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A NON-HYPOELLIPTIC DIRICHLET PROBLEM FROM STOCHASTICS

KARL DOPPEL and NIELS JACOB

To the memory of Rolf Nevanlinna

1. Introduction

1.1. In a recent paper [2] E. B. Dynkin solved a Dirichlet problem for a non-hypoelliptic differential operator $L(D)$ with methods of probability theory by extending some results of the theory of Markov processes.

This operator $L(D)$ is defined in the following way: Let k be a positive integer. For each $j\,(1\leqq j\leqq k)$ take $m_j\in N$ and set $n=m_1+\ldots+m_k$. With the notation

$$l_1 = 0, \qquad l_j = \sum_{i=1}^{j-1} m_i \quad (2 \leqq j \leqq k)$$

we set

(1.1) $$\Delta_j := \sum_{i=1}^{m_j} D^2_{l_j+i} \quad \text{with} \quad D_{l_j+i} = -\sqrt{-1}\frac{\partial}{\partial x_{l_j+i}}.$$

Thus, Δ_j denotes the Laplace operator which acts on functions defined on subsets of $\boldsymbol{R}^{m_j}\subset \boldsymbol{R}^n$. We define the differential operator $L(D)$ by

(1.2) $$L(D) := \Delta_1\ldots\Delta_k.$$

The operator $L(D)$ is not hypoelliptic for $k>1$. This fact can be seen already in the simplest case where $k=2$ and $m_1=m_2=1$. Then

$$L(D) = D_1^2D_2^2 = \frac{\partial^4}{\partial x_1^2\partial x_2^2}$$

and the polynomial associated with $L(D)$ is given by $P(\xi)=\xi_1^2\xi_2^2$. Since for $\xi_1=0$ and $\xi_2=n$ $(n\in \boldsymbol{N})$ we have

$$\frac{\partial^2}{\partial\xi_1^2}P(\xi) = 2n^2,$$

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it follows by a result of L. Hörmander (see [6], p. 99) that $L(D)$ is not hypoelliptic. The usual Sobolev space methods are not applicable to a study of Dirichlet problems for the operator $L(D)$.

1.2. In this paper we present a method which can be understood as a generalization of the Hilbert space methods used in the elliptic theory. Here we use "anisotropic" Sobolev spaces which correspond in a natural way to the operator $L(D)$. It turns out that in these anisotropic spaces the behaviour of the non-hypoelliptic operator $L(D)$ is quite similar to that of an elliptic operator in the usual Sobolev spaces.

For a domain G which is the Cartesian product of bounded open sets $G_j \subset \boldsymbol{R}^{m_j}$ $(n=m_1+\ldots+m_k)$ with sufficiently smooth boundaries ∂G_j we show that a generalized Dirichlet problem, for which the solution is searched in such an anisotropic space, has a unique solution. Further one gets for a class of data a regularity result for this solution.

We will mention that Rolf Nevanlinna expressed a similar idea in unpublished lectures given in Ann Arbor, Helsinki and Zürich. He indicated the possibility to construct for a differential operator a suitable bilinear form such that solving Dirichlet problems for this operator can be reduced to the problem of orthogonal projection in the sense of this bilinear form (cf. [8] and also [1]).

1.3. We give a slightly modified representation of $L(D)$. We denote by $\boldsymbol{N}_0^n$ the set of all ordered systems of n nonnegative integers (multi-indices). For $\alpha=(\alpha_1,\ldots,\alpha_n)\in\boldsymbol{N}_0^n$ we define its length as usual by $|\alpha|=\alpha_1+\ldots+\alpha_n$. For $\sigma,\tau\in\boldsymbol{N}_0^n$, $\sigma=(\sigma_1,\ldots,\sigma_n)$, $\tau=(\tau_1,\ldots,\tau_n)$, we set $\sigma\leqq\tau$ if $\sigma_i\leqq\tau_i$ for $1\leqq i\leqq n$. It is clear that with this definition $\boldsymbol{N}_0^n$ is a partially ordered set.

Let m_j and l_j $(1\leqq j\leqq k)$ be defined as in 1.1. For each j $(1\leqq j\leqq k)$ we define m_j multi-indices $\varepsilon_{t_j}\in\boldsymbol{N}_0^n$ of length 1, $|\varepsilon_{t_j}|=1$, having its only nonvanishing coordinate in the t_j-th position, $l_j+1\leqq t_j\leqq l_j+m_j$. Furthermore we set

$$\Gamma=\left\{\alpha \,\middle|\, \alpha\in\boldsymbol{N}_0^n,\ \alpha=\sum_{j=1}^{k}\varepsilon_{t_j}\right\} \tag{1.3}$$

and

$$2\Gamma=\{\beta \mid \beta\in\boldsymbol{N}_0^n,\ \beta=2\alpha,\ \alpha\in\Gamma\}.$$

Note that both Γ and 2Γ have $m_1\ldots m_k$ elements. Now we obtain the very useful expression for the differential operator $L(D)$:

$$L(D)=\sum_{\alpha\in\Gamma}D^{2\alpha}, \tag{1.4}$$

where we used the abbreviation

$$D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n}.$$

The authors want to thank E. B. Dynkin for leading their interest to these problems and I. S. Louhivaara for his suggestions while writing this paper.

2. The Hilbert spaces $H^{\Gamma}(G)$, $H_0^{\Gamma}(G)$ and $H^{(k)}(G)$

2.1. Let G be a bounded open set in $\boldsymbol{R}^n$. As usual, let $C^k(G)$, $k\in\boldsymbol{N}_0$, be the linear space of all complex valued functions u which are k times continuously differentiable in G. By $C_0^k(G)$ we denote the space of all functions $u\in C^k(G)$ each having a compact support in G. We write also $C_0^\infty(G)=\bigcap_{k\in\boldsymbol{N}_0}C_0^k(G)$. Further we define for $k\in\boldsymbol{N}_0$

$$C_*^k(G) := \{u|\ u\in C^k(G),\ D^\alpha u\in L^2(G),\ \alpha\in\boldsymbol{N}_0^n,\ |\alpha|\leqq k\}.$$

In $C_*^k(G)$ we associate with the operator $L(D)$ the sesquilinear form

$$B(u, v) := \sum_{\alpha\in\Gamma}\int_G \overline{D^\alpha u(x)}D^\alpha v(x)\,dx. \tag{2.1}$$

We define

$$(u, v)_\Gamma := B(u, v)+(u, v)_0, \tag{2.2}$$

where $(\cdot, \cdot)_0$ denotes the scalar product in $L^2(G)$,

$$(u, v)_0 = \int_G \overline{u(x)}v(x)\,dx. \tag{2.3}$$

Thus, we have a scalar product $(\cdot, \cdot)_\Gamma$ on $C_*^k(G)$ with the corresponding norm $\|\cdot\|_\Gamma$. The completion of $C_*^k(G)$ with respect to the scalar product (2.2) will be denoted by $H^\Gamma(G)$. One obtains $H^{2\Gamma}(G)$ analogously. For the closure of $C_0^\infty(G)$ in $H^\Gamma(G)$ we write $H_0^\Gamma(G)$. The elements of $H_0^\Gamma(G)$ are interpreted as functions with generalized homogenous boundary data (cf. Theorem 4).

In contrast to the usual Sobolev space theory we cannot conclude that the strong L^2-derivative $D^\tau u$ exists for $u\in H^\Gamma(G)$ and for each multi-index $\tau\in\boldsymbol{N}_0^n$ with $\tau\leqq\sigma$ for some $\sigma\in\Gamma$. But one can prove (cf. also [7])

Lemma 1. *Let $u\in H_0^\Gamma(G)$ be given. Then the strong L^2-derivative $D^\tau u$ exists for all multi-indices τ with $\tau\leqq\sigma$ for some $\sigma\in\Gamma$.*

Proof. For $\tau\in\Gamma$ the assertion is obvious. Let now $\tau\notin\Gamma$ such that $\tau\leqq\sigma\in\Gamma$. Since $G\subset\boldsymbol{R}^n$ is a bounded set, we can find a constant $d>0$ such that for each $x=(x_1, \dots, x_n)\in G$

$$\max_{1\leqq j\leqq n}|x_j|\leqq d$$

holds. For $\varphi\in C_0^\infty(G)$ we get by partial integration $(1\leqq j\leqq k)$

$$\|D^\tau\varphi\|_0^2 = \int_G |D^\tau\varphi(x)|^2\,dx = -\int_G x_j\frac{\partial}{\partial x_j}\big(\overline{D^\tau\varphi(x)}D^\tau\varphi(x)\big)\,dx$$

$$\leqq 2d\,\|D^{\tau+\varepsilon_j}\varphi\|_0\|D^\tau\varphi\|_0.$$

Hence one has

$$\|D^\tau\varphi\|_0\leqq 2d\,\|D^{\tau+\varepsilon_j}\varphi\|_0.$$

Since $\tau \leqq \sigma$, there exists a multi-index β with $\tau+\sigma=\beta$ and the result follows by iteration.

As an immediate consequence of the Poincaré inequality we have

Lemma 2. *On* $H_0^\Gamma(G)$ *the sesquilinear form* (2.1) *defines a scalar product equivalent to* (2.2).

Proof. It is sufficient to prove all estimates for elements of $C_0^\infty(G)$. By (2.2) we have for all $\varphi\in C_0^\infty(G)$

$$B(\varphi, \varphi) \leqq (\varphi, \varphi)_\Gamma.$$

Now by the Poincaré inequality (cf. [5], p. 33) we get for each multi-index $\varepsilon_j\in N_0^n$, $|\varepsilon_j|=1$ $(1\leqq j\leqq n)$

$$\|\varphi\|_0^2 \leqq c_1 \|D^{\varepsilon_j}\varphi\|_0^2,$$

where the constant c_1 depends only on G. Repeating this argument we get finally for each $\alpha\in\Gamma$

$$\|\varphi\|_0^2 \leqq c_\alpha \|D^\alpha \varphi\|_0^2$$

and by (2.1)

$$\|\varphi\|_0^2 \leqq cB(\varphi, \varphi),$$

which proves Lemma 2.

Thus we have on $H_0^\Gamma(G)$ a norm $|||\cdot|||_\Gamma$ defined by

$$|||\varphi|||_\Gamma := \big(B(\varphi, \varphi)\big)^{1/2}, \quad \varphi\in C_0^\infty(G), \tag{2.4}$$

equivalent to $\|\cdot\|_\Gamma$. This norm can be interpreted as the energetic norm associated with the differential operator $L(D)$.

2.2. Now let us suppose that the set $G\subset \boldsymbol{R}^n$ has

Property P. *The open bounded set* G *is the Cartesian product*

$$G = G_1\times\ldots\times G_k \tag{2.5}$$

of bounded open sets $G_j\subset \boldsymbol{R}^{m_j}$ $(1\leqq j\leqq k)$, $m_1+\ldots+m_k=n$, *with sufficiently smooth boundaries*[1] ∂G_j.

We will generalize a well-known theorem from the theory of Sobolev spaces.

Let $G_j'\subset \boldsymbol{R}^{m_j}$ $(1\leqq j\leqq k)$ be a bounded open set. If Φ_j is a diffeomorphism of class C^1 from $\bar{G}_j'$ onto $\bar{G}_j$ then the tensor product Φ,

$$\Phi := \Phi_1\otimes\ldots\otimes\Phi_k, \tag{2.6}$$

is a diffeomorphism of class C^1 from $\bar{G}'=\bar{G}_1'\times\ldots\times\bar{G}_k'$ onto $\bar{G}=\bar{G}_1\times\ldots\times\bar{G}_k$.

[1] All our considerations are valid if the boundaries ∂G_j are of class C^∞ (cf. e.g. [3], pp. 9—10).

By definition the Jacobi matrix is the direct sum of the linear mappings A_j: $\boldsymbol{R}^{m_j} \to \boldsymbol{R}^{m_j}$ $(1 \leqq j \leqq k)$ defined by

$$A_j = \begin{pmatrix} \dfrac{\partial x_{l_j+1}}{\partial y_{l_j+1}} & \cdots & \dfrac{\partial x_{l_j+m_j}}{\partial y_{l_j+1}} \\ \vdots & & \vdots \\ \dfrac{\partial x_{l_j+1}}{\partial y_{l_j+m_j}} & \cdots & \dfrac{\partial x_{l_j+m_j}}{\partial y_{l_j+m_j}} \end{pmatrix}, \tag{2.7}$$

where x_i denotes the i-th component of Φ. We remark that the elements of A_j are continuous functions on $\bar{G}'_j$.

Lemma 3. *Let G be a bounded open set with Property* P *and let Φ be defined by* (2.6). *Then for all $u \in H_0^\Gamma(G)$ the function*

$$u'(y) = u(\Phi(y)) \tag{2.8}$$

is in $H_0^\Gamma(G')$ and

$$c_1 \|u'\|_{\Gamma, G'} \leqq \|u\|_{\Gamma, G} \leqq c_2 \|u'\|_{\Gamma, G'} \tag{2.9}$$

holds, where the constants c_1, c_2 depend only on the diffeomorphism Φ.

Proof. By Lemma 2 inequality (2.9) is equivalent to

$$\tilde{c}_1 |||u'|||_{\Gamma, G'} \leqq |||u|||_{\Gamma, G} \leqq \tilde{c}_2 |||u'|||_{\Gamma, G'}. \tag{2.10}$$

Now by definition we get for each function (2.8)

$$|||u'|||^2_{\Gamma, G'} = \sum_{\alpha \in \Gamma} \int_{G'} |D_y^\alpha u'(y)|^2 \, dy.$$

For a fixed $\alpha \in \Gamma$ we have by (1.3)

$$\alpha = \varepsilon_{t_1} + \ldots + \varepsilon_{t_k},$$

where ε_{t_j} $(1 \leqq j \leqq k)$ has its only non-vanishing coordinate in the t_j-th position, $l_j + 1 \leqq t_j \leqq l_j + m_j$. Furthermore we get

$$D_y^\alpha u'(y) = (-\sqrt{-1})^k \frac{\partial^k u'(y)}{\partial y_{t_1} \cdots \partial y_{t_k}} \tag{2.11}$$

$$= (-\sqrt{-1})^k \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \frac{\partial x_{l_1+i_1}}{\partial y_{t_1}} \cdots \frac{\partial x_{l_k+i_k}}{\partial y_{t_k}}.$$

Thus, using the fact that the Jacobian of Φ is continuous on $\bar{G}'$, we get

$$|D_y^\alpha u'(y)| \leqq \hat{c}_\alpha \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \right|,$$

where the constant $\hat{c}_\alpha$ depends only on Φ and α. Now one has with another constant $\tilde{c}_\alpha$

$$|D_y^\alpha u'(y)|^2 \leqq \hat{c}_\alpha^2 \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \right| \right)^2$$

$$\leqq \tilde{c}_\alpha^2 \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \right|^2$$

and further

$$\|D_y^\alpha u'\|_{0,G'}^2 \leqq \tilde{c}_\alpha^2 \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \int_{G'} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \right|^2 dy. \tag{2.12}$$

We substitute $y = \Phi^{-1}(x)$. As the Jacobian of Φ^{-1} is continuous on $\bar{G}$ we find

$$\|D_y^\alpha u'\|_{0,G'}^2 \leqq c_\alpha^2 \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left\| \frac{\partial^k u}{\partial x_{l_1+i_1} \cdots \partial x_{l_k+i_k}} \right\|_{0,G}^2 \tag{2.13}$$

with a suitable constant c_α. In the summation on the right side of (2.13) only multi-indices which belong to Γ appear, wherefore we get

$$\|D_y^\alpha u'\|_{0,G'}^2 \leqq c_\alpha^2 |||u|||_{\Gamma,G}^2. \tag{2.14}$$

Because (2.14) holds for all multi-indices $\alpha \in \Gamma$, we can finally show that there exists a constant $\tilde{c}_1^{-1}$ depending only on Φ such that

$$|||u'|||_{\Gamma,G'} \leqq \tilde{c}_1^{-1} |||u|||_{\Gamma,G}$$

holds for all functions (2.8). By interchanging the roles of x and y we get the second inequality of (2.10).

To prove a theorem for functions in $H_0^\Gamma(G)$ which is quite similar to a theorem on the behaviour of the elements of $H_0^l(G)$ (cf. e.g. [5], p. 28), we define for each $j\,(1 \leqq j \leqq k)$

$$\Gamma^j := \left\{ \gamma \,\middle|\, \gamma \in N_0^n,\ \gamma = \sum_{\substack{i=1 \\ i \neq j}}^{k} \varepsilon_{t_i} \right\} \tag{2.15}$$

(for $\gamma \in \Gamma^j$ one has $|\gamma| = k-1$) and

$$\partial_j G := G_1 \times \ldots \times G_{j-1} \times \partial G_j \times G_{j+1} \times \ldots \times G_k.$$

Theorem 4. *Let G be a bounded open set with Property* P. *Then for each $u \in H_0^\Gamma(G) \cap C^{k-1}(\bar{G})$ the relation*

$$D^\beta u|_{\partial_j G} = 0 \tag{2.16}$$

holds for all multi-indices β with $\beta \leqq \gamma$ for some $\gamma \in \Gamma^j$ $(1 \leqq j \leqq k)$.

Proof. It is obviously enough to prove the result for $j=1$. For a point $(\xi, \eta) \in \partial G_1 \times G^{\sim}$, $G^{\sim} = G_2 \times \ldots \times G_k$, we can assume by Lemma 3 that the point $\xi\ (\in \partial G_1)$ and for a neighbourhood $U \subset \boldsymbol{R}^{m_1}$ of ξ the set $U \cap \partial G_1$ lie in the hypersurface $x_1=0$ and that $U\cap G_1$ lies in the halfspace $x_1\leqq 0$ [2]. In the Cartesian product $(U\cap\partial G_1)\times G^{\sim}$ we can find an n-dimensional cylinder S_h with height h, the base of which is an $(n-1)$-dimensional ball

$$B_{r_h}(\xi,\eta)=\left\{x\,\middle|\, x\in\boldsymbol{R}^n,\ x_1=0, \sum_{i=2}^{m_1}|x_i-\xi_i|^2+\sum_{i=m_1+1}^{n}|x_i-\eta_i|^2<r_h^2\right\}$$

in $U\cap\partial G_1$ in the hypersurface $x_1=0$, where the radius r_h is chosen such that the volume of S_h is equal to h^2. With sufficiently small values of h we have $S_h\subset G_1\times G^{\sim}$.

For each $\varphi\in C_0^\infty(G)$ one has

$$D^\beta\varphi(x)=0 \tag{2.17}$$

for all $x\in B_{r_h}(\xi,\eta)$ and all $\beta\in\boldsymbol{N}_0^n$. From (2.17) one has

$$D_x^\beta\varphi(x)=\int_0^{x_1}\frac{\partial}{\partial t}D_x^\beta\varphi(t,x_2,\dots,x_n)\,dt. \tag{2.18}$$

By the Cauchy—Schwarz inequality we get

$$|D_x^\beta\varphi(x)|^2\leqq h\int_0^h\left|\frac{\partial}{\partial t}D_x^\beta\varphi(t,x_2,\dots,x_n)\right|^2dt.$$

Integration with respect to x gives

$$\int_{S_h}|D_x^\beta\varphi(x)|^2\,dx\leqq h^2\int_{S_h}\left|\frac{\partial}{\partial x_1}D_x^\beta\varphi(x)\right|^2dx.$$

Since $\operatorname{vol}S_h=h^2$, we get

$$\frac{1}{\operatorname{vol}S_h}\int_{S_h}|D_x^\beta\varphi(x)|\,dx\leqq\int_{S_h}|D_x^{\beta+\varepsilon_1}\varphi(x)|^2\,dx \tag{2.19}$$

for $\varepsilon_1=(1,0,\dots,0)\in\boldsymbol{N}_0^n$.

Now take a multi-index β with $\beta\leqq\gamma$ for some $\gamma\in\Gamma^1$. Then one has $\beta+\varepsilon_1\leqq\gamma+\varepsilon_1\in\Gamma$. By Lemma 1 it follows that the right-hand integral tends to zero also for all elements $u\in H_0^\Gamma(G)$ if $h\to 0$, whence we have

$$\lim_{h\to 0}\frac{1}{\operatorname{vol}S_h}\int_{S_h}|D^\beta u(x)|^2\,dx=0.$$

If in addition $u\in C^{k-1}(\bar{G})$, we conclude by using the mean value theorem

$$D^\beta u(\xi,\eta)=0$$

for all $\eta\in G^{\sim}$ and for all $\beta\leqq\gamma\in\Gamma^1$. This proves Theorem 4.

[2] To reach this situation by a local C^1-diffeomorphism (cf. Lemma 3) we must have some regularity of the boundaries; in the special case $m_1=\ldots=m_k=1$ the following proof is always applicable without any transformation by a C^1-diffeomorphism.

2.3. In this connection we remark that partial integration is possible for functions of the set

$$X := H_0^\Gamma(G) \cap C^\infty(\bar{G}), \tag{2.20}$$

provided that the set G has Property P.

Lemma 5. *Let G be a bounded open set with Property P. Then*

$$\int_G \overline{D^\alpha u(x)} D^\alpha v(x)\, dx = \int_G \overline{D^{2\alpha} u(x)} v(x)\, dx$$

holds for all $u, v \in X$ and $\alpha \in \Gamma$.

2.4. In the case where the bounded open set G has Property P we introduce another Hilbert space denoted by $H^{(k)}(G)$. On X a scalar product is defined by

$$(u, v)_{(k)} := (L(D)u, L(D)v)_0 + (u, v)_0. \tag{2.21}$$

The Hilbert space $H^{(k)}(G)$ is defined as the completion of X with respect to the scalar product (2.21). It is clear that (2.21) gives the graph norm $\|\cdot\|_{(k)}$,

$$\|u\|_{(k)}^2 = \|L(D)u\|_0^2 + \|u\|_0^2 \tag{2.22}$$

on X.

We consider the densely defined linear operator L in the Hilbert space $L^2(G)$ given by

$$\begin{cases} D(L) := X (\subset L^2(G)), \\ Lu := L(D)u \quad \text{for all} \quad u \in X. \end{cases} \tag{2.23}$$

By partial integration one gets

$$(Lu, v)_0 = \int_G \overline{L(D)u(x)} v(x)\, dx = \int_G \overline{u(x)} L(D) v(x)\, dx = (u, g)_0 \tag{2.24}$$

for $u, v \in X$, with $L(D)v =: g =: L^* v$. Thus, we have $D(L^*) \supset X$, and the adjoint operator L^* is densely defined on $L^2(G)$. Hence the operator L is closable with the closure (smallest closed extension) $L^{\sim} = L^{**}$.

Theorem 6. *Let G be a bounded open set with Property* P. *Then the relation*

$$D(L^{\sim}) = H^{(k)}(G)$$

holds.

Proof. We denote the graphs of L and $L^{\sim}$ by

$$G(L) := \{(u, Lu) \mid u \in X\}$$

and

$$G(L^{\sim}) := \{(f, L^{\sim} f) \mid f \in D(L^{\sim})\},$$

respectively. On the other hand, it is well-known (cf. [10], p. 89) that $G(L^{\sim})$ coincides with the completion of $G(L)$ with respect to the graph norm (2.22),

$$G(L^{\sim}) = \overline{G(L)}^{\|\cdot\|_{(k)}}. \tag{2.25}$$

For an arbitrary $u \in H^{(k)}(G)$ there exists by definition a sequence $\{u_m\}_{m\in N} \subset X$ such that

$$\|u_m - u\|_0 \to 0 \quad \text{for} \quad m \to \infty \tag{2.26}$$

and $\{Lu_m\}_{m\in N}$ is a Cauchy sequence in $L^2(G)$, which implies the existence of a unique element $w \in L^2(G)$ such that

$$\|Lu_m - w\|_0 \to 0 \quad \text{for} \quad m \to \infty. \tag{2.27}$$

Because of (2.25) it follows that $(u, w) \in G(L^{\sim})$, i.e., $u \in D(L^{\sim})$ (and $w = L^{\sim}u$).

On the other hand, let $u \in D(L^{\sim})$ be given. By the definition of a closed operator there exists a sequence $\{u_m\}_{m\in N} \subset D(L) = X$ such that (2.26) and (2.27) hold with $w := L^{\sim}u$. By definition we have $u \in H^{(k)}(G)$.

Remark. 1. From Theorem 6 we obtain for the scalar product of $H^{(k)}(G)$ the expression

$$(u, v)_{(k)} = (L^{\sim} u, L^{\sim} v)_0 + (u, v)_0.$$

2. Let B be the sesquilinear from (2.1). By partial integration one has

$$B(u, \varphi) = (Lu, \varphi)_0 \quad \text{for all} \quad u \in X \quad \text{and} \quad \varphi \in C_0^\infty(G) \tag{2.28}$$

and by continuous extension

$$B(u, v) = (Lu, v)_0 \quad \text{for all} \quad u, v \in X.$$

By the Cauchy—Schwarz inequality we get

$$|B(u, v)| \leqq \|u\|_{(k)} \|v\|_0 \quad \text{for all} \quad u, v \in X$$

and especially

$$|||u|||_\Gamma^2 \leqq \|u\|_{(k)} \|u\|_0 \quad \text{for all} \quad u \in X. \tag{2.29}$$

2.5. Now we will show that the elements of $H^{(k)}(G)$ have the same boundary behaviour as the elements of $H_0^\Gamma(G)$:

$$H^{(k)}(G) \cap H_0^\Gamma(G) = H^{(k)}(G). \tag{2.30}$$

Take an element $v \in H^{(k)}(G)$. By the definition of $H^{(k)}(G)$ there exists a sequence $\{u_j\} \subset X = H_0^\Gamma(G) \cap C^\infty(\bar{G})$ with

$$\|u_j - v\|_{(k)}^2 = \|Lu_j - L^{\sim} v\|_0^2 + \|u_j - v\|_0^2 \to 0.$$

Especially $\{u_j\}$ is a Cauchy sequence in $H^{(k)}(G)$ and by (2.29) also in $H_0^\Gamma(G)$:

$$|||u_j - u_l|||_\Gamma \to 0.$$

Thus $\{u_j\}$ has a limit element v^* in $H_0^\Gamma(G)$:

$$|||u_j - v^*|||_\Gamma \to 0.$$

By Lemma 2 it follows $\|u_j - v^*\|_0 \to 0$. Hence we get $v = v^* \in H_0^\Gamma(G)$, which means $H^{(k)}(G) \subset H_0^\Gamma(G)$. This proves the assertion (2.30).

3. A generalized Dirichlet problem

3.1. Let G be a bounded open set with Property P. We have defined the sesquilinear form B on $C_*^k(G)$ by (2.1). Since $C_*^k(G)$ is dense in $H^\Gamma(G)$ and since the obvious estimate

$$|B(u, v)| \leqq c\|u\|_\Gamma \|v\|_\Gamma$$

holds for all $u, v \in C_*^k(G)$, this sesquilinear form can be extended continuously onto $H^\Gamma(G)$.

Using this sesquilinear form we can now formulate (analogously to the theory of strongly elliptic boundary value problems) a generalized Dirichlet problem. Starting from the Dirichlet problem of classical type

(3.1a) $$L(D)u = f \quad \text{in } G,$$

(3.1b) $$D^\alpha u = g_{\alpha, j} \quad \text{on } \partial_j G,$$

for all $\alpha \in N_0^n, \alpha \leqq \gamma$ with some $\gamma \in \Gamma^j$ $(1 \leqq j \leqq k)$ we pose the generalized problem as follows (cf. also [4]):

Problem 1. *For $f \in L^2(G)$ and $g \in H^\Gamma(G)$ find all $u \in H^\Gamma(G)$ such that*

(3.2) $$B(u, \varphi) = (f, \varphi)_0$$

holds for all $\varphi \in C_0^\infty(G)$ and the generalized boundary condition

$$v := u - g \in H_0^\Gamma(G)$$

is fulfilled.

From Lemma 2 it follows that the linear functional $l_{f,g}$ defined by

(3.3) $$l_{f,g}(\varphi) := (f, \varphi)_0 - B(g, \varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(G)$$

is bounded on $H_0^\Gamma(G)$. Thus, Problem 1 is equivalent to

Problem 2. *For $f \in L^2(G)$ and $g \in H^\Gamma(G)$ find all elements $v \in H_0^\Gamma(G)$ such that*

$$l_{f,g}(\varphi) = B(v, \varphi)$$

holds for all $\varphi \in C_0^\infty(G)$.

3.2. Now, we have

Theorem 7. *Problem 2 has a unique solution $v \in H_0^\Gamma(G)$.*

Proof. By Lemma 2 the sesquilinear form B is equivalent to the scalar product (2.2) on $H_0^r(G)$. By the Fréchet—Riesz representation theorem the result follows immediately.

Remark. If (in the case $f\in C^0(G)$ and $g\in H^r(G)\cap C^{k-1}(\bar{G})$) the function $u=v+g\in H^r(G)$ has more regularity, $u\in X$, and if G has Property P, the function u is also a classical solution of the differential equation (3.1a) in G and fulfils the boundary condition (3.1b) in the usual sense (cf. Theorem 4).

4. On the regularity of the solutions of the generalized

4.1. Let G be a bounded open set with Property P. Furthermore, let $f\in L^2(G)$ and an "admissible" boundary data $g\in H^r(G)$ be given. We call the boundary data $g\in H^r(G)$ admissible if the linear functional $l_{f,g}$ is also bounded on $L^2(G)$, and therefore by the Fréchet—Riesz theorem there exists an element $h\in L^2(G)$ such that the relation

$$l_{f,g}(\varphi)=(h,\varphi)_0 \tag{4.1}$$

holds for all $\varphi\in C_0^\infty(G)$.

First we prove, in the case where the function $h\in L^2(G)$ has the form

$$h=h_1\dots h_k \tag{4.2}$$

with $h_j\in L^2(G_j)$ $(1\leqq j\leqq k)$, a regularity result which is similar to that of [3] (pp. 46—68). At the second stage we will drop assumption (4.2) and take an arbitrary $h\in L^2(G)$.

We denote by $H_0^1(G_j)$ $(1\leqq j\leqq k)$ the usual Sobolev space of functions with generalized homogeneous boundary values and by $\|\cdot\|_{1,G_j}$ the norm in $H_0^1(G_j)$. For $1\leqq j\leqq k$ let B_j be the sesquilinear form

$$B_j(\varphi_j,\psi_j):=\int_{G_j}\sum_{i=1}^{m_j}\overline{D_{l_j+i}\varphi_j(x^{(j)})}D_{l_j+i}\psi_j(x^{(j)})\,dx^{(j)} \tag{4.3}$$

defined for $\varphi_j,\psi_j\in C_0^\infty(G_j)$ with $x^{(j)}:=(x_{l_j+1},\dots,x_{l_j+m_j})$ and $dx^{(j)}:=dx_{l_j+1}\dots dx_{l_j+m_j}$. This form gives a norm $|||\cdot|||_{1,G_j}$,

$$|||\varphi_j|||_{1,G_j}^2=B_j(\varphi_j,\varphi_j), \tag{4.4}$$

equivalent to $\|\cdot\|_{1,G_j}$.

By the theory of elliptic differential operators there exists for each j $(1\leqq j\leqq k)$ a unique $v_j\in H_0^1(G_j)$ such that

$$(h_j,\varphi_j)_{0,G_j}=B_j(v_j,\varphi_j) \tag{4.5}$$

holds for all $\varphi_j\in C_0^\infty(G_j)$.

Theorem 8. *Let G be a bounded open set with Property* P. *Let $f\in L^2(G)$ and an admissible boundary data $g\in H^{\Gamma}(G)$ be given. Furthermore assume that the element $h\in L^2(G)$ representing the functional $l_{f,g}$ of Problem* 2 *has the form* (4.2). *Then the unique solution of Problem* 2 *is the product of the unique solutions of the equations* (4.5).

Proof. A. First we prove by induction that the product $v=v_1\dots v_k$ of the solutions v_j of the equations (4.5) belongs to $H_0^{\Gamma}(G)$. For $k=1$ this is trivial because then the equality $H_0^1(G)=H_0^{\Gamma}(G)$ holds. For $l\in N$, $1\leqq l\leqq k$, we define

$$\Gamma_l = \left\{\alpha \,\middle|\, \alpha\in N_0^n,\ \alpha = \sum_{j=1}^{l} \varepsilon_{t_j}\right\}$$

(cf. (1.3)). Note that with this definition we have $\Gamma_k=\Gamma$.

Now let $G^\wedge:=G_1\times\dots\times G_{l-1}$, $l\leqq k$, and assume $\tilde{v}=v_1\dots v_{l-1}\in H_0^{l-1}(G^\wedge)$ with $v_j\in H_0^1(G_j)$, $1\leqq j\leqq l-1$. Thus, there exists a sequence $\{\tilde{\varphi}_m\}_{m\in N}$, $\tilde{\varphi}_m\in C_0^\infty(G^\wedge)$ such that

$$|||\tilde{v}-\tilde{\varphi}_m|||_{\Gamma_{l-1},G^\wedge}\to 0 \tag{4.6}$$

holds for $m\to\infty$. On the other hand, since $C_0^\infty(G_l)$ is dense in $H_0^1(G_l)$, we can find a sequence $\{\varphi_{l,m}\}_{m\in N}$, $\varphi_{l,m}\in C_0^\infty(G_l)$, such that

$$|||v_l-\varphi_{l,m}|||_{1,G_l}\to 0 \tag{4.7}$$

holds for $m\to\infty$. Now, if we put $G'=G^\wedge\times G_l$ $(l\leqq k)$,

$$|||\tilde{v}-\tilde{\varphi}_m|||_{\Gamma_{l-1},G^\wedge}\,|||v_l-\varphi_{l,m}|||_{1,G_l} = |||(\tilde{v}-\tilde{\varphi}_m)(v_l-\varphi_{l,m})|||_{\Gamma_l,G'}$$

$$\geqq |||\tilde{v}v_l-\tilde{\varphi}_m\varphi_{l,m}|||_{\Gamma_l,G'}-|||\tilde{\varphi}_m|||_{\Gamma_{l-1},G^\wedge}\,|||v_l-\varphi_{l,m}|||_{1,G_l}-|||\varphi_{l,m}|||_{1,G_l}\,|||\tilde{v}-\tilde{\varphi}_m|||_{\Gamma_{l-1},G^\wedge}.$$

Since $|||\tilde{\varphi}_m|||_{\Gamma_{l-1},G^\wedge}$ and $|||\varphi_{l,m}|||_{1,G_l}$ are bounded it follows by (4.6) and (4.7) that

$$|||\tilde{v}v_l-\tilde{\varphi}_m\varphi_{l,m}|||_{\Gamma_l,G'}\to 0$$

for $m\to\infty$.

B. Next we prove that $v=v_1\dots v_k$ solves the equation

$$B(v,\varphi) = (h_1\dots h_k,\varphi)_{0,G}$$

for all $\varphi\in C_0^\infty(G)$. For the functions $\varphi_j\in C_0^\infty(G_j)$ $(1\leqq j\leqq k)$ we have by (4.3) and (4.5)

$$B(v_1\dots v_k,\varphi_1\dots\varphi_k) = B_1(v_1,\varphi_1)\dots B_k(v_k,\varphi_k) \tag{4.8}$$

$$= (h_1,\varphi_1)_{0,G_1}\dots(h_k,\varphi_k)_{0,G_k} = (h_1\dots h_k,\varphi_1\dots\varphi_k)_{0,G}.$$

By [9] (Corollary 1 and 2 of Theorem 39.2, p. 409) the equation (4.8) can be extended continuously to all elements $\varphi \in C_0^\infty(G)$ instead of $\varphi_1 \ldots \varphi_k$. Thus v is the unique solution of Problem 2.

C. Assuming the boundaries ∂G_j $(1 \leqq j \leqq k)$ to be smooth enough we can apply the regularity theory for solutions of elliptic equations (cf. [3], pp. 46—68) to the functions $v_j \in H_0^1(G_j)$ $(1 \leqq j \leqq k)$ solving (4.5). Hence for smooth boundaries ∂G_j and $h_j \in L^2(G_j)$ we find $v_j \in H^2(G_j) \cap H_0^1(G_j)$ $(1 \leqq j \leqq k)$ and therefore

$$v_1 \ldots v_k = v \in H^{2\Gamma}(G) \cap H_0^\Gamma(G). \tag{4.9}$$

If in addition $h_j \in C^\infty(\bar{G}_j)$ $(1 \leqq j \leqq k)$, we have (again under the assumption of smooth boundaries) $v_j \in H_0^1(G_j) \cap C^\infty(\bar{G}_j)$ and therefore

$$v_1 \ldots v_k = v \in H_0^\Gamma(G) \cap C^\infty(\bar{G}) \tag{4.10}$$

(see [3], p. 68).

4.2. We will now examine the regularity of the solution of Problem 2 for arbitrary admissible data.

Theorem 9. *Let G be a bounded open set with Property* P. *Furthermore let $f \in L^2(G)$ and an admissible $g \in H^\Gamma(G)$ be given. Then the unique solution v of Problem 2 belongs to $H^{(k)}(G)$.*

Proof. Since g is admissible there exists an element $h \in L^2(G)$ such that the functional $l_{f,g}$ of Problem 2 has the representation $l_{f,g}(\varphi) = (h, \varphi)_0$. By [9] (Corollary 1 of Theorem 39.2, p. 409) there exists for h and for each $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$ a function $h_\varepsilon \in C_0^\infty(G)$ such that

$$\|h - h_\varepsilon\|_{0,G} < \varepsilon \tag{4.11}$$

holds, where h_ε has the form

$$h_\varepsilon := \sum_{(i_1, \ldots, i_k) \in \Omega} h_{i_1,\varepsilon} \ldots h_{i_k,\varepsilon} \tag{4.12}$$

with a suitable finite subset Ω of N^k and $h_{i_j,\varepsilon} \in C_0^\infty(G_j)$ $(1 \leqq j \leqq k)$.

For the functions $h_{i_j,\varepsilon}$ $(1 \leqq j \leqq k)$ we consider the unique solutions $v_{i_j,\varepsilon} \in H_0^1(G_j)$ of the generalized strongly elliptic Dirichlet problems

$$(h_{i_j,\varepsilon}, \varphi)_{0,G_j} = B_j(v_{i_j,\varepsilon}, \varphi_j) \quad \text{for all} \quad \varphi_j \in C_0^\infty(G_j). \tag{4.13}$$

From the elliptic theory we get $v_{i_j,\varepsilon} \in C^\infty(\bar{G}_j)$ (cf. [3], p. 68). The function

$$v_\varepsilon := \sum_{(i_1, \ldots, i_k) \in \Omega} v_{i_1,\varepsilon} \ldots v_{i_k,\varepsilon} \in H_0^\Gamma(G) \cap C^\infty(\bar{G}) = X = D(L)$$

is the unique solution of the equation

$$(h_\varepsilon, \varphi)_{0,G} = B(v_\varepsilon, \varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(G).$$

We denote the unique solution of Problem 2 again by $v \in H_0^r(G)$. We have

$$B(v - v_\varepsilon, \varphi) = (h - h_\varepsilon, \varphi)_{0,G} \quad \text{for all} \quad \varphi \in C_0^\infty(G).$$

The continuity of the sesquilinear form B in $H_0^r(G)$ implies (with $v - v_\varepsilon$ instead of φ)

$$|||v - v_\varepsilon|||_r^2 \leqq \|h - h_\varepsilon\|_0 \|v - v_\varepsilon\|_0 \leqq c\|h - h_\varepsilon\|_0 |||v - v_\varepsilon|||_r$$

with a positive constant c. We apply (4.11) and get

$$|||v - v_\varepsilon|||_r \leqq c\varepsilon.$$

Thus, because of the equivalence of the norms $\|\cdot\|_r$ and $|||\cdot|||_r$ on $H_0^r(G)$ we have

$$\|v - v_\varepsilon\|_0 \to 0$$

for $\varepsilon \to 0$, and by (4.11)

$$\|h - Lv_\varepsilon\|_0 = \|h - h_\varepsilon\|_0 \to 0.$$

Since $L^\sim$ is the closure of L we get finally

$$v \in D(L^\sim) = H^{(k)}(G).$$

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Freie Universität Berlin
Fachbereich Mathematik
Arnimallee 2—6
D 1000 Berlin 33
Bundesrepublik Deutschland

New addres of N. J.:
Hochschule der Bundeswehr Müncheu
Fachbereich Informatik
Institut für Mathematik
Werner—Heisenberg—Weg 39
D 8014 Neubiberg
Bundesrepublik Deutschland

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