jordan domain, whose boundary curve has modulus of continuity $O(1/\log(1/h))$, or onto the image of a parallel strip under a Möbius transform. I am greatly indebted to Professor Lehto and the reviewer who pointed out to me the Theorem of Gehring and Pommerenke.

6. Proof of Theorem 3 and its corollaries

Since $f' = -1/w_2^2$ and $g' = (1/f)' = 1/w_1^2$, Theorem 1 gives in both cases $|f'(z)| \leqq 1/v(|z|)$ and $|g'(z)| \leqq 1/v(|z|)$, respectively, and (41) shows that f and g have finite boundary values everywhere. Proceeding as in Duren's book [2], p. 75, we find

(54)
$$|f(e^{i\theta}) - f(e^{i\varphi})| \leqq \frac{r|\theta-\varphi|}{v(r)} + 2\int_r^1 \frac{dx}{v(x)},$$

where $r \in [1/2, 1)$ is arbitrary. The same holds true for g instead of f. Setting $r = 1-\eta$, where $1/2 \geqq \eta \geqq h = |\theta-\varphi|$, Theorem 3 follows from (54).

Proof of Corollary 1. Since

(55)
$$v(x) = \left(\frac{\sin\lambda x}{\lambda}\right)^2$$

solves (15) with $Q(x) \equiv \lambda^2$, Theorem 1 gives

(56)
$$|f'(z)| \leqq \left(\frac{\lambda}{\sin\lambda|z|}\right)^2$$

in $\delta \leqq |z| < 1$.

Proof of Corollary 2. One verifies easily that

(57)
$$v(x) = x^2(1-x^2)^\delta$$

satisfies (15), where Q is given by the right hand side of (46). Thus, Theorem 3 gives

(58)
$$\Omega(h) = O(h^{1-\delta}),$$

and Corollary 2 is proved.

Proof of Corollary 3. By assumption there exists a positive solution of (50). We may assume that y is even and normalized by $y(0)=1$ (otherwise we consider $(y(x)+y(-x))/2y(0)$). Then

$$v(x) = \left[y(x)\int_0^x \frac{dt}{y^2(t)}\right]^2 = x^2 - \frac{Q(0)}{3}x^4 + \dots \tag{59}$$

satisfies

$$vv'' = -2Q(x)v^2 + \tfrac{1}{2}v'^2. \tag{60}$$

Setting

$$Y(x) = \int_0^x \frac{dt}{y^2(t)}, \tag{61}$$

we find

$$\int_{1/2}^x \frac{dt}{v(t)} = \int_{1/2}^x \frac{Y'(t)}{Y^2(t)}\,dt < \frac{1}{Y(1/2)}, \quad \frac{1}{2} \leqq x < 1, \tag{62}$$

and so, by Theorem 3, f has continuous boundary values everywhere.

In the case $Q(x)=1/(1-x^2)^2$ we have $y(x)=\sqrt{1-x^2}$ and

$$v(x) = \frac{1}{4}(1-x^2)\left(\log\frac{1+x}{1-x}\right)^2. \tag{63}$$

Thus, Theorem 3 gives

$$\Omega(h) \leqq \frac{4}{((\log(1/h))^2} + \frac{8}{\log(1/h)}, \tag{64}$$

and Corollary 3 is proved.

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