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CONDENSER CAPACITIES AND REMOVABLE SETS IN $W^{1,p}$ [1)]

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Introduction. The purpose of this paper is to characterize the sets E in $\boldsymbol{R}^d$, $d \geqq 2$, closed and of Lebesgue measure zero, which are *removable* for the Sobolev space $W^{1,p}$, $1<p<\infty$, in the sense that

$$(*) \qquad W^{1,p}(\boldsymbol{R}^d \setminus E) = W^{1,p}(\boldsymbol{R}^d).$$

In [7], Vodop'janov and Gol'dshtein stated that $(*)$ holds if and only if E is a null set for *p-condenser capacities,* or equivalently (see Hesse [2]) a null set for p-extremal length. Here we present a short proof based on a so-called *strong-type capacitary estimate* (well known in potential theory) but with respect to condenser capacities. This is supplied in Theorem 7. In Theorem 9 we state and prove a somewhat more general version of Vodop'janov and Gol'dštein's result.

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1. Definition. In $\boldsymbol{R}^d$, $d \geqq 2$, let Ω be an arbitrary open set, and let $1<p<\infty$. We define $L^{1,p}(\Omega)$ as the class of (real-valued) distributions in Ω with partial derivatives of order one in $L^p(\Omega)$:

$$L^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega)\colon\ |\nabla u| \in L^p(\Omega)\}.$$

We drop Ω from the notation when $\Omega = \boldsymbol{R}^d$: $L^{1,p} = L^{1,p}(\boldsymbol{R}^d)$.

Equipped with the functional

$$(1) \qquad u \to \Big\{\int_\Omega |\nabla u|^p \, dx\Big\}^{1/p},$$

$L^{1,p}(\Omega)$ becomes a semi-normed space.

We remark that $C^\infty(\Omega) \cap L^{1,p}(\Omega)$ is dense in $L^{1,p}(\Omega)$, and that the quotient space $L^{1,p}(\Omega)/\{\text{constants}\}$ is a Banach space when Ω is connected. For these and other properties of $L^{1,p}$-spaces, we refer to Maz'ja [5, Kapitel 1].

[1)] Originally included as Chapter 9 in the author's preprint [4].

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2. The Sobolev space $W^{1,p}(\Omega)$ consists of all $u\in L^{1,p}(\Omega)$ such that in addition, $u\in L^p(\Omega)$. The functional

$$u \to \Big\{\int_\Omega |\nabla u|^p\,dx\Big\}^{1/p} + \Big\{\int_\Omega |u|^p\,dx\Big\}^{1/p} \tag{2}$$

makes $W^{1,p}(\Omega)$ a Banach space. In analogy with $L^{1,p}(\Omega)$, the $C^\infty(\Omega)$ members of $W^{1,p}(\Omega)$ form a dense subset (w.r.t. the norm (2)).

3. **Definition.** Let $E\subset \boldsymbol{R}^d$, $d\geqq 2$, be a closed set of zero Lebesgue measure. We say that E is *removable* for $L^{1,p}$ (or $W^{1,p}$) if $L^{1,p}(\complement E)=L^{1,p}$ (or $W^{1,p}(\complement E)=W^{1,p}$).

One may check that removability in $L^{1,p}$ and $W^{1,p}$ are equivalent concepts, and we will only treat the former case.

Since $m(E)=0$, removability of a set E means the following: Given a function $u\in L^{1,p}(\complement E)$, there is a function $\tilde u$ in $L^{1,p}$ such that $u=\tilde u$ a.e.

We remark that there is nothing sacred with $\boldsymbol{R}^d$ as the underlying set. One could equally well treat the problem of when $L^{1,p}(\Omega\setminus E)=L^{1,p}(\Omega)$ for some open set Ω, and the answer is completely analogous to our Theorem 9 below.

Before turning to the problem of removability, we need one more concept.

4. **Definition.** Let Ω be an open subset of $\boldsymbol{R}^d$, and let $A_0, A_1\subset\Omega$ be compact and disjoint sets. The $(1,p)$*-condenser capacity* of (A_0, A_1) with respect to Ω is defined as

$$\Gamma_p(A_0, A_1;\, \Omega) = \inf\Big\{\int_\Omega |\nabla u|^p\,dx\colon\; u=i \quad\text{on}\quad A_i,\; i=0,1\Big\}. \tag{3}$$

The functions u in question can be assumed continuous on Ω and we may also take the infimum over all u's such that $u\leqq 0$ on A_0 and $u\geqq 1$ on A_1. (See §6 below.)

We will also write Γ_p^Ω for the map $(A_0, A_1)\to\Gamma_p(A_0, A_1;\,\Omega)$ and we will omit mention of the set Ω when it equals $\boldsymbol{R}^d$: $\Gamma_p=\Gamma_p^{\boldsymbol{R}^d}$.

5. If $M_0, M_1\subset\Omega$ are two disjoint, not necessarily compact, sets, we define

$$\Gamma_p(M_0, M_1;\,\Omega) = \sup\,\{\Gamma_p(A_0, A_1;\,\Omega)\colon A_i\subset M_i,\, A_i\,\text{compact},\; i=0,1\}. \tag{4}$$

For a real-valued and continuous (say) function $u\in L^{1,p}(\Omega)$, we define

$$A^n_{j,k} = \{u<j+k\cdot 2^{-n}\}, \quad \textit{and} \tag{5}$$

$$M^n_{j,k} = \{u>j+(k+1)\cdot 2^{-n}\},\quad j,k,n\in\boldsymbol{Z},\quad n\geqq 1,\quad 0\leqq k\leqq 2^{n-1}.$$

With this definition, $\Gamma_p(A^n_{j,k}, M^n_{j,k};\,\Omega)$ makes sense.

6. It is well known that one can truncate functions in $L^{1,p}(\Omega)$. Furthermore, for any constant λ

$$\nabla\max(u,\lambda) = \nabla u\cdot 1_{\{u<\lambda\}}\ \text{a.e.}, \tag{6}$$

and similarly

(7) $$\nabla \min(u, \lambda) = \nabla u \cdot 1_{\{u>\lambda\}} \text{ a.e.}$$

Hence in both cases the semi-norm of the contracted function is no greater than that of u.

The following result is analogous to Theorem 2.1 in [4].

7. Theorem. *If* $\Lambda^n_{j,k}$ *and* $M^n_{j,k}$ *are defined as in* (5), *then*

(8) $$2^{-np} \cdot \sum_{j,k} \Gamma_p(\Lambda^n_{j,k}, M^n_{j,k}; \Omega) \leqq \text{const.} \int_\Omega |\nabla u|^p \, dx.$$

Proof. Assume first that u takes its values in $[0, 1]$. Let $\eta(t)=0$ on $(-\infty, 0)$, $\eta(t)=t$ on $[0, 1]$ and $\eta(t)=1$ on $[1, \infty)$.

For $k\in\boldsymbol{Z}$ we define $\eta_k(t)=\eta(t-k)$. Then $\|\eta_k'\|_\infty=1$ for all k. Also, define $F_k=\eta_k\circ(u\cdot 2^n)\in L^{1,p}(\Omega)$. Then $F_k=0$ on $\Lambda^n_{0,k}$ and $F_k=1$ on $M^n_{0,k}$, so

$$\Gamma_p(\Lambda^n_{0,k}, M^n_{0,k}; \Omega) \leqq \int_\Omega |\nabla F_k|^p.$$

Now, $\operatorname{supp}(\nabla F_k)\subset N_k=\{(k-\gamma)\cdot 2^{-n}<u<(k+1+\gamma)\cdot 2^{-n}\}$ (with obvious modifications when $k=0$ or 2^{n-1}) where $\gamma\in(0, \frac{1}{2})$, so that $\sum_k 1_{N_k}\leqq 2\cdot 1_\Omega$. Furthermore,

$$|\nabla F_k| \leqq \|\eta_k'\|_\infty \cdot |\nabla u| \cdot 2^n = 2^n \cdot |\nabla u|.$$

Hence we get

$$2^{-np} \cdot \sum_k \Gamma_p(\Lambda^n_{0,k}, M^n_{0,k}; \Omega) \leqq 2^{-np} \cdot \sum_k \int_\Omega |\nabla F_k|^p \, dx \leqq 2^{-np} \cdot \sum_k \int_{N_k} |\nabla u|^p \, dx$$

$$\leqq 2 \cdot \int_{\cup N_k} |\nabla u|^p \, dx \leqq 2 \cdot \int_\Omega |\nabla u|^p \, dx.$$

By (6) and (7) we may write $u=\sum_{j\in Z} u_j$ where $j\leqq u_j<j+1$ and $u_j\in L^{1,p}(\Omega)$. This gives us the desired result (8), after summation over j. □

We need one more concept.

8. Definition. A closed set $E\subset\boldsymbol{R}^d$ *is a null set for* Γ_p, abbreviated an $N\Gamma_p$-set, if $\Gamma_p=\Gamma_p^{\complement E}$.

Note that $m(E)=0$ is necessary in order that E be an $N\Gamma_p$-set. It is well known that $N\Gamma_p$-sets are precisely the null sets for p-extremal distance. See Hesse [2, Theorem 5.5], and also the early article [7] by Väisälä. We will not use this connection here.

— We can now prove

9. Theorem. *A set* E *is removable for* $L^{1,p}$ *if and only if it is an* $N\Gamma_p$*-set. More generally, if for each* $x\in E$ *there is a neighbourhood* U_x *of* x *and a constant* K_x *such that*

(9) $$\Gamma_p^{U_x} \leqq K_x \cdot \Gamma_p^{U_x\setminus E},$$

then E *is removable for* $L^{1,p}$.

Proof. It is enough to prove that E is removable if (9) holds. The rest is obvious. We first prove that if for an open set U,

$$\Gamma_p^U \leqq K \cdot \Gamma_p^{U \setminus E} \quad (K = \text{constant}), \tag{10}$$

then $L^{1,p}(U) = L^{1,p}(U \setminus E)$.

Choose $u \in L^{1,p}(U \setminus E)$. We may assume that $0 \leqq u \leqq 1$. Let V_k^n be the (unique) Γ_p^U-extremal of $(A_{0,k}^n, M_{0,k}^n)$:

$$\Gamma_p^U(A_{0,k}^n, M_{0,k}^n) = \int_U |\nabla V_k^n|^p \, dx.$$

Define $u_n = 2^{-n} \cdot \sum_k V_k^n \in L^{1,p}(U)$. Then u_n tends to u pointwise a.e. as n approaches infinity. Further,

$$\begin{aligned}
\int_U |\nabla u_n|^p \, dx &\leqq \text{const.}\, 2^{-np} \sum_k \int_U |\nabla V_k^n|^p \, dx \\
&= \text{const.}\, 2^{-np} \sum_k \Gamma_p^U(A_{0,k}^n, M_{0,k}^n) \\
&\leqq \text{const.}\, 2^{-np} \sum_k \Gamma_p^{U \setminus E}(A_{0,k}^n, M_{0,k}^n) \\
&\leqq \text{const.} \int_{U \setminus E} |\nabla u|^p \, dx,
\end{aligned}$$

where the first inequality follows from the fact that $\sum_k 1_{\operatorname{supp}(|\nabla V_k^n|)} \leqq 3$, the equality from the choice of V_k^n, the second inequality from (10), and the third inequality from Theorem 7.

A weak compactness argument gives us a sequence (u_n') in the convex hull of $(u_1, \ldots, u_n)$, which is strongly convergent in the semi-norm (1). It follows that $u \in L^{1,p}(U)$.

The general case now follows easily. Let $u \in L^{1,p}(\complement E)$. Multiplying u with a cut-off function if necessary, we may assume that u is compactly supported. Suppose $\operatorname{supp}(u) \cap E \subset \bigcup_1^N U_i$, where $U_i \in \{U_x, x \in E\}$. Then, from what we just proved, $u | U_1 \setminus E$ has a continuation to U_1, and it follows that $u \in L^{1,p}(\complement E \cup U_1)$. Starting anew with u, now considered as an element of $L^{1,p}(\complement E \cup U_1)$, we get $u \in L^{1,p}(\complement E \cup U_1 \cup U_2)$, and eventually $u \in L^{1,p}(\complement E \cup (\bigcup_1^N U_i))$, which proves that $u \in L^{1,p}(\complement E)$. $\square$

10. Remarks. 1. The proof follows the idea of Vodop'janov and Gol'dshtein [7]; what is needed in order to justify their argument is the estimate (6) of Theorem 7.

2. In [1], Hedberg uses condenser capacities as a means of characterizing removability in other function spaces than those considered here.

3. One can prove that $L^{1,p}$-removability implies $L^{1,p+\delta}$-removability for all $\delta > 0$. Also, defining $W^{m,p}$-removability in the natural way, $L^{1,p}$-(or perhaps better $W^{1,p}$-) removability implies $W^{m,p}$-removability. (Here $W^{m,p}(\Omega)$ is the

completion of $C^\infty(\Omega)$ w.r.t. the norm

$$\|u\| = \Big\{\sum_{|\alpha|\leqq m} \int_\Omega |D^\alpha u|^p\Big\}^{1/p},$$

for $m\in \mathbf{Z}^+$.) This follows from an argument similar to that of [3, Theorem 1, second part].

4. We note that if one had a strong-type capacitary estimate w.r.t. condenser capacities in $W^{m,p}(\Omega)$, then Theorem 9 could be carried over also to this situation. The problem is the absence of a continuous truncation operation.

5. It would be nice to find a situation in which a Dirichlet space (see [4]) version of Theorem 9 could be carried out. One has to find a proper way of defining a space $W(\Omega)$ ($\Omega\subset X$ open) corresponding to $W^{1,2}(\Omega)$ (in the case of $W^{1,2}$-removability), though.

6. Removability in $L^{1,p}$ can be formulated as a problem on removable singularities for certain PDEs. Consider the case $p=2$ and localize, i.e. consider $L^{1,2}(M)$ for $M=\Omega$ and $M=\Omega\setminus E$, where Ω is bounded. In analogy with [4, §6], we write

$$L^{1,2}(\Omega\setminus E) = H(\Omega\setminus E)\oplus L_0^{1,2}(\Omega\setminus E),$$

where $L_0^{1,2}(\Omega\setminus E)$ is the closure of $C_0^\infty(\Omega\setminus E)$ in the $L^{1,2}(\Omega\setminus E)$-semi-norm. Then $H(\Omega\setminus E)=L_0^{1,2}(\Omega\setminus E)^\perp=\{u\in L^{1,2}(\Omega\setminus E)\colon \Delta u=0 \text{ in } \Omega\setminus E\}$, by Weyl's lemma.

Thus E is removable for $L^{1,2}$ if and only if it is a removable singularity for harmonic functions of class $L^{1,2}(\Omega\setminus E)$. When $p\neq 2$ one gets a similar result about the equation

$$\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0.$$

7. We note that Theorem 9 gives an easy proof for the in a way obvious (and well-known, see [7]) fact that a removable set E does not separate, i.e. $\dim E\leqq d-2$ where $\dim$ refers to the topological dimension.

8. It is easily seen that E is removable if E is of zero $(d-1)$-dimensional Hausdorff measure: $\Lambda_{d-1}(E)=0$. (One can argue as in [3, Theorem 1].) Forming the Cartesian product of suitable sets, this observation can be used to obtain removable sets of any Hausdorff-dimension strictly less than d.

We mention also that it follows from the theory of quasi-conformal mappings that in the case $d=2$, there is a linear set E of positive linear measure, which is an $N\Gamma_2$-set, (Olli Martio, personal communication).

9. We refer to [6] for applications to quasi-conformal mappings (the case $p=d$).

Added in proof: In addition to the references already mentioned, we have become aware of the work of H. Yamamoto. In [8, Theorem 2], Yamamoto gives yet another proof of Vodop'janov and Gol'dštein's result.

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