A NON-HYPOELLIPTIC DIRICHLET PROBLEM FROM STOCHASTICS

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To the memory of Rolf Nevanlinna

1. Introduction


This operator $L(D)$ is defined in the following way: Let $k$ be a positive integer. For each $j (1 \leq j \leq k)$ take $m_j \in \mathbb{N}$ and set $n=m_1+\ldots+m_k$. With the notation

$$l_1 = 0, \quad l_j = \sum_{i=1}^{j-1} m_i \quad (2 \leq j \leq k)$$

we set

$$\Delta_j := \sum_{i=1}^{m_j} D_{j+i}^2 \quad \text{with} \quad D_{j+i} = -\sqrt{-1} \frac{\partial}{\partial x_{j+i}}.$$ (1.1)

Thus, $\Delta_j$ denotes the Laplace operator which acts on functions defined on subsets of $\mathbb{R}^n$. We define the differential operator $L(D)$ by

$$L(D) := \Delta_1 \ldots \Delta_k.$$ (1.2)

The operator $L(D)$ is not hypoelliptic for $k>1$. This fact can be seen already in the simplest case where $k=2$ and $m_1=m_2=1$. Then

$$L(D) = D_1^2 D_2^2 = \frac{\partial^4}{\partial x_1^2 \partial x_2^2}$$

and the polynomial associated with $L(D)$ is given by $P(\xi) = \xi_1^2 + \xi_2^2$. Since for $\xi_1=0$ and $\xi_2=n$ ($n \in \mathbb{N}$) we have

$$\frac{\partial^2}{\partial \xi_1^2} P(\xi) = 2n^2,$$

it follows by a result of L. Hörmander (see [6], p. 99) that $L(D)$ is not hypoelliptic. The usual Sobolev space methods are not applicable to a study of Dirichlet problems for the operator $L(D)$.

1.2. In this paper we present a method which can be understood as a generalization of the Hilbert space methods used in the elliptic theory. Here we use “anisotropic” Sobolev spaces which correspond in a natural way to the operator $L(D)$. It turns out that in these anisotropic spaces the behaviour of the non-hypoelliptic operator $L(D)$ is quite similar to that of an elliptic operator in the usual Sobolev spaces.

For a domain $G$ which is the Cartesian product of bounded open sets $G_j \subset \mathbb{R}^{m_j}$ ($n=m_1+\ldots+m_k$) with sufficiently smooth boundaries $\partial G_j$ we show that a generalized Dirichlet problem, for which the solution is searched in such an anisotropic space, has a unique solution. Further one gets for a class of data a regularity result for this solution.

We will mention that Rolf Nevanlinna expressed a similar idea in unpublished lectures given in Ann Arbor, Helsinki and Zürich. He indicated the possibility to construct for a differential operator a suitable bilinear form such that solving Dirichlet problems for this operator can be reduced to the problem of orthogonal projection in the sense of this bilinear form (cf. [8] and also [1]).

1.3. We give a slightly modified representation of $L(D)$. We denote by $N^n_0$ the set of all ordered systems of $n$ nonnegative integers (multi-indices). For $x=(x_1,\ldots,x_n) \in N^n_0$ we define its length as usual by $|x|=x_1+\ldots+x_n$. For $\sigma,\tau \in N^n_0$, $\sigma=(\sigma_1,\ldots,\sigma_n)$, $\tau=(\tau_1,\ldots,\tau_n)$, we set $\sigma \equiv \tau$ if $\sigma_i \equiv \tau_i$ for $1 \leq i \leq n$. It is clear that with this definition $N^n_0$ is a partially ordered set.

Let $m_j$ and $l_j$ ($1 \leq j \leq k$) be defined as in 1.1. For each $j$ ($1 \leq j \leq k$) we define multi-indices $e_j \in N^n_0$ of length 1, $|e_j|=1$, having its only nonvanishing coordinate in the $t_j$-th position, $l_j+1 \leq e_j \leq l_j+m_j$. Furthermore we set

\begin{equation}
\Gamma = \left\{ x \mid x \in N^n_0, \quad x = \sum_{j=1}^{k} e_{t_j} \right\}
\end{equation}

and

\begin{equation}
2\Gamma = \left\{ \beta \mid \beta \in N^n_0, \quad \beta = 2x, \quad x \in \Gamma \right\}.
\end{equation}

Note that both $\Gamma$ and $2\Gamma$ have $m_1\ldots m_k$ elements. Now we obtain the very useful expression for the differential operator $L(D)$:

\begin{equation}
L(D) = \sum_{x \in \Gamma} D^{2x},
\end{equation}

where we used the abbreviation

\begin{equation}
D^x = D_{t_1}^{x_1} \ldots D_{t_k}^{x_k}.
\end{equation}

The authors want to thank E. B. Dynkin for leading their interest to these problems and I. S. Louhivaara for his suggestions while writing this paper.
2. The Hilbert spaces $H^r(G)$, $H^r_0(G)$ and $H^{(k)}(G)$

2.1. Let $G$ be a bounded open set in $\mathbb{R}^n$. As usual, let $
abla^k(G)$, $k\in\mathbb{N}_0$, be the linear space of all complex valued functions $u$ which are $k$ times continuously differentiable in $G$. By $\nabla^k_0(G)$ we denote the space of all functions $u\in\nabla^k(G)$ each having a compact support in $G$. We write also $\nabla^\infty_0(G)=\bigcap_{k\in\mathbb{N}_0}\nabla^k_0(G)$. Further we define for $k\in\mathbb{N}_0$

$$\nabla^k_0(G) := \{u| \ u\in\nabla^k(G), \ D^a u\in L^2(G), \ a\in\mathbb{N}_0^n, \ |x| \equiv k\}.$$ 

In $\nabla^k_0(G)$ we associate with the operator $L(D)$ the sesquilinear form

$$B(u, v) := \sum_{\xi}\int_G \overline{D^\xi u(x)} D^\xi v(x) \, dx.$$

We define

$$u, v)_r := B(u, v) + (u, v)_0,$$

where $(\cdot, \cdot)_0$ denotes the scalar product in $L^2(G)$,

$$u, v)_0 = \int_G \overline{u(x)} v(x) \, dx.$$

Thus, we have a scalar product $(\cdot, \cdot)_r$ on $\nabla^k_0(G)$ with the corresponding norm $\|\cdot\|_r$. The completion of $\nabla^k_0(G)$ with respect to the scalar product (2.2) will be denoted by $H^r(G)$. One obtains $H^{2r}(G)$ analogously. For the closure of $\nabla^\infty_0(G)$ in $H^r(G)$ we write $H^{(r)}_0(G)$. The elements of $H^{(r)}_0(G)$ are interpreted as functions with generalized homogenous boundary data (cf. Theorem 4).

In contrast to the usual Sobolev space theory we cannot conclude that the strong $L^2$-derivative $D^\tau u$ exists for $u\in H^r(G)$ and for each multi-index $\tau\in\mathbb{N}_0^n$ with $\tau\equiv\sigma$ for some $\sigma\in\Gamma$. But one can prove (cf. also [7])

**Lemma 1.** Let $u\in H^r_0(G)$ be given. Then the strong $L^2$-derivative $D^\tau u$ exists for all multi-indices $\tau$ with $\tau\equiv\sigma$ for some $\sigma\in\Gamma$.

**Proof.** For $\tau\in\Gamma$ the assertion is obvious. Let now $\tau \in \Gamma$ such that $\tau \equiv \sigma \in \Gamma$. Since $G \subset \mathbb{R}^n$ is a bounded set, we can find a constant $d>0$ such that for each $x=(x_1, \ldots, x_n)\in G$

$$\max_{1\leq j \leq n} |x_j| \leq d$$

holds. For $\varphi\in C_0^\infty(G)$ we get by partial integration ($1\leq j \leq k$)

$$\|D^\tau \varphi\|_0^2 = \int_G |D^\tau \varphi(x)|^2 \, dx = -\int_G \sum_{j=1}^n \frac{\partial}{\partial x_j} (D^\tau \varphi(x)D^\tau \varphi(x)) \, dx$$

$$\leq 2d \|D^{\tau+\varepsilon_j} \varphi\|_0 \|D^\tau \varphi\|_0.$$ 

Hence one has

$$\|D^\tau \varphi\|_0 \leq 2d \|D^{\tau+\varepsilon_j} \varphi\|_0.$$
Since $\tau \equiv \sigma$, there exists a multi-index $\beta$ with $\tau + \sigma = \beta$ and the result follows by iteration.

As an immediate consequence of the Poincaré inequality we have

Lemma 2. On $H_0^2(G)$ the sesquilinear form (2.1) defines a scalar product equivalent to (2.2).

Proof. It is sufficient to prove all estimates for elements of $C_0^\infty(G)$. By (2.2) we have for all $\varphi \in C_0^\infty(G)$

$$B(\varphi, \varphi) \equiv (\varphi, \varphi)_R.$$ 

Now by the Poincaré inequality (cf. [5], p. 33) we get for each multi-index $\varepsilon_j \in \mathbb{N}_0^n$, $|\varepsilon_j| = 1$ ($1 \leq j \leq n$)

$$\|\varphi\|_0^2 \leq c_1 \|D^\varepsilon_j \varphi\|_0^2,$$

where the constant $c_1$ depends only on $G$. Repeating this argument we get finally for each $\alpha \in \Gamma$

$$\|\varphi\|_0^2 \leq c_2 \|D^\alpha \varphi\|_0^2$$

and by (2.1)

$$\|\varphi\|_0^2 \leq cB(\varphi, \varphi),$$

which proves Lemma 2.

Thus we have on $H_0^2(G)$ a norm $||| \cdot |||_R$ defined by

(2.4) $$|||\varphi|||_R := (B(\varphi, \varphi))^{1/2}, \quad \varphi \in C_0^\infty(G),$$

equivalent to $\| \cdot \|_R$. This norm can be interpreted as the energetic norm associated with the differential operator $L(D)$.

2.2. Now let us suppose that the set $G \subset \mathbb{R}^n$ has

Property P. The open bounded set $G$ is the Cartesian product

(2.5) $$G = G_1 \times \ldots \times G_k$$

of bounded open sets $G_j \subset \mathbb{R}^{m_j}$ ($1 \leq j \leq k$), $m_1 + \ldots + m_k = n$, with sufficiently smooth boundaries\(^1\) $\partial G_j$.

We will generalize a well-known theorem from the theory of Sobolev spaces.

Let $G_j' \subset \mathbb{R}^{m_j}$ ($1 \leq j \leq k$) be a bounded open set. If $\Phi_j$ is a diffeomorphism of class $C^1$ from $\bar{G}_j$ onto $\bar{G}_j$ then the tensor product $\Phi$,

(2.6) $$\Phi := \Phi_1 \otimes \ldots \otimes \Phi_k,$$

is a diffeomorphism of class $C^1$ from $\bar{G}' = \bar{G}_1' \times \ldots \times \bar{G}_k'$ onto $\bar{G} = \bar{G}_1 \times \ldots \times \bar{G}_k$.

\(^1\) All our considerations are valid if the boundaries $\partial G_j$ are of class $C^m$ (cf. e.g. [3], pp. 9—10).
By definition the Jacobi matrix is the direct sum of the linear mappings \( A_j : \mathbb{R}^{m_j} \to \mathbb{R}^{m_j} \) (1 \( \leq j \leq k \)) defined by

\[
A_j = \begin{pmatrix}
\frac{\partial x_{l_j+1}}{\partial y_{j+1}} & \cdots & \frac{\partial x_{l_j+m_j}}{\partial y_{j+1}} \\
\frac{\partial x_{l_j+1}}{\partial y_{j+1}} & \cdots & \frac{\partial x_{l_j+m_j}}{\partial y_{j+1}} \\
\vdots & \vdots & \vdots \\
\frac{\partial x_{l_j+1}}{\partial y_{j+1}} & \cdots & \frac{\partial x_{l_j+m_j}}{\partial y_{j+1}}
\end{pmatrix},
\]

where \( x_i \) denotes the \( i \)-th component of \( \Phi \). We remark that the elements of \( A_j \) are continuous functions on \( \mathcal{G}'_j \).

**Lemma 3.** Let \( G \) be a bounded open set with Property P and let \( \Phi \) be defined by (2.6). Then for all \( u \in H^1_0(G) \) the function

\[
(2.8) \quad u'(y) = u(\Phi(y))
\]

is in \( H^1_0(G') \) and

\[
(2.9) \quad c_1 \| u' \|_{r,G'} \leq \| u \|_{r,G} \leq c_2 \| u' \|_{r,G'}
\]

holds, where the constants \( c_1, c_2 \) depend only on the diffeomorphism \( \Phi \).

**Proof.** By Lemma 2 inequality (2.9) is equivalent to

\[
(2.10) \quad \tilde{c}_1 \| u' \|_{r,G'} \leq \| u \|_{r,G} \leq \tilde{c}_2 \| u' \|_{r,G'}.
\]

Now by definition we get for each function (2.8)

\[
\| u' \|_{r,G'} = \sum_{\alpha \in \Gamma} \int_{\mathcal{G}'} |D_\alpha^r u'(y)|^2 \, dy.
\]

For a fixed \( \alpha \in \Gamma \) we have by (1.3)

\[
\alpha = \epsilon_{t_1} + \ldots + \epsilon_{t_k},
\]

where \( \epsilon_{t_j} \) (1 \( \leq j \leq k \)) has its only non-vanishing coordinate in the \( t_j \)-th position, \( l_j+1 \equiv t_j \equiv l_j+m_j \). Furthermore we get

\[
(2.11) \quad D_\alpha^r u'(y) = (-\sqrt{-1})^k \frac{\partial^k u'(y)}{\partial y_{t_1} \ldots \partial y_{t_k}}
\]

\[
= (-\sqrt{-1})^k \sum_{i_1=1}^{m_1} \ldots \sum_{i_k=1}^{m_k} \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+1} \ldots \partial x_{l_k+1}} \frac{\partial x_{l_1+i_1}}{\partial y_{t_1}} \ldots \frac{\partial x_{l_k+i_k}}{\partial y_{t_k}}.
\]

Thus, using the fact that the Jacobian of \( \Phi \) is continuous on \( \mathcal{G}'_j \), we get

\[
|D_\alpha^r u'(y)| \leq \tilde{c}_a \sum_{i_1=1}^{m_1} \ldots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{l_1+i_1} \ldots \partial x_{l_k+i_k}} \right|,
\]
where the constant $c_x$ depends only on $\Phi$ and $\alpha$. Now one has with another constant $c_x$

$$|D^ju'(y)|^2 \leq c_x^2 \left( \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u(\Phi(y))}{\partial x_{i_1+i_1} \cdots \partial x_{i_k+i_k}} \right| \right)^2$$

and further

$$\|D^ju'\|_{0,G}^2 \leq c_x^2 \left( \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \int_G \left| \frac{\partial^k u(\Phi(y))}{\partial x_{i_1+i_1} \cdots \partial x_{i_k+i_k}} \right|^2 dy \right).$$

We substitute $y=\Phi^{-1}(x)$. As the Jacobian of $\Phi^{-1}$ is continuous on $\bar{G}$ we find

$$\|D^ju'\|_{0,G}^2 \leq c_x^2 \left( \sum_{i_1=1}^{m_1} \cdots \sum_{i_k=1}^{m_k} \left| \frac{\partial^k u}{\partial x_{i_1+i_1} \cdots \partial x_{i_k+i_k}} \right|^2 \right).$$

with a suitable constant $c_x$. In the summation on the right side of (2.13) only multi-indices which belong to $\Gamma$ appear, wherefore we get

$$\|D^ju'\|_{0,G}^2 \leq c_x^2 \|u\|_{l,G}^2.$$
surface $x_1=0$ and that $U \cap G_1$ lies in the halfspace $x_1 \leq 0$ a. In the Cartesian product $(U \cap \partial G_1) \times G^-$ we can find an $n$-dimensional cylinder $S_h$ with height $h$, the base of which is an $(n-1)$-dimensional ball

$$B_{r_h}(\bar{z}, \eta) = \left\{ x \mid x \in \mathbb{R}^n, x_1 = 0, \sum_{i=2}^{n_1} |x_i - \xi_i|^2 + \sum_{i=n_1+1}^{n} |x_i - \eta_i|^2 < r_h^2 \right\}$$

in $U \cap \partial G_1$ in the hypersurface $x_1=0$, where the radius $r_h$ is chosen such that the volume of $S_h$ is equal to $h^2$. With sufficiently small values of $h$ we have $S_h \subset G_1 \times G^-$. 

For each $\varphi \in C_0^\infty(G)$ one has

$$(2.17) \quad D^\beta \varphi(x) = 0$$

for all $x \in B_{r_h}(\bar{z}, \eta)$ and all $\beta \in \mathbb{N}_0^n$. From (2.17) one has

$$(2.18) \quad D^\beta \varphi(x) = \int_0^{x_1} \frac{\partial}{\partial t} D^\beta \varphi(t, x_2, ..., x_n) \, dt.$$ 

By the Cauchy—Schwarz inequality we get

$$|D^\beta \varphi(x)|^2 \leq h \int_0^h \left| \frac{\partial}{\partial t} D^\beta \varphi(t, x_2, ..., x_n) \right|^2 \, dt.$$ 

Integration with respect to $x$ gives

$$\int_{S_h} |D^\beta \varphi(x)|^2 \, dx \leq h^2 \int_{S_h} \left| \frac{\partial}{\partial x_1} D^\beta \varphi(x) \right|^2 \, dx.$$ 

Since $\text{vol} \, S_h = h^2$, we get

$$(2.19) \quad \frac{1}{\text{vol} \, S_h} \int_{S_h} |D^\beta \varphi(x)| \, dx \equiv \int_{S_h} |D^\beta + \epsilon_1 \varphi(x)|^2 \, dx$$

for $\epsilon_1 = (1, 0, ..., 0) \in \mathbb{N}_0^n$. 

Now take a multi-index $\beta$ with $\beta \equiv \gamma$ for some $\gamma \in \Gamma^1$. Then one has $\beta + \epsilon_1 \equiv \gamma + \epsilon_1 \in \Gamma$. By Lemma 1 it follows that the right-hand integral tends to zero also for all elements $u \in H^1_0(G)$ if $h \to 0$, whence we have

$$\lim_{h \to 0} \frac{1}{\text{vol} \, S_h} \int_{S_h} |D^\beta u(x)|^2 \, dx = 0.$$ 

If in addition $u \in C^{k-1}(\bar{G})$, we conclude by using the mean value theorem

$$D^\beta u(\bar{z}, \eta) = 0$$

for all $\eta \in G^-$ and for all $\beta \equiv \gamma \in \Gamma^1$. This proves Theorem 4.
2.3. In this connection we remark that partial integration is possible for functions of the set

\( X := H_0^2(G) \cap C^\infty(G) \),

provided that the set \( G \) has Property P.

**Lemma 5.** Let \( G \) be a bounded open set with Property P. Then

\[
\int_G \overline{D^2 u(x)} D^2 v(x) \, dx = \int_G \overline{D^2 u(x)} v(x) \, dx
\]

holds for all \( u, v \in X \) and \( \alpha \in \Gamma \).

2.4. In the case where the bounded open set \( G \) has Property P we introduce another Hilbert space denoted by \( H^{(b)}(G) \). On \( X \) a scalar product is defined by

\[
(u, v)_{(k)} := (L(D) u, L(D) v)_0 + (u, v)_0.
\]

The Hilbert space \( H^{(b)}(G) \) is defined as the completion of \( X \) with respect to the scalar product (2.21). It is clear that (2.21) gives the graph norm \( \| \cdot \|_{(k)} \),

\[
\| u \|_{(k)}^2 = \| L(D) u \|_0^2 + \| u \|_0^2
\]
on \( X \).

We consider the densely defined linear operator \( L \) in the Hilbert space \( L^2(G) \) given by

\[
D(L) := X(\subset L^2(G)),
\]

\[
Lu := L(D) u \quad \text{for all } u \in X.
\]

By partial integration one gets

\[
(Lu, v)_0 = \int_G \overline{L(D) u(x)} v(x) \, dx = \int_G \overline{u(x) L(D) v(x)} \, dx = (u, g)_0
\]

for \( u, v \in X \), with \( L(D)v =: g =: L^* v \). Thus, we have \( D(L^*) \supset X \), and the adjoint operator \( L^* \) is densely defined on \( L^2(G) \). Hence the operator \( L \) is closable with the closure (smallest closed extension) \( L^* = L^{**} \).

**Theorem 6.** Let \( G \) be a bounded open set with Property P. Then the relation

\[
D(L^*) = H^{(b)}(G)
\]

holds.

**Proof.** We denote the graphs of \( L \) and \( L^* \) by

\[
G(L) := \{(u, Lu) | u \in X\}
\]

and

\[
G(L^*) := \{(f, L^* f) | f \in D(L^*)\},
\]
respectively. On the other hand, it is well-known (cf. [10], p. 89) that $G(L^-)$ coincides with the completion of $G(L)$ with respect to the graph norm (2.22),

$$G(L^-) = \overline{G(L)}^{\|\cdot\|_{G}}. \tag{2.25}$$

For an arbitrary $u \in H^{(k)}(G)$ there exists by definition a sequence $\{u_m\}_{m \in \mathbb{N}} \subseteq X$ such that

$$\|u_m - u\|_0 \to 0 \quad \text{for} \quad m \to \infty \tag{2.26}$$

and $\{L u_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^2(G)$, which implies the existence of a unique element $w \in L^2(G)$ such that

$$\|L u_m - w\|_0 \to 0 \quad \text{for} \quad m \to \infty. \tag{2.27}$$

Because of (2.25) it follows that $(u, w) \in G(L^\infty)$, i.e., $u \in D(L^-)$ (and $w = L^- u$).

On the other hand, let $u \in D(L^-)$ be given. By the definition of a closed operator there exists a sequence $\{u_m\}_{m \in \mathbb{N}} \subseteq D(L^-) = X$ such that (2.26) and (2.27) hold with $w := L^- u$. By definition we have $u \in H^{(k)}(G)$.

Remark. 1. From Theorem 6 we obtain for the scalar product of $H^{(k)}(G)$ the expression

$$(u, v)_{(k)} = (L^- u, L^- v)_0 + (u, v)_0. \tag{2.28}$$

2. Let $B$ be the sesquilinear from (2.1). By partial integration one has

$$B(u, \varphi) = (L u, \varphi)_0 \quad \text{for all} \quad u \in X \quad \text{and} \quad \varphi \in C^\infty_0(G) \tag{2.29}$$

and by continuous extension

$$B(u, v) = (L u, v)_0 \quad \text{for all} \quad u, v \in X. \tag{2.30}$$

By the Cauchy—Schwarz inequality we get

$$|B(u, v)| \leq \|u\|_{(k)} \|v\|_0 \quad \text{for all} \quad u, v \in X$$

and especially

$$\|u\|^2_\ell \leq \|u\|_{(k)} \|u\|_0 \quad \text{for all} \quad u \in X. \tag{2.31}$$

2.5. Now we will show that the elements of $H^{(k)}(G)$ have the same boundary behaviour as the elements of $H^\ell_0(G)$:

$$H^{(k)}(G) \cap H^\ell_0(G) = H^{(k)}(G). \tag{2.32}$$

Take an element $v \in H^{(k)}(G)$. By the definition of $H^{(k)}(G)$ there exists a sequence $\{u\} \subseteq X = H^\ell_0(G) \cap C^\infty(\overline{G})$ with

$$\|u_j - v\|_{(k)}^2 = \|L u_j - L^- v\|_0^2 + \|u_j - v\|_0^2 \to 0.$$  

Especially $\{u\}$ is a Cauchy sequence in $H^{(k)}(G)$ and by (2.29) also in $H^\ell_0(G)$:

$$\|u_j - u\|_\ell \to 0.$$
Thus \( \{u_j\} \) has a limit element \( v^* \) in \( H^r_0(G) \):
\[
\|u_j - v^*\|_r \to 0.
\]
By Lemma 2 it follows \( \|u_j - v^*\|_0 \to 0 \). Hence we get \( v = v^* \in H^r_0(G) \), which means \( H^{(k)}(G) \subset H^r_0(G) \). This proves the assertion (2.30).

3. A generalized Dirichlet problem

3.1. Let \( G \) be a bounded open set with Property P. We have defined the sesquilinear form \( B \) on \( C^k_*(G) \) by (2.1). Since \( C^k_*(G) \) is dense in \( H^r(G) \) and since the obvious estimate
\[
|B(u, v)| \leq c \|u\|_r \|v\|_r
\]
holds for all \( u, v \in C^k_*(G) \), this sesquilinear form can be extended continuously onto \( H^r(G) \).

Using this sesquilinear form we can now formulate (analogously to the theory of strongly elliptic boundary value problems) a generalized Dirichlet problem. Starting from the Dirichlet problem of classical type

\( (3.1a) \quad L(D)u = f \quad \text{in} \quad G, \)
\( (3.1b) \quad D^\alpha u = g_{z, j} \quad \text{on} \quad \partial J G, \)
for all \( \alpha \in \mathbb{N}_0^n, \alpha \equiv \gamma \) with some \( \gamma \in \Gamma^j \) (\( 1 \equiv j \equiv k \)) we pose the generalized problem as follows (cf. also [4]):

Problem 1. For \( f \in L^2(G) \) and \( g \in H^r(G) \) find all \( u \in H^r(G) \) such that
\( (3.2) \quad B(u, \varphi) = (f, \varphi)_0 \)
holds for all \( \varphi \in C^m_0(G) \) and the generalized boundary condition
\[ v := u - g \in H^r_0(G) \]
is fulfilled.

From Lemma 2 it follows that the linear functional \( l_{f, g} \) defined by
\( (3.3) \quad l_{f, g}(\varphi) := (f, \varphi)_0 - B(g, \varphi) \quad \text{for all} \quad \varphi \in C^m_0(G) \)
is bounded on \( H^r_0(G) \). Thus, Problem 1 is equivalent to

Problem 2. For \( f \in L^2(G) \) and \( g \in H^r(G) \) find all elements \( v \in H^r_0(G) \) such that
\[ l_{f, g}(\varphi) = B(v, \varphi) \]
holds for all \( \varphi \in C^m_0(G) \).

3.2. Now, we have

Theorem 7. Problem 2 has a unique solution \( v \in H^r_0(G) \).
Proof. By Lemma 2 the sesquilinear form $B$ is equivalent to the scalar product (2.2) on $H^1_0(G)$. By the Fréchet—Riesz representation theorem the result follows immediately.

Remark. If (in the case $f \in C^0(G)$ and $g \in H^1(G) \cap C^{k-1}(G)$) the function $u = v + g \in H^1(G)$ has more regularity, $u \in X$, and if $G$ has Property P, the function $u$ is also a classical solution of the differential equation (3.1a) in $G$ and fulfills the boundary condition (3.1b) in the usual sense (cf. Theorem 4).

4. On the regularity of the solutions of the generalized

4.1. Let $G$ be a bounded open set with Property P. Furthermore, let $f \in L^2(G)$ and an “admissible” boundary data $g \in H^1(G)$ be given. We call the boundary data $g \in H^1(G)$ admissible if the linear functional $l_{f,g}$ is also bounded on $L^2(G)$, and therefore by the Fréchet—Riesz theorem there exists an element $h \in L^2(G)$ such that the relation

(4.1) $l_{f,g}(\varphi) = (h, \varphi)_0$

holds for all $\varphi \in C_0^\infty(G)$.

First we prove, in the case where the function $h \in L^2(G)$ has the form

(4.2) $h = h_1 \ldots h_k$

with $h_j \in L^2(G_j)$ ($1 \leq j \leq k$), a regularity result which is similar to that of [3] (pp. 46—68). At the second stage we will drop assumption (4.2) and take an arbitrary $h \in L^2(G)$.

We denote by $H^1_0(G_j)$ ($1 \leq j \leq k$) the usual Sobolev space of functions with generalized homogeneous boundary values and by $\| \cdot \|_{1,G_j}$ the norm in $H^1_0(G_j)$. For $1 \leq j \leq k$ let $B_j$ be the sesquilinear form

(4.3) $B_j(\varphi_j, \psi_j) := \int_{G_j} \sum_{i=1}^{m_j} \bar{D}_{i,j+1}(\varphi_j(x^{(j)}))D_{i,j+1}(\psi_j(x^{(j)})) \, dx^{(j)}$

defined for $\varphi_j, \psi_j \in C_0^\infty(G_j)$ with $x^{(j)} := (x_{i,j+1}, \ldots, x_{i,j+m_j})$ and $dx^{(j)} := dx_{i,j+1} \ldots dx_{i,j+m_j}$. This form gives a norm $\| \cdot \|_{1,G_j}$, equivalent to $\| \cdot \|_{1,G_j}$.

By the theory of elliptic differential operators there exists for each $j$ ($1 \leq j \leq k$) a unique $v_j \in H^1_0(G_j)$ such that

(4.5) $(h_j, \varphi_j)_{0,G_j} = B_j(v_j, \varphi_j)$

holds for all $\varphi_j \in C_0^\infty(G_j)$. 

Theorem 8. Let $G$ be a bounded open set with Property P. Let $f \in L^2(G)$ and an admissible boundary data $g \in H^1(G)$ be given. Furthermore assume that the element $h \in L^2(G)$ representing the functional $l_{r, g}$ of Problem 2 has the form (4.2). Then the unique solution of Problem 2 is the product of the unique solutions of the equations (4.5).

Proof. A. First we prove by induction that the product $v=v_1...v_k$ of the solutions $v_j$ of the equations (4.5) belongs to $H_0^1(G)$. For $k=1$ this is trivial because then the equality $H_0^1(G) = H_0^1(G)$ holds. For $l \in \mathbb{N}$, $1 \leq l \leq k$, we define

$$\Gamma_l = \left\{ \alpha \in \mathbb{N}_0^l, \alpha = \sum_{j=1}^l e_j \right\}$$

(cf. (1.3)). Note that with this definition we have $\Gamma_k = \Gamma$.

Now let $G^* := G_1 \times \ldots \times G_{l-1}$, $l \leq k$, and assume $v=v_1...v_{l-1} \in H_0^{l-1}(G^*)$ with $v_j \in H_0^1(G_j)$, $1 \leq j \leq l-1$. Thus, there exists a sequence $\{ \tilde{\varphi}_m \}_{m \in \mathbb{N}}$, $\tilde{\varphi}_m \in C_0^\infty(G^*)$ such that

$$|||\tilde{v} - \tilde{\varphi}_m|||_{l^{-1}, G^*} \to 0$$

holds for $m \to \infty$. On the other hand, since $C_0^\infty(G_i)$ is dense in $H_0^1(G_i)$, we can find a sequence $\{ \varphi_{1,m} \}_{m \in \mathbb{N}}$, $\varphi_{1,m} \in C_0^\infty(G_i)$, such that

$$|||v_1 - \varphi_{1,m}|||_{1, G_1} \to 0$$

holds for $m \to \infty$. Now, if we put $G' = G^* \times G_1$ ($l \leq k$),

$$|||\tilde{v} - \tilde{\varphi}_m|||_{l^{-1}, G^*} |||v_1 - \varphi_{1,m}|||_{1, G_1} = |||(\tilde{v} - \tilde{\varphi}_m)(v_1 - \varphi_{1,m})|||_{l, G^*} \leq |||\tilde{v} - \tilde{\varphi}_m|||_{l^{-1}, G^*} |||v_1 - \varphi_{1,m}|||_{1, G_1} = |||\varphi_{1,m}|||_{1, G_1} |||v_1 - \tilde{\varphi}_m|||_{l^{-1}, G^*}$$

Since $|||\tilde{\varphi}_m|||_{l^{-1}, G^*}$ and $|||\varphi_{1,m}|||_{1, G_1}$ are bounded it follows by (4.6) and (4.7) that

$$|||\tilde{v} - \tilde{\varphi}_m|||_{l^{-1}, G^*} \to 0$$

for $m \to \infty$.

B. Next we prove that $v=v_1...v_k$ solves the equation

$$B(v, \varphi) = (h_1...h_k, \varphi)_{0,G}$$

for all $\varphi \in C_0^\infty(G)$. For the functions $\varphi_j \in C_0^\infty(G_j)$ ($1 \leq j \leq k$) we have by (4.3) and (4.5)

$$B(v_1...v_k, \varphi_1...\varphi_k) = B_1(v_1, \varphi_1)...B_k(v_k, \varphi_k)$$

$$\quad = (h_1, \varphi_1)_{0,G_1}... (h_k, \varphi_k)_{0,G_k} = (h_1...h_k, \varphi_1...\varphi_k)_{0,G}.$$
By [9] (Corollary 1 and 2 of Theorem 39.2, p. 409) the equation (4.8) can be extended continuously to all elements \( \varphi \in C_0^\infty (G) \) instead of \( \varphi_1 \ldots \varphi_k \). Thus \( v \) is the unique solution of Problem 2.

C. Assuming the boundaries \( \partial G_j \ (1 \leq j \leq k) \) to be smooth enough we can apply the regularity theory for solutions of elliptic equations (cf. [3], pp. 46—68) to the functions \( v_j \in H^1_0 (G_j) \ (1 \leq j \leq k) \) solving (4.5). Hence for smooth boundaries \( \partial G_j \) and \( h_j \in L^2 (G_j) \) we find \( v_j \in H^2 (G_j) \cap H^1_0 (G_j) \ (1 \leq j \leq k) \) and therefore

\[
(4.9) \quad v_1 \ldots v_k = v \in H^{2r} (G) \cap H^r_0 (G).
\]

If in addition \( h_j \in C_0^\infty (\overline{G}_j) \ (1 \leq j \leq k) \), we have (again under the assumption of smooth boundaries) \( v_j \in H^1_0 (G_j) \cap C_0^\infty (\overline{G}_j) \) and therefore

\[
(4.10) \quad v_1 \ldots v_k = v \in H^r_0 (G) \cap C_0^\infty (\overline{G})
\]
(see [3], p. 68).

4.2. We will now examine the regularity of the solution of Problem 2 for arbitrary admissible data.

**Theorem 9.** Let \( G \) be a bounded open set with Property P. Furthermore let \( f \in L^2 (G) \) and an admissible \( g \in H^r (G) \) be given. Then the unique solution \( v \) of Problem 2 belongs to \( H^{r} (G) \).

**Proof.** Since \( g \) is admissible there exists an element \( h \in L^2 (G) \) such that the functional \( l_{f,g} \) of Problem 2 has the representation \( l_{f,g} (\varphi) = (h, \varphi)_0 \). By [9] (Corollary 1 of Theorem 39.2, p. 409) there exists for \( h \) and for each \( \varepsilon \in R, \varepsilon > 0 \) a function \( h_\varepsilon \in C_0^\infty (G) \) such that

\[
(4.11) \quad \| h - h_\varepsilon \|_{0, G} < \varepsilon
\]
holds, where \( h_\varepsilon \) has the form

\[
(4.12) \quad h_\varepsilon := \sum_{(i_1, \ldots, i_k) \in \Omega} h_{i_1, \varepsilon} \ldots h_{i_k, \varepsilon}
\]

with a suitable finite subset \( \Omega \) of \( N^k \) and \( h_{i_j, \varepsilon} \in C_0^\infty (G_j) \ (1 \leq j \leq k) \).

For the functions \( h_{i_j, \varepsilon} \ (1 \leq j \leq k) \) we consider the unique solutions \( v_{i_j, \varepsilon} \in H^1_0 (G_j) \) of the generalized strongly elliptic Dirichlet problems

\[
(4.13) \quad (h_{i_j, \varepsilon}, \varphi)_0, G_j = B_j (v_{i_j, \varepsilon}, \varphi_j) \quad \text{for all} \quad \varphi_j \in C_0^\infty (G_j).
\]

From the elliptic theory we get \( v_{i_j, \varepsilon} \in C_0^\infty (\overline{G}_j) \) (cf. [3], p. 68). The function

\[
v_\varepsilon := \sum_{(i_1, \ldots, i_k) \in \Omega} v_{i_1, \varepsilon} \ldots v_{i_k, \varepsilon} \in H^r_0 (G) \cap C_0^\infty (\overline{G}) = X = D (L)
\]
is the unique solution of the equation

\[(h_{\varepsilon}, \varphi)_{0,G} = B(v_{\varepsilon}, \varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(G).\]

We denote the unique solution of Problem 2 again by \(v \in H^r_0(G)\). We have

\[B(v - v_{\varepsilon}, \varphi) = (h - h_{\varepsilon}, \varphi)_{0,G} \quad \text{for all} \quad \varphi \in C_0^\infty(G).\]

The continuity of the sesquilinear form \(B\) in \(H^r_0(G)\) implies (with \(v - v_{\varepsilon}\) instead of \(\varphi\))

\[||v - v_{\varepsilon}||^2_r \equiv \|h - h_{\varepsilon}\|_0\|v - v_{\varepsilon}\|_0 \leq c\|h - h_{\varepsilon}\|_0\|v - v_{\varepsilon}\|_r\]

with a positive constant \(c\). We apply (4.11) and get

\[||v - v_{\varepsilon}||_r \leq c\varepsilon.\]

Thus, because of the equivalence of the norms \(\|\cdot\|_r\) and \(||\cdot||_r\) on \(H^r_0(G)\) we have

\[\|v - v_{\varepsilon}\|_0 \to 0\]

for \(\varepsilon \to 0\), and by (4.11)

\[\|h - Lv_{\varepsilon}\|_0 = \|h - h_{\varepsilon}\|_0 \to 0.\]

Since \(L^-\) is the closure of \(L\) we get finally

\[v \in D(L^-) = H^{(k)}(G).\]

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