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ON MEROMORPHIC FUNCTIONS CONTINUOUS ON THE STOÏLOW BOUNDARY

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Introduction

In this paper we study meromorphic functions on an open Riemann surface which extend continuously to the Stoïlow ideal boundary (*MC*-functions). Our main concern is how to classify the boundary elements into "essential" and "inessential" points from the point of view of *MC*-functions. It goes without saying that for most boundary elements the problem is banal: they are simply too "large" to tolerate nonconstant *MC*-functions nearby. For example, all boundary elements of positive harmonic measure are out of the question [16, p. 265]. To exclude trivialities, we confine ourselves to "admissible" points, i.e., to elements which have neighborhoods carrying nonconstant *MC*-functions.

Chapter 1 is devoted to topological properties of *MC*-functions. In particular, we exhibit the close relationship between the openness of extended functions and their covering properties. In Chapter 2, we propose a definition for removable and essential boundary points. It turns out that removability can be characterized as well in topological as in algebraic and analytic terms: via the openness of extended functions, via the field property of *MC*-functions and via a certain function-theoretic null-class, respectively. As an application, we give a solution to a problem proposed by Ozawa, concerning certain classification principles for Riemann surfaces [10, p. 751]. In Chapter 3 we give conditions, in terms of cluster sets attached to the ideal boundary, which guarantee continuous extension of the functions involved. As a very special case we obtain a recent result of Ishchanov [5].

We note in conclusion that some of the problems discussed in the present paper have been touched, although from a somewhat different point of view, in our earlier works [6] and [7].

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1. Topological properties of MC-functions

1.1. Let W be an open Riemann surface, and let V be a subregion of W with compact (possibly empty) relative boundary $\partial_W V$. Then V is said to be an *end* of W. We often assume, as we may without loss of generality, that $\partial_W V$ consists of a finite number of piecewise analytic closed curves. The (Kerékjártó-) Stoïlow ideal boundary of W is denoted by β and the relative Stoïlow boundary of V (see [15, p. 366]) by β_V. The usual topological operations (closure $A \mapsto \bar{A}$, boundary $A \mapsto \partial A$ etc.) are to be taken with respect to the compactified space $W \cup \beta$ (or $V \cup \beta_V$). The class of analytic or meromorphic functions on V is denoted by $A(V)$ or $M(V)$, respectively. The subclass of $A(V)$ (resp. $M(V)$) consisting of functions which have a finite (resp. finite or infinite) limit at every relative ideal boundary element is denoted by $AC(V)$ (resp. $MC(V)$). Whenever f is a function of class AC or MC, we let f^* stand for the extension of f to the (relative) ideal boundary. We say that V is an *admissible* end if $MC(V)$ contains nonconstant functions. A boundary element $p \in \beta$ is called admissible provided there is an admissible end V with $p \in \beta_V$.

Let V be an end of W with nice boundary, and suppose that $f \in AC(V \cup \partial_W V)$ is nonconstant. Assuming that $z \in \mathbf{C} \setminus f(\partial_W V)$, the *index* of z with respect to $f(\partial_W V)$ is defined to be

$$i(z;\, f(\partial_W V)) = (2\pi)^{-1} \int_{\partial_W V} d \arg (f(p) - z)$$

and the *valence* function, as usual,

$$\nu_{f|V}(z) = \sum_{\substack{f(p)=z \\ p \in V}} n(p;\, f),$$

where $n(p; f)$ denotes the multiplicity of f at p. Then we have

Lemma 1. *Suppose that* $z \in \mathbf{C} \setminus (f(\partial_W V) \cup f^*(\beta_V))$. *Then*

$$\nu_{f|V} = i(z;\, f(\partial_W V)).$$

Proof. Fix $z_0 \in \mathbf{C} \setminus (f(\partial_W V) \cup f^*(\beta_V))$, and denote by d the distance between $\{z_0\}$ and $f(\partial_W V) \cup f^*(\beta_V)$. For every $p \in \beta_V$ choose an open neighborhood U_p such that ∂U_p is contained in V and $f^*(U_p) \subset D(f^*(p), d/2) = \{z \in \mathbf{C} \,|\, |z - f^*(p)| < d/2\}$. From the open covering $\{U_p | p \in \beta_V\}$ of β_V pick out a finite subcovering $\{U_{p_1}, \ldots, U_{p_k}\}$. Let (V_n) be a relative exhaustion of $V \cup \partial_W V$ such that the components of $V \setminus V_n$ are noncompact. Since $F = f^{-1}(z_0) \cup (\bigcup_{i=1}^k \partial U_{p_i})$ is a compact subset of V, there is a positive integer n_0 such that $F \subset V_n$ for $n \geqq n_0$. Given any component C of $V \setminus V_{n_0}$, there is $i \in \{1, \ldots, k\}$ such that $C \subset U_{p_i}$. Let $B_1, \ldots, B_m$ be the components of $\partial_W V_{n_0} \cap V$. We infer that each $f(B_i)$ is contained in some $D(f^*(p_j), d/2)$, $j = 1, \ldots, k$. Thus the winding number of $f(B_i)$ with respect to z_0 is 0 for each i. We conclude from the argument principle that

$$\nu_{f|V}(z_0) = i(z_0;\, f(\partial_W V)). \quad \square$$

Remark. The above result holds true even if $z\in f(\partial_W V)\setminus f^*(\beta_V)$ provided $i(z; f(\partial_W V))$ and $\nu_{f|V}$ are given a suitable interpretation in case $z\in f(\partial_W V)$ (see [12]).

1.2. Fix $p_0\in\beta$ and assume that p_0 has an admissible neighborhood, i.e., there is an admissible end V such that $p_0\in\beta_V$. By passing to a subend and performing a preliminary linear fractional transformation, we obtain the situation where $V\cup\partial_W V$ carries a nonconstant AC-function f with $f^*(p_0)\notin f(\partial_W V)$.

1°. Assume first that $f^*(\beta_V)$ is nowhere dense in $\boldsymbol{C}$. Denote by G the component of $\boldsymbol{C}\setminus f(\partial_W V)$ which contains $f^*(p_0)$. By Lemma 1, $n=i(z; f(\partial_W V))>0$ for $z\in G$, and there is an open neighborhood $U\subset\boldsymbol{C}$ of $f^*(p_0)$ such that for each $z\in U\setminus f^*(\beta_V)$ $\nu_{f|V}(z)=n$. Now let $V=V_1\supset V_2\supset\ldots V_j\supset\ldots$ be a determining sequence of p_0. By applying the argument above for each j, we get a decreasing sequence of positive integers $(n_j=i(f^*(p_0); f(\partial_W V)))$. The limit

$$n(p_0; f^*) = \lim_{j\to\infty} n_j > 0$$

is called the *multiplicity* or the *local degree* of f^* at p_0 (cf. [4, p. 301]). It is clear that $n(p_0; f^*)$ is independent of the choice of (V_j). Also, it is obvious that

$$\text{(A)} \qquad \nu_{f^*|V}(z) = \sum_{\substack{f^*(p)=z \\ p\in V\cup\beta_V}} n(p; f^*) = i(z; f(\partial_W V))$$

for every $z\in\boldsymbol{C}\setminus f(\partial_W V)$ (note that the procedure given before applies to each $p\in\beta_V$).

2°. Assume then that $f^*(\beta_{V'})$ has interior points for every subend V' of V for which $p_0\in\beta_{V'}$. Fix such a V', and let (V_n') be a relative exhaustion of V'. Let F_n denote the closed set $f^*(\beta_{V'})\setminus f(V'\setminus\overline{V_n'})$, $n=1, 2, \ldots$. By continuity, $f^*(\beta_{V'})\subset\overline{f(V'\setminus\overline{V_n'})}$, so that F_n is a nowhere dense subset of $f^*(\beta_{V'})$ for each n. But clearly $f^*(\beta_{V'})\setminus\bigcup_{n=1}^{\infty}F_n\subset\{z\in\boldsymbol{C}|\nu_{f|V}(z)=\infty\}$. In other words, $\nu_{f|V}$ becomes infinite in a set residual in $f^*(\beta_{V'})$. Therefore, given any neighborhood U of p_0, we can find a sequence of points (z_n) in $\boldsymbol{C}$ such that $z_n\to f^*(p_0)$ and $f^{-1}(z_n)\cap U$ is infinite for each n. Hence there is no reason to define the local degree for f^* at p_0 in this situation.

1.3. Let X and Y be topological spaces, and let $f\colon X\to Y$ be a continuous mapping. Then f is said to be *open* if $f(U)$ is open in Y for every open set U in X. It is *quasiopen,* provided that for any $y\in f(X)$ and any open set U in X containing a compact component of $f^{-1}(y)$, y is an interior point of $f(U)$. Further, f is *light* if every point-inverse $f^{-1}(y)$, $y\in Y$, is totally disconnected, and f is *discrete* if each point-inverse is discrete, i.e., consists of isolated points. Clearly, a mapping is open provided it is both quasiopen and light. For mappings of reasonably nice spaces — as is the case in this paper — quasiopenness can be characterized by the condition $\partial f(U)\subset f(\partial U)$ for each relatively compact open set U in X [20, p. 112].

We are now ready to state some useful results concerning the behavior of MC-functions at the ideal boundary.

Theorem 1. *Let W be an open Riemann surface, let V be an admissible end of W with nice boundary, and let $g\in MC(V\cup\partial_W V)$ be nonconstant. Let f stand for $g|V$. Then the following statements are equivalent:*

(1) $f^*\colon V\cup\beta_V\to\hat{C}=C\cup\{\infty\}$ *is open.*
(2) f^* *is quasiopen.*
(3) $f^*(\beta_V)$ *is nowhere dense in* $\hat{C}$.
(4) $f^*(\beta_V)$ *is totally disconnected.*
(5) ν_f *is bounded.*
(6) $\nu_f(z)$ *is finite for each* $z\in\hat{C}$.
(7) f^* *is discrete.*

Proof. Since f^* is light, we immediately have (1)$\Leftrightarrow$(2).

(3)$\Rightarrow$(1): Suppose U is an open set in $V\cup\beta_V$ and $z_0\in f^*(U)$. If $(f^*)^{-1}(z_0)\cap V$ is nonempty, z_0 belongs to the interior of $f^*(U)$ by the openness of f. So assume that $z_0=f^*(p)$ for some $p\in\beta_V\cap U$. Then choose a subend V' of V such that $p\in\beta_{V'}$, $V'\cup\beta_{V'}\subset U$, $f^*(p)\notin f(\partial_W V')$ and $f^*(\overline{V'})\neq\hat{C}$. Next, pick out a linear fractional transformation φ such that $h=\varphi\circ(f|V'\cup\partial_W V')$ is bounded. Plainly $h\in AC(V'\cup\partial_W V')$. Lemma 1 now applies to h. Thus, letting G denote the component of $C\setminus h(\partial_W V')$ that contains $h^*(p)=\varphi(z_0)$, we have $i(z;\, h(\partial_W V'))=m>0$ for $z\in G$; further, $G\subset h^*(V'\cup\beta_{V'})$. We infer that z_0 is an interior point of $f^*(U)$. It follows that $f^*(U)$ is open in $\hat{C}$.

(1)$\Rightarrow$(4): Suppose that f^* gives an open mapping into $\hat{C}$. Let $p_0\in\beta_V$ be arbitrary. By the compactness of β_V, it suffices to find a subend V' of V, with $p_0\in\beta_{V'}$, such that $f^*(\beta_{V'})$ is totally disconnected. Therefore, we may again limit ourselves to the case that $h=\varphi\circ(f|V'\cup\partial_W V')$ belongs to $AC(V'\cup\partial_W V')$, φ being a linear fractional mapping. Assume now that C is a component of $h^*(\beta_{V'})$. Fix a point $z_0\in\partial C$. Modifying V' slightly, we may assume that $z_0\notin h(\partial_W V')$. Let G denote the component of $C\setminus h(\partial_W V')$ that contains z_0. It follows from the openness of h^* that $i(z_0;\, h(\partial_W V'))=m>0$ (Lemma 1).

Next, choose $r>0$ such that the disc $D(z_0, r)\subset G$, and set $B=\partial C\cap D(z_0, r)$. Let $z\in B$. We claim that $(h^*)^{-1}(z)$ contains at most m points. Indeed, assuming that we can find $m+1$ points $p_1, \ldots, p_{m+1}$ in $(h^*)^{-1}(z)$, we can also find mutually disjoint open neighborhoods U_i of p_i, $i=1, \ldots, m+1$. But then $\bigcap_{i=1}^{m+1} h^*(U_i)\cap G$ is an open neighborhood of z and hence contains points from $C\setminus h^*(\beta_{V'})$. Given such a point z', the inverse image $h^{-1}(z')$ contains at least $m+1$ points (in V'), whereas Lemma 1 gives $\nu_{h|V'}(z')=m$. We are led to a contradiction.

Thus, $h^*|(h^*)^{-1}(B)$ is discrete. Since it also defines, as is readily seen, an open mapping onto B, we may apply [2, Lemma 2.1]. It follows that the topological dimension of $h^*(\beta_{V'}\cap(h^*)^{-1}(B))=B$ is 0. We conclude that C reduces to a singleton. This proves the implication (1)$\Rightarrow$(4).

Since the implication (4)$\Rightarrow$(3) is trivial, we have now settled the equivalence of conditions (1) to (4).

(4)$\Rightarrow$(5): The number of the components of $\hat{C}\setminus(g(\partial_W V)\cup f^*(\beta_V))$ is finite. In each of them ν_f is finite and constant. The desired conclusion now follows from the lower semicontinuity of ν_f.

(6)$\Rightarrow$(3): Suppose, for the moment, that $f^*(\beta_V)$ has interior points. The argument of Section 2.1 then yields the result that ν_f is infinite in a residual part of $f^*(\beta_V)$. This contradicts (6).

(4)$\Rightarrow$(7): Since discreteness is a local property, we may again refer to the results of Section 1.2 about AC-functions: (7) is indeed a direct consequence of formula (A).

The implications (5)$\Rightarrow$(6) and (7)$\Rightarrow$(6) being trivial, the proof is complete. $\square$

Corollary 1. *Let W be an open Riemann surface, let β be the ideal boundary of W and suppose that $f\in MC(W)$ is nonconstant. Then either*

(a) *$f^*(\beta)$ is totally disconnected, in which case $\nu_{f^*}(z)=\sum_{f^*(p)=z} n(p; f^*)$ is finite and constant, or*

(b) *the interior of $f^*(\beta)$ is nonempty, and the set $\{z\in\hat{C}|\nu_f(z)=\infty\}$ is residual in $f^*(\beta)$.*

The next result, a direct consequence of Lemma 1, was given and utilized in [6].

Corollary 2. *Let W and β be as above, and let $f\in AC(W)$. Then $f^*(\beta)=f^*(W\cup\beta)$.*

The following corollary provides a generalization of Stoïlow's uniqueness theorem [17, p. 124].

Corollary 3. *Let W be an open Riemann surface, and let V be an end of W such that the set β_V is infinite. Suppose that $f\in MC(V)$ and $f^*(p)=0$ for every $p\in\beta_V$. Then f vanishes identically on V.*

To give an example of nontrivial AC-functions, take a compact totally disconnected set $E\subset C$ such that $m(E)$, the two-dimensional Lebesgue measure of E, is positive and set

$$f(z)=\iint_E \frac{1}{\zeta-z}\,dm_\zeta.$$

It turns out that f belongs to $AC(\hat{C}\setminus E)$; for details see e.g. [3, p. 79—80].

Remark. We point out that all the results given in this chapter are of a purely topological character. In particular, they remain valid if the analyticity of mappings is replaced by interiority (in the sense of Stoïlow). Of course, the requirement that boundary elements be admissible can then be dropped.

2. Essential and removable boundary elements

2.1. We begin with some terminology. Let E be a proper closed subset of $\hat{C}$. Then E is said to be of class N_C if, for each domain $G\subset\hat{C}$ with $E\subset G$, every function $G\to C$ continuous on G and analytic on $G\setminus E$ is actually analytic all over G. The subclass of N_C constituted by the totally disconnected elements of N_C is denoted by N_C'. It is known that every closed set $E\subset\hat{C}$ of σ-finite linear mesure is of class N_C and, on the other hand, no set whose Hausdorff dimension exceeds 1 is of class N_C (see e.g. [3]). In the natural way (see [6, p. 308]), the classes N_C and N_C' can be generalized for arbitrary Riemann surfaces.

Let V denote an end of an open Riemann surface W. We say that V satisfies the *absolute AC-maximum principle* if for each subend V' of V and for each $f\in AC(V'\cup\partial_W V')$

$$\sup\{|f(p)|\,|\,p\in V'\cup\partial_W V'\}=\max\{|f(p)|\,|\,p\in\partial_W V'\}.$$

THEOREM 2. *Let W be an open Riemann surface, let β be the ideal boundary of W, and let $p\in\beta$ be an admissible boundary point. Then the following properties are equivalent:*

(1) *There is an end $V\subset W$ with $p\in\beta_V$ such that for every subend V' of V and for every nonconstant $f\in MC(V')$, f^* defines an open mapping $V'\cup\beta_{V'}\to\hat{C}$.*

(2) *There is an end $V\subset W$ with $p\in\beta_V$ which satisfies the absolute AC-maximum principle.*

(3) *There is an end $V\subset W$ with $p\in\beta_V$ such that for every subend V' of V, $MC(V')$ constitutes a field.*

(4) *There is an end $V\subset W$ with $p\in\beta_V$ such that for every subend V' of V and for every $f\in MC(V')$, $f^*(\beta_{V'})$ belongs to N_C'.*

(5) *There is an end $V\subset W$ with $p\in\beta_V$ and a nonconstant function f in $MC(V)$ such that $f^*(\beta_V)$ belongs to N_C'.*

Proof. (1)⇒(2): Suppose that $V\subset W$ fulfils the hypotheses of (1). Let V' be a subend of V, and let $f\in AC(V'\cup\partial_W V')$ be nonconstant. Since $V'\cup\beta_{V'}$ is an open set in $V\cup\beta_V$, $f^*(V'\cup\beta_{V'})$ is open in C. Therefore f attains its maximum at a point on $\partial_W V'$.

(2)⇒(1): Suppose $V\subset W$ satisfies the absolute AC-maximum principle, and for some subend $V'\subset V$ and some nonconstant $f\in MC(V')$ f^* fails to be open. By the equivalence (1)⇔(2) in Theorem 1, we can find a relatively compact open set $U\subset V'\cup\beta_{V'}$ such that $\partial f^*(U)\not\subset f^*(\partial U)$. Pick out a point $z_0\in\partial f^*(U)\setminus f^*(\partial U)$. It is clear that $z_0=f^*(p_0)$ for some $p_0\in U\cap\beta_{V'}$. Now choose an end $V''\subset U$ such that $p_0\in\beta_{V''}$. Plainly, $z_0\notin f(\partial_W V'')$. Let z_1 stand for a point in $C\setminus f^*(\overline{V}'')$ such that $|z_1-z_0|<\min\{|z_1-z|\,|\,z\in f(\partial_W V'')\}$ (we may assume that $f^*(\overline{V}'')$ lies in C). Denote by g the function $p\mapsto(z_1-f(p))^{-1}$, $p\in V''\cup\partial_W V''$. It is clear that

$g \in AC(V'' \cup \partial_W V'')$ and $|g^*(p_0)| > \max \{|g(p)| \mid p \in \partial_W V''\}$. We have obtained the desired contradiction.

(1)⇒(5): Let V be an admissible end with $p \in \beta_V$, and let $f \in MC(V \cup \partial_W V)$ be nonconstant. Reducing V and performing a preliminary linear fractional transformation, we may assume that f belongs to $AC(V \cup \partial_W V)$. Since f^* gives an open mapping of $V \cup \beta_V$ into $\mathbf{C}$, it follows from Theorem 1 that $f^*(\beta_V)$ is totally disconnected. Thus we may arrange $f^*(\beta_V) \cap f(\partial_W V) = \emptyset$.

Let n stand for $\max \{i(z;\, f(\partial_W V)) \mid z \in \mathbf{C} \setminus f(\partial_W V)\}$. Then $\nu_{f|V}(z)$ is bounded by n, in view of formula (A) in Section 1.2. Let E_i denote $\{z \in f^*(\beta_V) \mid \nu_{f|V}(z) \leqq i\}$, $i = 0, \ldots, n-1$; then $E_{n-1} = f^*(\beta_V)$. Since f is open, each E_i is closed. Moreover, we claim that each E_i belongs to N'_C.

Assume that E_0 is not of class N'_C. By a standard application of Cauchy's integral formula (see e.g. [6, Lemma 4]), we can find a nonconstant function g in $AC(\hat{\mathbf{C}} \setminus E_0)$. Clearly $g \circ f$ belongs to $AC(V)$. Since $g^*(E_0) \subset (g \circ f)^*(\beta_V)$, and $g^*(E_0)$ contains interior points (see Corollary 2 to Theorem 1), we conclude by Theorem 1 that $(g \circ f)^*$ is not open. This contradicts (1). Suppose next that, for some i, E_i is of class N'_C, and fix a point $z_0 \in E_{i+1} \setminus E_i$. Choose a neighborhood U of z_0 such that ∂U is an analytic Jordan curve with $\partial U \cap (f^*(\beta_V) \cup f(\partial_W V)) = \emptyset$ and $f^{-1}(U)$ contains j $(j \leqq i+1)$ relatively compact mutually disjoint Jordan regions V_k in V such that each $z \in U$ has exactly $i+1$ antecedents in $\bigcup_{k=1}^{j} V_k$ (with due account of multiplicities). Then $f | V \setminus \bigcup_{k=1}^{j} V_k$ assumes no value in $E_{i+1} \cap U$. Now $E_{i+1} \cap U$ must be of class N'_C, for otherwise — reproducing the argument given above — we would again arrive at a contradiction with (1). Since z_0 was arbitrary, and belonging to N'_C is a local property (see e.g. [6, p. 308]), we infer $E_{i+1} \in N'_C$. It follows that $E_{n-1} = f^*(\beta_V)$ is of class N'_C.

(5)⇒(4): Suppose $V \subset W$ is an end which carries a nonconstant MC-function f_0 with $f_0^*(\beta_V) \in N'_C$. Let V' be a subend of V. Modifying V' slightly we obtain $f_0(\partial_W V') \cap f_0^*(\beta_{V'}) = \emptyset$. Let G be a component of $\hat{\mathbf{C}} \setminus f_0(\partial_W V')$ such that $f_0^*(\beta_{V'}) \cap G \neq \emptyset$. Then $\nu_{f_0|V'}$ is finite and constant, say n, in $G \setminus f_0^*(\beta_{V'})$. Assume that $f \in MC(V')$ is nonconstant. By an argument familiar from the context of compact Riemann surfaces, it can be shown that f satisfies on $f_0^{-1}(G \setminus f_0^*(\beta_{V'}))$ an identity

$$f^n + \textstyle\sum_{i=1}^{n} (a_i \circ f_0) f^{n-i} = 0,$$

where $a_1, \ldots, a_n$ are meromorphic functions on $G \setminus f_0^*(\beta_{V'})$. Arguing as in [6, p. 309], it can be shown that for each i, a_i admits a meromorphic extension over $f_0^*(\beta_{V'}) \cap G$ (this is the point where use is made of the assumption $f_0^*(\beta_V) \in N'_C$). Henceforth we regard each a_i as defined and meromorphic all over G.

Denote by $\tilde{G}$ the Riemann surface of the relation

$$P(z, w) = w^n + \textstyle\sum_{i=1}^{n} a_i(z) w^{n-i} = 0, \quad z \in G,$$

i.e., the totality of pairs (z, w_z), where $z \in G$ and w_z is a function element with center z and associated with the equation $P(z, w) = 0$. Note that $\tilde{G}$ is a finite union of connected Riemann surfaces. The functions $c\colon (z, w_z)\mapsto z$ and $v\colon (z, w_z)\mapsto w_z(z)$ are meromorphic on $\tilde{G}$. Obviously, $f^*\big((f_0^*)^{-1}(f_0^*(\beta_{V'})\cap G)\cap\beta_{V'}\big)\subset v\big(c^{-1}(f_0^*(\beta_{V'})\cap G)\big)$. Hence by [6, Lemma 2], $f^*\big((f_0^*)^{-1}(f_0^*(\beta_{V'})\cap G)\cap\beta_{V'}\big)$ is of class N'_C. Being a finite union of sets of this kind, $f^*(\beta_{V'})$ also belongs to N'_C.

(5)⇒(3): Suppose V and $f\in MC(V)$ satisfy (5). Fix $p\in\beta_V$. As in [6, Theorem 8], there is a subend $V'\subset V$ with $p\in\beta_{V'}$ and an analytic function $f_0\in AC(V')\colon V'\to D=\{z\in C \mid |z|<1\}$ such that, given any $g\in MC(V')$, one can find a unique $h\in M(D)$ satisfying $g=h\circ f_0$. Making use of this composition, we can readily obtain the conclusion.

(3)⇒(2): Suppose there is a subend V' of V and a function $f\in AC(V'\cup\partial_W V')$ with $\max\{|f(p)| \mid p\in\partial_W V'\}<\max\{|f^*(p)| \mid p\in\bar{V}'\}=r$. Pick out a point $p_0\in\beta_{V'}$ such that $|f^*(p_0)|=r$. Let φ be a conformal mapping of the disc $D(0, r)$ onto the half-strip $\{z\in C \mid \operatorname{Re} z<0,\ |\operatorname{Im} z|<1\}$ such that $f^*(p_0)$ corresponds to the point $-\infty$. It is clear that the functions $h=\exp(\varphi\circ f)$ and $g=\exp\big((1-i)\cdot\varphi\circ f\big)$ belong to $AC(V'\cup\partial_W V')$ (exp stands for $z\mapsto e^z$); moreover, $h(p)$ and $g(p)\to 0$ as $p\to p_0$ in V'. Now choose a sequence of points (p_n) in V' such that $p_n\to p_0$ and $\operatorname{Re}\big((\varphi\circ f)(p_n)\big)=-n$ for large n. Then $|h(p_n)|=\exp(-n)$ and $|g(p_n)|=\exp\big(-n+\operatorname{Im}((\varphi\circ f)(p_n))\big)$. Hence $|g(p_n)/h(p_n)|=\exp\big(\operatorname{Im}((\varphi\circ f)(p_n))\big)$, whence $\exp(-1)\leqq|g(p_n)/h(p_n)|\leqq\exp(1)$ for large n. We conclude that

$$(*)\qquad \lim_{\substack{p\to p_0\\ p\in V'}} (g/h)(p)\neq 0,\ \infty.$$

Similarly, a simple calculation yields $\arg g(p_n)=\operatorname{Im}\big((\varphi\circ f)(p_n)\big)-\operatorname{Re}\big((\varphi\circ f)(p_n)\big)$ and $\arg h(p_n)=\operatorname{Im}\big((\varphi\circ f)(p_n)\big)$. Thus, $\arg h(p_n)$ remains bounded, while $\arg g(p_n)$ varies unboundedly as $n\to\infty$. Hence $\arg(g/h)(p_n)$ also varies unboundedly as $n\to\infty$. But this state of affairs is in apparent contradiction with $(*)$. The implication follows.

The remaining implication (4)⇒(1) follows immediately from Theorem 1. □

In view of the preceding theorem, it seems reasonable to make the following definition (cf. [7, p. 320]):

Definition. Let $p\in\beta$ be an admissible boundary element. Then p is said to be *(AC-)removable* if there is an end V with $p\in\beta_V$ and a nonconstant function $f\in MC(V)$ such that $f^*(\beta_V)$ is of class N'_C. Otherwise p is called *essential.* A closed subset β' of β is said to be removable if each element of β' is removable.

It is clear that the removable boundary points constitute a relatively open subset of β.

Suppose W is parabolic. Then every admissible boundary point is removable; indeed, given any end $V\subset W$ and any $f\in MC(V)$, $f^*(\beta_V)$ is of logarithmic capacity zero (see [9]). More generally, the same is true of Riemann surfaces satisfying the absolute AB-maximum principle ([12], [10], [6]); this case can be characterized by the relation $f^*(\beta_V)\in N_B$ [6, p. 304] (for N_B and other standard null-classes see [1] or [15, Chapter II]). It is to be noted that there are even parabolic surfaces which entirely lack admissible boundary elements [4, p. 298].

2.2. Our next theorem describes the class of globally defined MC-functions in case β is removable.

Theorem 3. *Let W be an open Riemann surface, and suppose that the ideal boundary β of W is removable. Then either*

(a) $MC(W)=\mathbf{C}$, *or*

(b) *$MC(W)$ is a field algebraically isomorphic to the field of rational functions on a compact Riemann surface W', which is uniquely determined up to a conformal equivalence. Moreover, the isomorphism is induced by an analytic mapping of W into W'.*

Proof. Suppose that $MC(W)$ contains a nonconstant function f. By definition and by Theorem 2, each boundary element $p\in\beta$ has a neighborhood U_p with ∂U_p in W such that $f^*(\beta\cap U_p)$ is of class N'_C. By compactness, we can pick out $U_{p_1},\ldots,U_{p_n}$ such that $\beta\subset\bigcup_{i=1}^n U_{p_i}$. Hence $f^*(\beta)\subset\bigcup_{i=1}^n f^*(\beta\cap U_{p_i})$, whence $f^*(\beta)$ is of class N'_C. The theorem now follows from [6, Theorem 6]. □

A local counterpart to the preceding theorem is

Theorem 4. *Let W be an open Riemann surface with ideal boundary β, and suppose that $p\in\beta$ is removable. Then there is an end V of W with $p\in\beta_V$ and an AC-function f_0: $V\to D=\{z\in\mathbf{C}\,\big|\,|z|<1\}$ such that, given any $f\in MC(V)$, one can find a unique $g\in M(D)$(=*the class of meromorphic functions on D) *satisfying $f=g\circ f_0$. Accordingly, $MC(V)$ is isomorphic to the field $M(D)$.*

Proof. See [6, Theorem 8]. □

Suppose next that an end $V\subset W$ has finite genus. Then V can be imbedded conformally in a compact Riemann surface U^*. Therefore β_V can be realized as a subset of U^*. Thus, it makes sense to ask what β_V looks like near a removable boundary point. An answer is given by

Theorem 5. *Let W be an open Riemann surface, and let $V\subset W$ be an end of finite genus. Suppose that $\partial_W V$ is a finite union of analytic Jordan curves and β_V is removable. Then there exists a finite Riemann surface V^* and a compact subset $E\subset V^*$ of class N'_C such that V is conformally equivalent to $V^*\setminus E$. Further, V^* is uniquely determined up to a conformal equivalence.*

Proof. Let $p\in\beta_V$, and choose a planar end $V'\subset V$ such that $p\in\beta_{V'}$, $\partial_W V'$ is a Jordan curve, and $MC(V')$ contains a nonconstant function f. Assume also that V' satisfies condition (4) in Theorem 2. By [11, Theorem 3], we can find a Jordan domain $G\subset\mathbf{C}$, a compact totally disconnected set $F\subset G$ and a sense-preserving homeomorphism φ: $V'\to G\setminus F$. Further, $\beta_{V'}$ and F being totally disconnected, φ admits a homeomorphic extension φ^*: $V'\cup\beta_{V'}\to G$. Consider the continuous mapping $g=f^*\circ(\varphi^*)^{-1}\colon G\to\hat{C}$. It is clear that g is light. Also, $g|G\setminus F$ is open and sense-preserving, and $g(F)=f^*(\beta_{V'})$ is totally disconnected (of class N'_C in fact). Hence by [18, Theorem 9], g is light and open on G. By Stoïlow's theorem [17, p. 121], there is a plane domain G', a sense-preserving homeomorphism $\psi\colon G\to G'$ and a meromorphic function h on G' such that $g=h\circ\psi$. As in [7, p. 319], we see that $\psi\circ\varphi$ defines a conformal mapping $V'\to G'\setminus\psi(F)$. Further, by condition (4) in Theorem 2 $(\psi\circ\varphi^*)(\beta_{V'})$ is of class N'_C.

Altogether, for each $p\in\beta_V$ there is an open neighborhood $U_p\subset V\cup\beta_V$ of p and a homeomorphism Φ_p of U_p onto a plane domain such that $\Phi_p|U_p\setminus\beta_V$ is conformal and $\Phi_p(\beta_V\cap U_p)$ is of class N'_C. The very definition of N'_C implies that the transition mappings $\Phi_p\circ\Phi_q^{-1}\colon \Phi_q(U_p\cap U_q)\to\Phi_p(U_p\cap U_q)$ are actually conformal. Accordingly, $V\cup\beta_V$ can be given a conformal structure, compatible with that of V, which makes $V\cup\beta_V$ a finite Riemann surface. Clearly, β_V is of class N'_C in $V^*=V\cup\beta_V$ (see [6, p. 308]). Hence we may set $E=\beta_V$; the inclusion mapping $i\colon V\to V^*$ defines the desired conformal homeomorphism $V\to V^*\setminus E$.

To prove the uniqueness, suppose that the pairs (V_1^*, E_1) and (V_2^*, E_2) have the required properties. Let $\varphi_1\colon V\to V_1^*\setminus E_1$ and $\varphi_2\colon V\to V_2^*\setminus E_2$ denote the related conformal homeomorphisms. Then $\varphi=\varphi_2\circ\varphi_1^{-1}$ maps $V_1^*\setminus E_1$ conformally onto $V_2^*\setminus E_2$. Since E_1 and E_2 are totally disconnected, φ admits a homeomorphic extension $\varphi^*\colon V_1^*\to V_2^*$. Finally, E_1 being of class N'_C in V_1^*, φ^* is conformal throughout V_1^*. □

Remark 1. As appears from the proof, the uniqueness of V^* follows already from the requirement that the set E, the realization of the ideal boundary, be totally disconnected. It should be noted that there are realizations of removable boundaries which contain proper continua. This state of affairs derives from the fact that there are sets of class N'_C which do not belong to N_{SB}.

Remark 2. Suppose $f\in MC(V)$ under the hypotheses of the preceding theorem, and let φ map V conformally onto $V^*\setminus E$ with E in N'_C. Then $f\circ\varphi^{-1}\in MC(V^*\setminus E)$ and, since $E\in N'_C$, $(f\circ\varphi^{-1})^*=f^*\circ(\varphi^{-1})^*$ is meromorphic in V^*. In this sense, f can be continued to be "meromorphic" on the ideal boundary.

Our next theorem gives a criterion to recognize the situation described above.

Theorem 6. *Let W be an open Riemann surface, and let $V\subset W$ be an end whose relative boundary $\partial_W V$ consists of a finite number of closed analytic curves. Suppose $AC(V\cup\partial_W V)$ separates the points of $V\cup\partial_W V$, and for each $f\in AC(V\cup\partial_W V)$ and for each subend V' of V*

$$\max\{|f(p)|\,|\,p\in\partial_W V'\}=\sup\{|f(p)|\,|\,p\in V'\cup\partial_W V'\}. \tag{1}$$

Then there exists a finite Riemann surface V^* and a compact subset $E\subset V^*$ of class N'_C such that V is conformally equivalent to $V^*\setminus E$; V^* is uniquely determined up to a conformal equivalence.

Proof. Since $AC(V\cup\partial_W V)$ is point-separating, and each $f\in AC(V\cup\partial_W V)$ attains its maximum on $\partial_W V$, it follows from a theorem of Royden [13, Theorem 3] that V has finite genus.

Let $f\in AC(V\cup\partial_W V)$, and let $K\subset V\cup\beta_V$ be a compact set. By assumption and by the total disconnectedness of β_V, it is readily seen that $|f^*(p_0)|\leqq \max\{|f^*(p)|\,|p\in\partial K\}$ for each $p_0\in K$. Therefore, taken as an algebra of functions defined in $V\cup\beta_V$, $AC(V\cup\partial_W V)$ constitutes a maximum modulus algebra in the sense of [8]. Hence by [8, Theorem 1], every $f\in AC(V\cup\partial_W V)$ defines a quasiopen mapping $f^*\colon V\cup\beta_V\to\boldsymbol{C}$. By Theorem 1, $f^*(\beta_V)$ is totally disconnected.

Fix a nonconstant $f\in AC(V\cup\partial_W V)$. Combining [11, Theorem 3], [18, Theorem 9] and Stoïlow's theorem as in the proof of the preceding theorem, we infer that there exist a finite Riemann surface V^*, a compact totally disconnected set $E\subset V^*$ and a conformal homeomorphism $\varphi\colon V\to V^*\setminus E$. Of course, V^* can be taken as a subregion of a compact Riemann surface $\tilde{V}$. Suppose E fails to be of class N_C' in V^* (and in $\tilde{V}$). By [6, Lemma 4], we can then find a nonconstant function g in $AC(\tilde{V}\setminus E)$. Further, by Corollary 2 to Theorem 1, $g^*(V^*)\subset g^*(\tilde{V})=g^*(E)=(g\circ\varphi)^*(\beta_V)$. Hence $g\circ\varphi$, albeit a member of $AC(V\cup\partial_W V)$, by the reflection principle, does not attain its maximum on $\partial_W V$. This contradicts (1).

The uniqueness of V^* is proven as in the preceding theorem. □

Remark. It seems possible that Theorem 6 remains valid even if condition (1) is imposed only on V. Actually, a result of this sort holds for the algebra of bounded analytic functions on V, as shown by Wermer [19] and Royden [13].

2.3. A Riemann surface W is said to satisfy the *absolute AB-maximum principle,* briefly $W\in\mathcal{M}_B$, if

$$\sup\{|f(p)|\,|p\in V\cup\partial_W V\}=\max\{|f(p)|\,|p\in\partial_W V\}$$

for every end $V\subset W$ and for every $f\in AB(V\cup\partial_W V)$ (=the class of bounded analytic functions on $V\cup\partial_W V$) (see [12], [10]). Further, W is said to belong to the class $\mathcal{D}_B$ if, for every end $V\subset W$, the cluster set $\mathrm{Cl}(f;\beta_V)$ of every $f\in AB(V\cup\partial_W V)$ attached to β_V is totally disconnected [10]. Finally, W is said to belong to the class $\mathcal{A}_B$ provided that $AB(V\cup\partial_W V)\subset AC(V\cup\partial_W V)$ for every end $V\subset W$ [10]. As to the inclusion relations between these classes, it is immediate that $\mathcal{D}_B\subset\mathcal{A}_B$; further, the work of Royden [12] readily brings in the equality $\mathcal{M}_B=\mathcal{D}_B$ (see [6]). In the next theorem, it will be shown that $\mathcal{A}_B=\mathcal{D}_B$ also. This settles a problem proposed by Ozawa [10, p. 751].

Theorem 7. *The three classes defined above coincide:*

$$\mathcal{M}_B=\mathcal{D}_B=\mathcal{A}_B$$

Proof. It remains to prove that $\mathcal{A}_B\subset\mathcal{D}_B$. Assume the contrary, and let W be a Riemann surface in $\mathcal{A}_B\setminus\mathcal{D}_B$. By definition, there exist an end $V\subset W$ and a function $f\in AB(V\cup\partial_W V)=AC(V\cup\partial_W V)$ such that $f^*(\beta_V)$ contains proper continua.

Hence by Theorem 1, $f^*|V\cup\beta_V$ is not open. By Theorem 2, we can then find a subend V' of V and a function g in $AC(V'\cup\partial_W V')$ such that $\max\{|g(p)|\,|\,p\in\partial_W V'\}<r=\max\{|g^*(p)|\,|\,p\in\bar{V}'\}=\max\{|g^*(p)|\,|\,p\in\beta_{V'}\}$. Pick out a point $p_0\in\beta_{V'}$ such that $|g^*(p_0)|=r$.

Let φ be a conformal mapping of the disc $D(0, r)$ onto the strip domain bounded by the lines $\operatorname{Re} z=0$ and $\operatorname{Re} z=1$ such that $g^*(p_0)$ corresponds to the point $+\infty i$. Clearly, $\exp(\varphi\circ g)$ belongs to $AB(V'\cup\partial_W V')$. Yet $\exp(\varphi\circ g)$ fails to have a limit as $p\to p_0$ in V'. This contradiction completes the proof. □

Corollary. *Let W be an open Riemann surface. Then W satisfies the absolute AB-maximum principle if and only if $AB(V\cup\partial_W V)=AC(V\cup\partial_W V)$ for every end $V\subset W$.*

Remark. In case W is planar or, more generally, has finite genus, the equality $\mathscr{A}_B=\mathscr{M}_B$ follows immediately from a result of Rudin. In fact, whenever $p\in\partial W$ is an essential boundary point in the sense of Rudin [14, p. 333], there exists an analytic function f in $AB(W)$, bounded by 1, such that $\operatorname{Cl}(f; p)$, the cluster set of f at p, equals $\overline{D(0,1)}$ [14, Theorem 14].

3. A condition for continuity

3.1. Let V be an end of an open Riemann surface W, and let f be a nonconstant bounded analytic function on V. Suppose that no $\operatorname{Cl}(f; p)$, $p\in\beta_V$, separates the plane (note that each $\operatorname{Cl}(f; p)$ is connected) and $\operatorname{Cl}(f; \beta_V)=\bigcup_{p\in\beta_V}\operatorname{Cl}(f; p)$ is nowhere dense in C. We say that $p\in\beta_V$ is a *generalized antecedent* of a point $z\in C$ with respect to f provided there is a sequence of points (p_n) in V such that $p_n\to p$ and $f(p_n)\to z$ as $n\to\infty$; in other words, p is a generalized antecedent of z if and only if $z\in\operatorname{Cl}(f; p)$.

Let V' be a subend of V such that $\partial_W V'$ is contained in V and consists of a finite number of piecewise analytic closed curves. We will need the following generalization of Lemma 1.

Lemma 2. *Suppose that $z\in C\setminus(f(\partial_W V')\cup\operatorname{Cl}(f; \beta_{V'}))$. Then*

$$\nu_{f|V'}(z) = i(z; f(\partial_W V')).$$

Proof. Fix $z_0\in C\setminus(f(\partial_W V')\cup\operatorname{Cl}(f; \beta_{V'}))$, and denote by d the distance between $\{z_0\}$ and $f(\partial_W V')\cup\operatorname{Cl}(f; \beta_{V'})$. For every $p\in\beta_{V'}$ choose an open neighborhood U_p such that $\partial U_p\subset V'$ and z_0 can be joined to ∞ by an arc in $C\setminus\overline{f(U_p\cap V')}$; this is possible because $C\setminus\operatorname{Cl}(f; p)$ is assumed to be connected. From the open covering $\{U_p|p\in\beta_{V'}\}$ of $\beta_{V'}$ pick out a finite subcovering $\{U_{p_1}, \ldots, U_{p_k}\}$. Let (V_n') be a relative exhaustion of $V'\cup\partial_W V'$ such that the components of $V'\setminus V_n'$ are noncompact. Since $F=\bigcup_{i=1}^k \partial U_{p_i}$ is a compact subset of V', there is a positive integer n_0 such that $F\subset V_n'$ for $n\geqq n_0$. Let $n\geqq n_0$. For every component C of $V'\setminus V_n'$

there is $i\in\{1, \dots, k\}$ such that $C\subset U_{p_i}$. Let $B_1, \dots, B_m$ be the components of $\partial_W V_n' \cap V'$. Since each $f(B_j)$ is contained in some $\overline{f(U_{p_i}\cap V')}$, the winding number of $f(B_j)$ with respect to z_0 is 0 for each j. It follows from the argument principle that $\nu_{f|V'}(z_0)=i(z_0;\ f(\partial_W V'))$. □

We will extend the notion of local degree to the generalized antecedents. So fix $p_0\in\beta_V$, and let $z_0\in \mathrm{Cl}\,(f;\ p_0)$. Let V' be a subend of V with $p_0\in\beta_{V'}$ such that $\partial_W V'\subset V$ and $z_0\notin f(\partial_W V')$. Since every neighborhood of z_0 contains points z from $C\setminus \mathrm{Cl}\,(f;\ \beta_V)$ with $f^{-1}(z)\cap V'\neq\emptyset$, $i(z_0;\ f(\partial_W V'))>0$ by the preceding lemma. It also appears from Lemma 2 that $i(z_0;\ f(\partial_W V'))\geqq i(z_0;\ f(\partial_W V''))$ whenever $V''\subset V'$. Thus, it is reasonable to set

$$n(p_0, z_0;\ f) = \inf\{i(z_0;\ f(\partial_W V'))\},$$

where V' runs over all subends of V with $p_0\in\beta_{V'}$ and $z_0\notin f(\partial_W V')$. It is clear that $n(p_0, z_0;\ f)>0$. Further, the definition gives rise to the formula

$$\text{(B)} \qquad \sum\nolimits_{p\in f^{-1}(z)\cap V'} n(p;\ f)+\sum\nolimits_{\substack{p\in\beta_{V'} \\ z\in \mathrm{Cl}(f;\,p)}} n(p, z;\ f) = i(z;\ f(\partial_W V'))$$

for every $z\in C\setminus f(\partial_W V')$.

Remark. We point out a consequence for future use: Let $p_0\in\beta_V$, let V' be a subend of V with $p_0\in\beta_{V'}$, and let $z_0\in \mathrm{Cl}\,(f;\ p_0)$. Then there is an open neighborhood $U_{z_0}\subset C$ of z_0 such that $\nu_{f|V'}(z)\geqq n(p_0, z_0;f)$ for every $z\in U_{z_0}\setminus \mathrm{Cl}\,(f;\ \beta_{V'})$.

3.2. We are going to show, roughly speaking, that meromorphic functions with meager cluster sets on an admissible end admit continuous extension to the ideal boundary. Besides, we obtain a condition for removability of the ideal boundary.

Theorem 8. *Let W be an open Riemann surface, and let V be an admissible end of W. Suppose f is a nonconstant meromorphic function on V such that* $\mathrm{Cl}\,(f;\ \beta_V)$ *is of class N_C and no* $\mathrm{Cl}\,(f;p)$, $p\in\beta_V$, *separates the plane. Then f admits a continuous extension to β_V; a fortiori, $f^*(\beta_V)$ belongs to N_C'. Accordingly, β_V is removable.*

Proof. Since the problem is local and $\mathrm{Cl}\,(f;\ \beta_V)$ is nowhere dense, we may assume, passing to a subend and performing an auxiliary linear fractional mapping, that f is bounded on $V\cup\partial_W V$. Similarly, we can find a nonconstant bounded function g in $MC(V)$.

As the first step, we will prove that $g^*(\beta_V)$ is totally disconnected. Let $z_0\in \mathrm{Cl}\,(f;\ \beta_V)$. Modifying V slightly, we obtain $z_0\notin f(\partial_W V)$. Let m stand for the positive integer $i(z_0;f(\partial_W V))$, and denote by G the component of $C\setminus f(\partial_W V)$ that contains z_0. Set $U=f^{-1}(G\setminus \mathrm{Cl}\,(f;\ \beta_V))$. By virtue of Lemma 2, g satisfies on U an identity

$$g^m+\sum\nolimits_{i=1}^m (a_i\circ f)\, g^{m-i} = 0,$$

where $a_1, \dots, a_m$ are bounded analytic functions on $G\setminus \mathrm{Cl}\,(f;\ \beta_V)$ (cf. the proof of the implication (5)⇒(4) in Theorem 2). We proceed to show that each a_i can be continued to be analytic all over G. So fix $z \in \mathrm{Cl}(f;\beta_V)\cap G$ for a while, and let (z_n) be a sequence of points in $G\setminus \mathrm{Cl}(f;\beta_V)$ such that $z_n \to z$. Let $p_1, \ldots, p_k$ or $q_1, \ldots, q_{k'}$ be the antecedents or the generalized antecedents of z, respectively. Suppose $U_1, \ldots, U_k, U_{k+1}, \ldots, U_{k+k'}$ are mutually disjoint open neighborhoods (in $V\cup\beta_V$) of the points $p_1, \ldots, q_{k'}$ such that ∂U_j lies in V for each j. By formula B and the ensuing remark, all antecedents of z_n lie in $\bigcup_{j=1}^{k+k'} U_j$ for large n. It follows that $a_1(z_n)\to \sum_{j=1}^{k} n(p_j;f)g(p_j)+\sum_{j=1}^{k'} n(q_j, z; f)g^*(q_j)$. We conclude that a_1 admits a continuous extension to $\mathrm{Cl}(f;\beta_V)\cap G$. Further, by the assumption concerning $\mathrm{Cl}(f;\beta_V)$, a_1 can be regarded as analytic all over G. Obviously, a similar reasoning applies to $a_2, \ldots, a_m$.

Denote by $\tilde{G}$ the Riemann surface of the relation

$$(1) \qquad P(z, w) = w^m + \sum_{i=1}^{m} a_i(z) w^{m-i} = 0, \quad z\in G.$$

Note that the number of the components of $\tilde{G}$ is at most m. The mappings $c\colon (z, w_z)\mapsto z$ and $v\colon (z, w_z)\mapsto w_z(z)$ are analytic on $\tilde{G}$ (cf. the proof of Theorem 2). Choose $r_{z_0}>0$ such that $\overline{D(z_0, r_{z_0})}\subset G$. Then $E_{z_0}=v(c^{-1}(\mathrm{Cl}(f;\beta_V)\cap \overline{D(z_0, r_{z_0})}))$ is a compact and nowhere dense subset of $\boldsymbol{C}$.

Now let z_0 vary over $\mathrm{Cl}(f;\beta_V)$. From the open covering $\{D(z, r_z)\,|\, z\in \mathrm{Cl}(f;\beta_V)\}$ of $\mathrm{Cl}(f;\beta_V)$ pick out a finite subcovering $\{D(z_1, r_{z_1}), \ldots, D(z_s, r_{z_s})\}$. We will complete the first part of the proof by showing that $g^*(\beta_V)\subset \bigcup_{i=1}^{s} E_{z_i}$. So let $p\in\beta_V$, and choose a sequence of points (p_n) in $V\setminus f^{-1}(\mathrm{Cl}(f;\beta_V))$ such that $p_n\to p$. We may assume, passing to a subsequence, that $f(p_n)\to z\in \mathrm{Cl}(f;p)$. Now $z\in D(z_i, r_{z_i})$ for some $i\in\{1, \ldots, s\}$. It is clear that $(f(p_n), g(p_n))$ satisfies a relation of type (1), say $P(f(p_n), g(p_n))=0$, for large n. By continuity, the same holds for $(z, g^*(p))$, i.e., $P(z, g^*(p))=0$. But this implies that $g^*(p)\in v(c^{-1}(z))$. Hence $g^*(p)\in E_{z_i}$. Thus $g^*(\beta_V)\subset\bigcup_{i=1}^{s} E_{z_i}$. So by Theorem 1, $g^*(\beta_V)$ is totally disconnected as was asserted. We note that the removability of β_V could now be readily established by observing that $E_{z_i}\in N_C$ for each i. Of course, this also follows from the claim we are going to prove next: that $\mathrm{Cl}(f;\beta_V)$ is totally disconnected.

The set $g^*(\beta_V)$ being totally disconnected, it obviously suffices to prove that $\mathrm{Cl}(f;\beta_V)\cap\overline{D(z_0, r_{z_i})}\subset c(v^{-1}(g^*(\beta_V)\cap E_{z_i}))$ for each $i\in\{1, \ldots, s\}$; of course, c and v here stand for the center mapping and the value mapping associated with the point z_i. So fix $i\in\{1, \ldots, s\}$, and let $z\in \mathrm{Cl}(f;\beta_V)\cap\overline{D(z_i, r_{z_i})}$. Pick out a point $p\in\beta_V$ such that $z\in \mathrm{Cl}(f;p)$. Then we can find a sequence of points (p_n) in V such that $p_n\to p$ and $f(p_n)\to z$. Clearly, we may also assume that $f(p_n)\notin \mathrm{Cl}(f;\beta_V)$ for all n. Now, $(f(p_n), g(p_n))$ satisfies a relation of type (1), say $P(f(p_n), g(p_n))=0$, for large n. By continuity, the same is true of $(z, g^*(p))$. Hence there is a function element w_z with center z associated with the relation $P=0$ such that $v(z, w_z)=g^*(p)$. But this means that $z\in c(v^{-1}(g^*(p))\subset c(v^{-1}(g^*(\beta_V)\cap E_{z_i}))$, whence the claim follows. Now, by conncetedness, each $\mathrm{Cl}(f;p)$ reduces to a singleton. Therefore f extends to a continuous mapping of $V\cup\beta_V$. $\square$

Corollary 1. *Let W be an open Riemann surface, and let E be a closed, totally disconnected subset of W. Suppose f is a meromorphic function on $W\setminus E$ such that the closed parts of* $\mathrm{Cl}(f;E)$ *are of class N_C and no* $\mathrm{Cl}(f;p)$, $p\in E$, *separates the plane. Then f can be continued to be meromorphic on W.*

Proof. Fix $p\in E$, and choose a relatively compact region V in W such that $p\in V$ and $E\cap\partial V=\emptyset$. Certainly $V\setminus E$ is then an admissible end of $W\setminus E$. It follows from the preceding theorem, in view of the remarks following Theorem 5, that f can be extended to be meromorphic in a neighborhood of p. Since p was arbitrary, the proof is complete. □

The next result is due to Ishchanov [5].

Corollary 2. *Let G be a plane domain, and let E be a closed totally disconnected subset of G. Suppose f is an analytic function on $G\setminus E$ such that* $\mathrm{Re}\, f$ *admits a continuous extension to E and takes a constant value there. Then f can be extended to be analytic on G.*

Proof. By assumption, we have a continuous function h on G and a constant c with $h|G\setminus E=\mathrm{Re}\, f$ and $h(p)=c$ for all $p\in E$. Plainly, this implies $\mathrm{Cl}(f;p)\subset L=\{z\in \boldsymbol{C}|\mathrm{Re}\, z=c\}\cup\{\infty\}$ for each $p\in E$.

Fix $p_0\in E$ and let $G'\subset G$ be a Jordan domain with $p_0\in G'$, $\partial G'\subset G$ analytic and $\partial G'\cap E=\emptyset$. Pick out a finite point $z_0\in L\setminus f(\partial G')$, and choose $r>0$ such that $D(z_0,r)\cap f(\partial G')=\emptyset$. Then $v_{f|G'\setminus E}$ is finite and constant, say n', in $D(z_0,r)\cap\{z\in \boldsymbol{C}|\mathrm{Re}\, z<c\}$; similarly, there is a nonnegative integer n'' such that $v_{f|G'\setminus E}(z)=n''$ for $z\in D(z_0,r)\cap\{z\in \boldsymbol{C}|\mathrm{Re}\, z>c\}$. It is now readily seen that there are at most $n'+n''$ points p_i, $i=1,\ldots,k$, in $E\cap G'$ such that $z_0\in\mathrm{Cl}(f;p_i)$. Thus for each $p\in E\cap G'\setminus\{p_1,\ldots,p_k\}$ $\mathrm{Cl}(f;p)$ is a proper subset of L. By Corollary 1, f admits a meromorphic extension f^* to $G'\setminus\{p_1,\ldots,p_k\}$. But p_i, $i=1,\ldots,k$, being now isolated singularities, f^* can be taken as meromorphic all over G'. Since the arising of poles evidently contradicts the hypothesis of the corollary, the assertion is proved. □

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