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ITERATION OF EXPONENTIAL FUNCTIONS

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1. Introduction

For any entire or rational function f in the complex plane define the sequence of iterates (f^n), $n \in N$, by $f^0(z) \equiv z$, $f^n = f^{n-1} \circ f$, $n \geqq 1$. The Fatou—Julia set $F(f)$ is the complement of the maximal open set $C(f)$ in which (f^n) is a normal family. The Fatou—Julia theory of iteration [14, 15, 17] attempts to analyze the way in which $F(f)$ divides the plane and to consider the various possible limit functions for convergent subsequences of (f^n) in the components of $C(f)$.

The fixed points of f are of great importance in this study. The value z_1 is a fixed point of exact order (or period) p if $f^p(z_1) = z_1$, $f^k(z_1) \neq z_1$ for $k < p$. In this case the values z_1, $f(z_1) = z_2, \ldots, f(z_{p-1}) = z_p$ form a cycle of fixed points of order p, such that $f(z_p) = z_1$. By definition $(f^p)'(z_1)$ is called the multiplier of z_1 and one finds that all fixed points of a cycle have the same multiplier. If the multiplier of z_1 has modulus less than 1 (greater than 1) then z_1, and also the cycle $z_1, \ldots, z_p$, are called attractive (repulsive); in the attractive case each z_i belongs to a different component D_i of $C(f)$, such that $\bigcup_{i=1}^{p} D_i$ contains at least one singularity of the inverse f^{-1} of f (c.f. Section 2, property X).

We shall study the case when $f(z) = e^{az}$ and a is an arbitrary complex parameter. Some aspects of this study are very old. For example, Euler [13] considered the convergence of the infinite exponential

$$b^{b^{b^{\cdot^{\cdot^{\cdot}}}}},$$

which, if we put $b = e^a$, can be regarded as the convergence of the sequence

$$w_n = f^n(0), \quad f(z) = e^{az}, \quad n = 1, 2, \ldots. \tag{1}$$

Euler found the range of real b (or a) for which (1) converges. For complex a, apart from some exceptional cases, the sequence (1) converges for a in a domain D_1 bounded by a cardioid (Section 3, Theorem 1).

The study of the sequence (1) is no isolated curiosity, but is fundamental to the understanding of many aspects of the iteration. If we wish to know which limit func-

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tions can arise in components of $C(e^{az})$ then the set of singularities of the inverse functions f^{-n} plays an important part (Section 2, properties XI and XII), and these are just the values in (1).

As the parameter a moves outside D_1 the convergence of (1) shows an interesting complicated bifurcation. The plane contains for every positive integer p open sets $\hat{D}_p$ such that for a in $\hat{D}_p$ the function $f(z)=e^{az}$ has an attractive cycle of period p. For different p, q the sets $\hat{D}_p, \hat{D}_q$ are disjoint. Every component D_p of a $\hat{D}_p$ is unbounded, except for $D_1=\hat{D}_1$ which is the cardioid region mentioned above. There is a single component D_2; for $p>2$ there are infinitely many D_p. Any D_p is tangent to many D_{pk}, $k=2, 3, \ldots$. All D_p are simply-connected. The relations are indicated for small values of a and p in the figures. The existence and properties of D_p are discussed in Sections 4—7.

For a in any D_p the sequence $(f^n(0))$ splits into p periodic convergent subsequences $n=mp+j$, $0\leqq j<p$, $1\leqq m<\infty$, each convergent to one of the fixed points of an attractive cycle. For values of a in the same D_p the iteration of the functions e^{az} will show similar features.

Recently D. Sullivan [22, 23] has completed the analysis of Fatou and Julia, at least for rational functions, by proving the non-existence of wandering domains. A wandering domain for f is a component D of $C(f)$ such that $f^n(D)\cap f^m(D)=\emptyset$ for all $n>m\geqq 1$. Sullivan's proof does not apply to entire functions in general, which may indeed have wandering domains [7]. However, Sullivan's method can be adapted to show (see Section 8), Theorem 6:

For $a\neq 0$ the function e^{az} has no wandering domains.

This result is useful in simplifying the discussion of the possible limit functions which can occur. Some consequences are noted in Section 9. In particular the constant ∞ is never a limit in a component of $C(e^{az})$.

One may also ask when there are no limit functions, in the sense that $F(f)=C$. It has long been known that this can occur for rational functions and recently M. Misiurewicz [19] proved Fatou's conjecture that $F(e^z)=C$. It is interesting that many examples occur in the exponential class; in particular this is the case for all real $a>1/e$ and also for a set S of values which lie in the boundaries of the D_p and such that any arc of any ∂D_p contains a non-countable dense subset in S.

We also determine all cases when $C(e^{az})$ is connected and give some account of the iteration theory for the whole class e^{az}.

It may be noted that our work has many parallels with that of Douady and Hubbard [12], who examine the iteration of the family $g(z)=z^2+c$, where c is a complex parameter. The main cause of this similarity is that the classes z^2+c and e^{az} each have inverse functions with precisely one (finite) singularity and thus form the rational or transcendental classes which we may expect to have the simplest iteration theory.

There are of course many differences between the cases; for example the iterative behaviour of z^2+c is essentially the same for all large values of c.

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2. Results from iteration theory

We need a number of properties of the Fatou—Julia set defined in the introduction. Where not otherwise stated they are proved for entire functions in Fatou [15]. It is assumed throughout that f is a non-linear entire function.

I. *$C(f)$ is open. $F(f)$ is perfect and non-empty.*

II. *$C(f)$ and $F(f)$ are completely invariant under f in the sense that if $z\in C(f)$ then $f(z)\in C(f)$, and if, further, $f(w)=z$, then $w\in C(f)$.*

III. *For any positive integer p, $F(f)=F(f^p)$.*

IV. *An attractive fixed point of any order belongs to $C(f)$; a fixed point whose multiplier m satisfies $|m|>1$ or $m=$ a root of unity, belongs to $F(f)$.*

V. *In $[0,2\pi]$ there is a class K of "centrum numbers" such that for any f, if f has a fixed point z_1 of order p and multiplier $m=e^{i\theta}$, where $\theta\in K$, then $z_1\in C(f)$. Moreover in this case there is a function $S(z)$ analytic near z_1 with $S'(z_1)=1$, $S(z_1)=0$ such that $S\circ f^p\circ S^{-1}(z)=e^{i\theta}z$ near 0, and f^p is univalent in the component of $C(f)$ which contains z_1. The class K has measure 2π.*

C. L. Siegel [21] showed that K includes all θ such that θ/π is not a Liouville number. H. Rüssmann [20] showed that K/π also includes certain Liouville numbers.

VI. *The repulsive fixed points are dense in $F(f)$* [4].

There exists at most one value α such that the set of equations $f^n(z)=\alpha$, $n\in N$, has in all a finite set of solutions. If such an α exists it is called Fatou-exceptional and f has the form $f(z)=\alpha+(z-\alpha)^k e^{g(z)}$, where $k\geqq 0$ and g is entire.

VII. *Given any $z\in F(f)$ and any w different from the Fatou exceptional point if this exists, there is a sequence z_{n_k}, $n_k\in N$, such that $f^{n_k}(z_{n_k})=w$, $z_{n_k}\to z$, $n_k\to\infty$.*

VIII. *Given $z\in F(f)$, N an open neighbourhood of z, K any compact plane set which does not contain the Fatou exceptional point if there is one, then there exists n_0 such that for all $n>n_0$ we have $f^n(N)\supset K$.*

IX. *If in a component D of $C(f)$ some subsequence of (f^n) converges to a finite limit function, then D is simply-connected.*

X. *If α is an attractive fixed point (of order* 1*) of f, then the component D of $C(f)$ which contains α also contains a singularity of the inverse function f^{-1}. The sequence $f^n \to \alpha$ in D.*

If α has multiplier 1 *so that f has an expansion*

$$f(z) = \alpha + (z-\alpha) + a_{m+1}(z-\alpha)^{m+1} + \dots, \quad a_{m+1} \neq 0,$$

then $\alpha \in F(f)$ is on the boundary of m components of $C(f)$, in which $f^n \to \alpha$ and each component contains a singularity of f^{-1}.

If α has a multiplier which is a primitive q-th root of unity then α is on the boundary of one or more cycles of q-domains $D_1, \dots, D_q$, which are permuted cyclically by f and in which $f^n \to \alpha$; each such cycle contains a singularity of f^{-1}.

Similarly if $\alpha_1, \dots, \alpha_p$ are the points of an attractive cycle, then each α_i belongs to a different component D_i of $C(f)$ and at least one of the components contains a singularity of f^{-1}.

This was proved by Fatou [14] for rational functions and his proof applies also to the entire case.

XI. *Let S be the set of finite singularities of the inverse of the non-linear entire function f and let $E = \bigcup_{n=0}^{\infty} f^n(S)$. Set $L = \bar{E} \cup \{\infty\}$.*

If L has an empty interior and connected complement, then no sequence (f^{n_k}) has a non-constant limit function in any component of $C(f)$ [6].

XII. *If L is defined as in* XI, *then any constant limit function of an (f^{n_k}) in a component of $C(f)$ belongs to L* [6].

3. Simple convergence of $f^n(0)$. The domain D_1

Suppose that e^{az} has an attractive fixed point z_1 of order 1. Then $z_1 = e^{az_1}$. Putting $t = az_1$, we have $z_1 = e^t$ and $a = te^{-t}$; the multiplier of z_1 is $ae^{az_1} = az_1 = t$. Thus we have shown that

$$D_1 \overset{\text{def}}{=} \{a;\, e^{az} \text{ has an attractive fixed point of order } 1\}$$

may also be described by

$$D_1 = \{a;\, a = te^{-t} \text{ for some complex } t \text{ with } |t| < 1\}.$$

Thus D_1 is the interior of a cardioid. See Figure 1.

We may note that the case $a=0$ is included, corresponding to the constant function 1 and has multiplier 0. No other fixed point of any order can have multiplier 0.

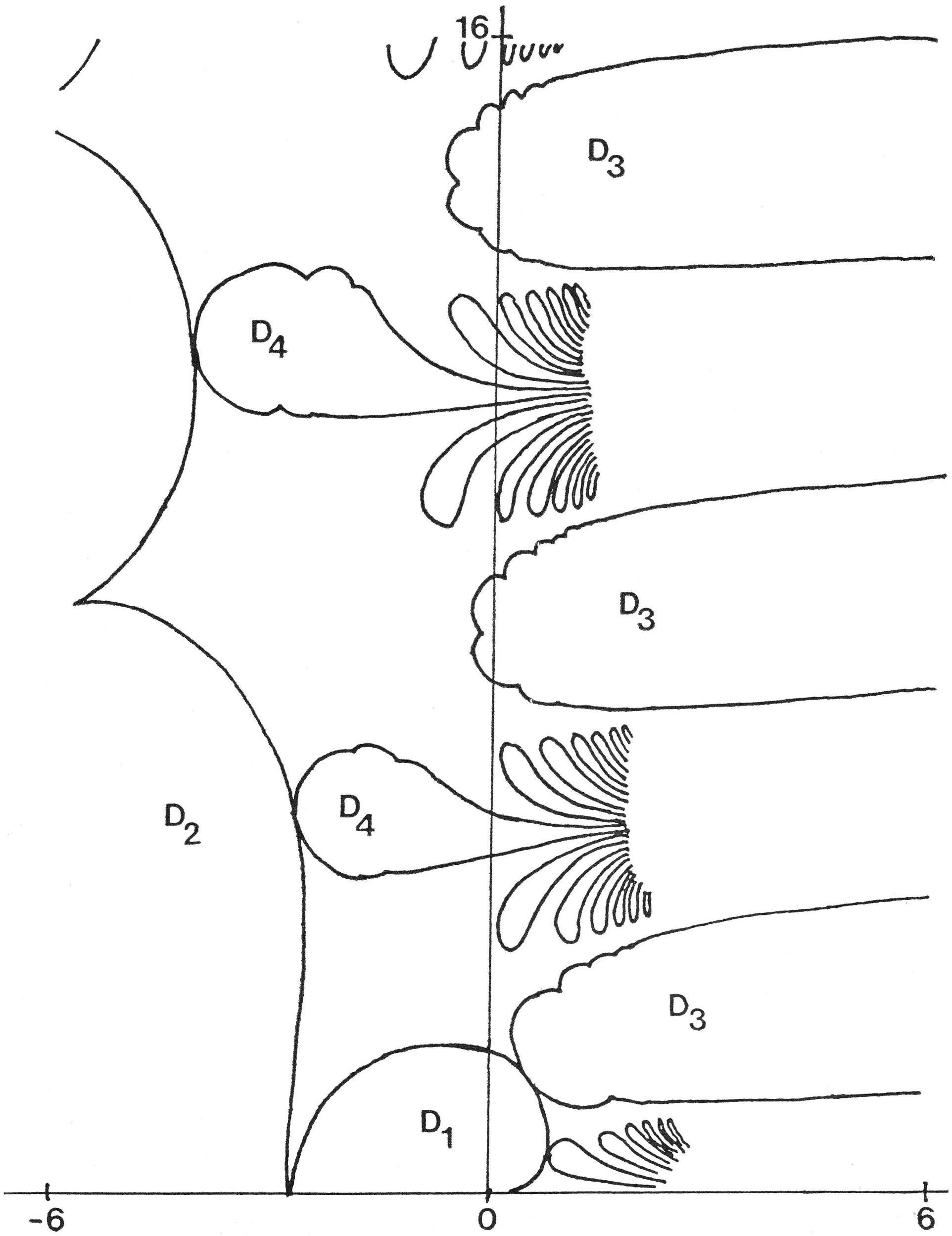

Figure 1

Convergence of (1) $w_n = f^n(0)$.

If w_n is constant for $n \geqq n_0$ we call the convergence terminating. This does indeed occur for a countable set of values of a, as described in detail in [9]. It will follow from Theorem 2 below that terminating convergence of w_n cannot occur for any a in the closure $\bar{D}_1$ of D. Using this remark we may state Theorems 1 and 3 of [9] in the slightly stronger form

Theorem 1. *If* $a = te^{-t}$, $|t| < 1$ *or* $t =$ *a root of unity then the sequence* (1) *converges to* e^t. *If* $t = e^{2\pi i\theta}$ *where* θ *is a centrum number, in particular if* θ *is a non-Liouville number, then* (1) *diverges.*

If (1) *converges then either* $a \in \bar{D}_1$ *or* a *is one of the countable set of values, all of which lie outside* $\bar{D}_1$, *which lead to terminating convergence.*

The centrum case is of measure 2π on the circle. It will follow from Theorem 7, Corollary 1, that (1) cannot satisfy $\lim w_n = \infty$ in the centrum case.

Remark. The convergence of (1) in the case $|t| < 1$ is an easy consequence of Section 2, property X and the fact; that t is the multiplier of the fixed point.

Next generalize the notion of terminating convergence: The sequence (1) has terminating convergence of period $q \geqq 1$ if there is some k such that $w_k = w_{k+q}$.

Theorem 2. *Suppose that the sequence* (1) *has the property of terminating convergence of period* $q \geqq 1$ *for some* $a \neq 0, 1$. *Then for every* $p \in N$ *every fixed point of order* p *of* f *is repulsive.*

Lemma 3.1. (See e.g. [6].) *The finite singularities of the inverses* $(f^n)^{-1}$ *of* f^n *are all transcendental and are* $0, w_1, \ldots, w_{n-1}$.

Proof of the theorem. Suppose that for some $q \geqq 1$, $k \geqq 1$ one has $w_k = w_{k+q}$. Then the different members of (1) are $w_1, \ldots, w_k, \ldots, w_{k+q-1}$ (all non-zero) of which $w_k, \ldots, w_{k+q-1}$ are fixed points of order q. We take $p \geqq 1$ and β such that $f^p(\beta) = \beta$ and have to show that $|(f^p)'(\beta)| > 1$. Note that $\beta \neq 0$.

We remark that in the special case when $a = 2l\pi i$, l a non-zero integer, we have terminating convergence with $q = 1$, $k = 1$, $w_1 = 1$. If also $p = 1$, $\beta = 1$ we have $|f'(\beta)| = 2|l|\pi > 1$ so the result holds in this special case by direct calculation. Consider any other case.

There is a disc D of centre β and positive radius d such that D contains neither 0 nor any member of the sequence (1), except that in the special case when $\beta = w_{k+i}$, $0 \leqq i < q$ the centre β belongs to (1). We have the expansion

$$f^p(z) = \beta + A(z-\beta) + B(z-\beta)^s + \ldots, \tag{2}$$

$$AB \neq 0, \quad s \geqq 2.$$

There is thus a branch $z=F(w)$ of the inverse function of $w=f^p(z)$ which is analytic in a neighbourhood of β and satisfies

(3)
$$F(w)=\beta+A^{-1}(w-\beta)+b(w-\beta)^s+\dots,$$
$$b=-B/A^{s+1}.$$

For any positive integer n the n-th iterate $F^n(w)$ is given by

(4)
$$F^n(w)=\beta+A^{-n}(w-\beta)+\dots$$

which is an analytic branch of the inverse of $w=f^{np}(z)$ near β. Since the only possible singularities of F^n are 0 and members of (1) it follows that F^n is analytic in D for all n.

The next stage is to show that (F^n) is a normal family in D. Suppose that $F^n(w)=0$ for some w in D. Then $w=f^{np}(0)=w_{np}\in D$ so that w_{np} can only be β. But $F^n(\beta)=\beta\neq 0$ and so F^n never takes the value 0 in D. Suppose next that $F^n(w)=1$ for w in D so that $w=w_{np+1}\in D$, and again $w=\beta$. If $q=1$ then β is a fixed point of order 1 which means that $\beta=e^{a\beta}$ and also $1=F^n(\beta)=\beta=e^{a\beta}$. Thence $\beta=1, a=2l\pi i$; but this case has been excluded by our preliminary discussion. Thus if $q=1$ the functions F^n omit 0, 1 in D and form a normal family.

If $q>1$ we again have $F^n(w)\neq 1$ in D except in the case when $w=\beta=1$. But in this case we have $\beta=w_1$ and further $f(\beta)=w_2$ is a different member of (1). If $F^m(w)=w_2$, $w\in D$, then $w=f^{mp}(w_2)=f^{mp+2}(0)=w_{mp+2}$ so that $w=\beta$. But $F^m(\beta)=\beta\neq w_2$. Thus in all cases (F^n) omits two values $\{0,1\}$ or $\{0,f(\beta)\}$ in D and forms a normal family in D.

It remains to show that $|A|>1$ holds in (2), by eliminating the other possibilities. If $|A|<1$ the non-normality of (4) is obvious. If A is a root of unity, say $A^q=1$, then (4) gives

$$F^q(w)=\beta+(w-\beta)+\gamma(w-\beta)^t+\dots,\quad t>1,\ \gamma\neq 0.$$

Then for all positive integers n

$$F^{nq}(w)=\beta+(w-\beta)+\gamma n(w-\beta)^t+\dots,$$

so that $F^{nq}(\beta)=\beta$ but $(d^t/dw^t)F^{nq}=t!\,\gamma n$ at $w=\beta$, which contradicts the normality of F^n in D.

We therefore suppose that $|A|=1$ but that A is not a root of unity. We select a sequence (n_k) of integers such that $n_k\to\infty$, $A^{n_k}\to 1$. By (4) and normality there is a subsequence, which we may assume to be n_k, such that F^{n_k} converges locally uniformly in D to a function

(5)
$$\psi(w)=\beta+(w-\beta)+\textstyle\sum_{n=2}^{\infty}a_n(w-\beta)^n.$$

In a neighbourhood of β, which we may take to be independent of n_k, we have $F(F^{n_k})=F^{n_k}(F)$, and hence $F(\psi)=\psi(F)$.

Suppose that l is the smallest value of n such that a_n is non-zero in (5). Equating coefficients of $(w-\beta)^l$ in the expansion of $F(\psi)=\psi(F)$ shows that $A^{-l}=A^{-1}$, which is impossible, since A is not a root of unity. Thus all $a_n=0$ and $\psi(w)\equiv w$.

If Δ is the disc $|w-\beta|\leqq d/2$ inside D then for some fixed $d'>0$ and all large n_k the set $F^{n_k}(\Delta)$ contains the disc Δ': $|w-\beta|<d'$. Hence $f^{pn_k}(\Delta')\subset\Delta$. Thus the functions f^{pn_k} are normal in Δ' and $f^{pn_k}(\beta)=\beta$, $(f^{pn_k})'(\beta)=A^{n_k}\to 1$. We may select a subsequence of the f^{pn_k} locally uniformly convergent in Δ' to a non-constant limit function φ such that $\varphi(\beta)=\beta$, $\varphi'(\beta)=1$. From Section 2, property VIII it follows that $\Delta'\subset C(f)$. Now the set L of Section 2, property XI is by our assumptions and Lemma 3.1 a finite set, so that by the property XI there are no non-constant limit functions in $C(f)$, which contradicts the result just proved about φ. We conclude that A cannot have the form assumed and the proof that $|A|>1$ holds is complete.

4. Domains D_p tangent to D_1

Theorem 3. *For each integer $p\geqq 2$ and for each primitive p-th root η of unity there is a domain D_p which lies outside D_1 but is tangent to D_1 at $a_0=\eta e^{-\eta}$, such that for a in D_p the function e^{az} has an attractive fixed point of order p.*

Proof. If $a_0=\eta e^{-\eta}$ then $\xi=e^{\eta}$ is a fixed point of f with multiplier $f'(\xi)=\eta$. A calculation (see e.g. [3, Theorem 2]) shows that

$$f^p(z) = \xi+(z-\xi)+\sum_{n=kp+1}^{\infty} A_n(z-\xi)^n, \quad A_{kp+1}\neq 0.$$

$C(f)$ contains kp domains each with ξ as a boundary point and by Section 2, property X each contains a singularity of f^{-p}, which by Lemma 3.1 has at most p singularities. Thus $k=1$.

It is convenient to work with the parameter t such that $a=te^{-t}$, rather than with a itself. We have to find a region D' in $|t|>1$ and tangent to the unit circumference at $t=\eta$, such that, for $t\in D'$, $a=te^{-t}$, e^{az} has an attractive fixed point of order p.

Now for any t near η we have

$$f(z) = \xi+t(z-\xi)+\sum_2^{\infty} a_n(t)(z-\xi)^n$$

$$f^p(z) = \xi+t^p(z-\xi)+\sum_2^{\infty} A_n(t)(z-\xi)^n,$$

where $\xi=e^t$ and $A_2(\eta)=\ldots=A_p(\eta)=0$, $A_{p+1}(\eta)\neq 0$. Writing $F(t,z)=f^p(z)$ we find that when $t=\eta$, $z=\xi=e^\eta$ we have

$$\frac{\partial^2 F}{\partial t \partial z} = p\eta^{p-1} \neq 0, \tag{6}$$

$$\frac{\partial^j F}{\partial z^j}\begin{cases} = 1, & j=1, \\ = 0, & j=2, \dots, p, \\ \neq 0, & j=p+1, \end{cases} \tag{7}$$

and

$$\frac{\partial F}{\partial t} = 0. \tag{8}$$

We examine those fixed points of f which "coincide" as a multiple $(p+1)$-fold solution $z=e^\eta$ of $f^p(z)-z=0$ when $a=\eta e^{-\eta}$, that is we consider the solutions of $F(t, z)-z=0$ for z as a function of t near $t=\eta$, $z=e^\eta$. Putting $z-e^\eta=Z$, $t-\eta=T$ we have

$$F(t, z)-z = \sum_{m,n=0}^{\infty} A_{m,n} Z^m T^n = 0,$$

where

$$A_{0,0} = A_{1,0} = \dots = A_{p,0} = 0,\ A_{p+1,0} \neq 0$$
$$A_{1,1} = p\eta^{p-1} \neq 0, \quad A_{1,0} = 0.$$

Thus the Newton polygon for the problem has 2 sides, one of which joins (1, 1) with $(p+1, 0)$, with slope $-1/p$. There is an expansion

$$Z = c_1 T^{1/p} + c_2 T^{2/p} + \dots, \quad c_1 \neq 0, \tag{9}$$

which represents p of the desired solutions near $T=0$. The coefficient c_1 is determined from

$$A_{1,1} c_1 + A_{p+1,0} c_1^{p+1} = 0. \tag{10}$$

For the p fixed points given by (9) the multiplier is

$$\frac{\partial}{\partial z} f^p(z) = 1 + \frac{\partial F}{\partial z} = 1 + (p+1) A_{p+1,0} Z^p + \dots + A_{1,1} T + \dots$$
$$= 1 + \{(p+1) A_{p+1,0} c_1^p + A_{1,1}\} T + \text{higher powers of } T,$$

after substituting (9). Using (10) we see that

$$(p+1) A_{p+1,0} c_1^p + A_{1,1} = -pA_{1,1} = -p^2 \eta^{p-1}.$$

Thus

$$\frac{\partial f^p(z)}{\partial z} = 1 - \frac{p^2(t-\eta)}{\eta} + \dots$$

and there is a region D' of the t-plane outside $|t|>1$ and bounded by a curve which is tangent to the circle $|t|=1$ at $t=\eta$ such that for $t \in D'$ all the p fixed points determined by (9) have multiplier less than one in modulus. These fixed points must then form a single attractive cycle. The theorem is proved.

At $t=\eta$ there are $(p+1)$ coincident fixed points e^η. The remaining one is of course the fixed point which has been attractive for $|t|<1$ and becomes repulsive as t passes outside the unit disc.

5. General properties of D_p

Theorem 3 shows that for each p the set $\{a; f(z)=e^{az}$ has an attractive cycle of order $p\}\neq\emptyset$. Henceforth D_p shall denote a component of this set, whether or not it is tangent to D_1.

It follows from Section 2, property X that if $a\in D_p$, for some $p\geqq 1$, then for $k=1, 2, \ldots, p$ each of the sequences $f^{np+k}(0)$ is convergent, as $n\to\infty$. In particular

$$\bar{D}_p \cap D_q = \emptyset, \quad \text{if} \quad p \neq q.$$

Lemma 5.1. *In D_p the sequence $(f^{np}(0))$, $n\in N$, converges locally uniformly to one of the attractive fixed points $z(a)$, which is analytic in a.*

Proof. Take a fixed $a_0\in D_p$ and denote $e^{a_0 z}$ by $f_0(z)$. Since 0 is in the component of $C(f_0)$ which contains one of the points $z_0=z(a_0)$ of the attractive p-cycle we have by Section 2, property X $\lim f_0^{np}(0)=z_0$ as $n\to\infty$. Put $\sigma=|(f_0^p)'(z_0)|<1$. Take a fixed ϱ such that $\sigma<\varrho<1$. Choose a disc $\Delta=\{z;\ |z-z_0|<d\}$, $d>0$, such that $|(f_0^p)'(\Delta)|<\varrho$, so that $f_0^p(\Delta)\subset\Delta'=\{z;\ |z-z_0|<\varrho d\}$. There is a positive integer q such that $f_0^{pq}(0)\in\Delta'$. By continuity there is a neighbourhood U of a_0 such that

(i) $U\subset D_p$,

(ii) $f^{pq}(0)\in\Delta$, $a\in U$,

(iii) $f^p(\Delta)\subset\Delta$, $a\in U$.

Thus for $a\in U$ and any $n\in\mathbf{N}$ we have $f^{(n+q)p}(0)\in\Delta$ by (ii) and (iii). Hence $f^{(n+q)p}(0)$ is a normal family of analytic functions of a in U, convergent pointwise to some value $z(a)$ of the attractive p-cycle for a by (i). The convergence must then be locally uniform.

Since $f^n(0)$ is an entire function of a we have:

Corollary. *D_p is simply-connected.*

Lemma 5.2. *Suppose that the fixed point $z(a)$ of order p of e^{az} remains analytic on the open arc γ in the a-plane ending at $a_0\neq 0$. Then provided $(f^p)'(z(a))$ remains bounded on γ one cannot have $z(a)\to\infty$ as $a\to a_0$ on γ.*

Proof. If $z(a)=f^p(z(a))=f^{p-k}(f^k(z(a)))\to\infty$ as $a\to a_0$ then $f^k(z(a))\to\infty$, $k=1, 2, \ldots, p$. But then $(f^p)'(z(a))=a^p f^p(z(a))f^{p-1}(z(a))\ldots f(z(a))\to\infty$, against assumption.

From Lemma 5.2 it follows that not only is the $z(a)$ of Lemma 5.1 analytic in D_p, but further that as a approaches a finite boundary point b of D_p the relation $\limsup |(f^p)'(z(a))| \leqq 1$ implies $z(a)$ does not have a transcendental singularity at b. In fact $z(a)$ remains analytic at b unless $(\partial/\partial z)(f^p(z)-z)=(f^p)'(z)-1$ becomes zero, in which case $z(a)$ may have a branch point.

Define the function $M(a)=M(D_p, a)$ given by

$$M(a) = (f^p)'(z(a)), \tag{11}$$

which is thus analytic in D_p and which remains analytic at the finite boundary points of D_p, except for at most algebraic singularities where $M(a)=1$. The boundary consists of arcs of level curves $|M(a)|=1$.

Lemma 5.3. *For $p>1$ any D_p is unbounded.*

We note that $M(a)$ in (11) never takes the value 0 for any $a \in D_p$. If D_p is bounded it is a compact simply-connected domain bounded by arcs of $|M(a)|=1$. Now $\partial M(D_p) \subset M(\partial D_p) \subset \{M;\ |M|=1\}$. This implies that $0 \in M(D_p)$, against assumption, and the lemma is proved.

Finally we note that in general the boundary curves of D_p have cusps at points where $M(a)=1$. We carry out the calculations for D_2, remarking that it will be shown in the next section that there is only one D_2, namely the one found in Section 4, which is tangent to D_1 at $a=-e$.

Lemma 5.4. *Suppose that $a_0 \in \partial D_2$, $a_0 \neq -e$, $M(a_0)=1$. Then ∂D_2 has a cusp at $a=a_0$.*

Proof. Let $z(a)$ be one of the two attractive fixed points of f^2. Then $f(z(a))=z_1(a)$ is the other and as $a \to a_0$ we have $z(a) \to \xi$ (say) and $z_1(a) \to \eta = f(\xi)$. Further $(f^2)'(\xi)=1$. If for $a=a_0$ we have $\xi=\eta$, then $f'(\xi)=\pm 1$. In either case then a_0 is on the boundary of D_1. The case $f'(\xi)=1$ corresponds to $a_0=e^{-1}$, $t=1$ in the notation of Section 3. Two first order fixed points of f coincide at $\xi=\eta=e$ but there is no cycle of period 2 near these. The case $f'(\xi)=-1$ corresponds to $t=-1$, $a_0=-e$, which has been excluded by hypothesis.

Thus in the above argument we have $\xi \neq \eta$ and also $a_0 \xi \neq -1$, since $a_0 \xi = -1$ implies $a_0 \eta = -1$ and $\xi = \eta$.

At $a=a_0$ we have, near $z=\xi$,

$$f^2(z) = \xi + (z-\xi) + a_2(z-\xi)^2 + \dots,$$

and a similar expansion near $z=\eta$. Since $(f^2)^{-1}$ has only two singularities, there is by Section 2, property X just one component of $C(f)$ near ξ in which $f^{2n} \to \xi$ and containing one singularity of f^{-2}, and another similar component near η. This implies that $a_2 \neq 0$.

If $F(a,z)=f^2(z)-z$ we have at $a=a_0$, $z=\xi$ that

$$F=0,\quad \frac{\partial F}{\partial z}=0,\quad \frac{\partial^2 F}{\partial z^2}=2a_2,\quad \frac{\partial F}{\partial a}=\xi\eta(1+a_0\xi)=\mu\neq 0.$$

Putting $z-\xi=Z$, $a-a_0=A$ this gives an expansion

$$0=F(a,z)=\mu A+a_2Z^2+bAZ+\dots,$$

which has a local solution near $A=0$ of

$$Z=(-\mu/a_2)^{1/2}A^{1/2}+\sum_{n=2}^{\infty}c_nA^{n/2}.$$

This leads to

$$(f^2)'(z(a))=1+\frac{\partial F}{\partial Z}=1+\lambda(a-a_0)^{1/2}+\dots$$

where $\lambda=2a_2(-\mu/a_2)^{1/2}$. Thus the boundary of D_2 near a_0 has the same form as the level set $|1+\lambda(a-a_0)^{1/2}|=1$, that is a curve with a cusp at a_0 and D_2 fills out the angle of 2π bounded on one side by this cusp.

Remark. A calculation similar to that of Theorem 3 shows that at a point a of ∂D_p where the $M(a)$ of (11) is a primitive q-th root of unity, $q>1$, there is a D_{pq} tangent to D_p.

The situation is illustrated for $p\leqq 6$ by Figure 1. Note for instance the regions D_2, D_3, D_4, D_5 tangent to D_1, the D_4 tangent to D_2 and the D_6 tangent to D_2 or to the D_3. Various cusps on ∂D_2 and the ∂D_3 also show up clearly.

These figures were obtained by testing for the various cases with a computer, and not all parts of the regions D_p, $p=4, 5, 6$, were plotted. In the missing parts, which are in any case too slender to plot accurately, the terms $f^n(0)$ become too large for the computer to handle.

6. The single domain D_2

Theorem 4. *There is a single D_2 which is an unbounded domain tangent to D_1 at $a=-e$ and lying in the left half-plane. The boundary of D_1 is a single curve which for large $a=X+iY$ has the asymptotic form $Y=\pm e^{-X/2}(1+o(1))$, as $-X\to+\infty$*

Proof. Consider a fixed D_2 and denote by $\xi=z_1(a)$, $\eta=z_2(a)$ the attractive fixed points of $f(z)=e^{az}$ in D_2. Write $M(a)=(f^2)'(\xi)$ as in (11). Each boundary component is an unbounded level curve $|M(a)|=1$ and two of these curves cannot intersect (see Lemmas 5.1 (Corollary) and 5.3).

Writing $s=a\xi$, $t=a\eta$ gives $se^s=te^t$; $st=M(a)$. For any a in D_2 we have $|st|<1$ so that one of s, t, say t, satisfies $|t|<1$. Since we^w is univalent in $|w|<1$ we then have $|s|>1$. By continuity of η, t in D_2 one has $|t|<1$, $|s|>1$ for all a in D_2.

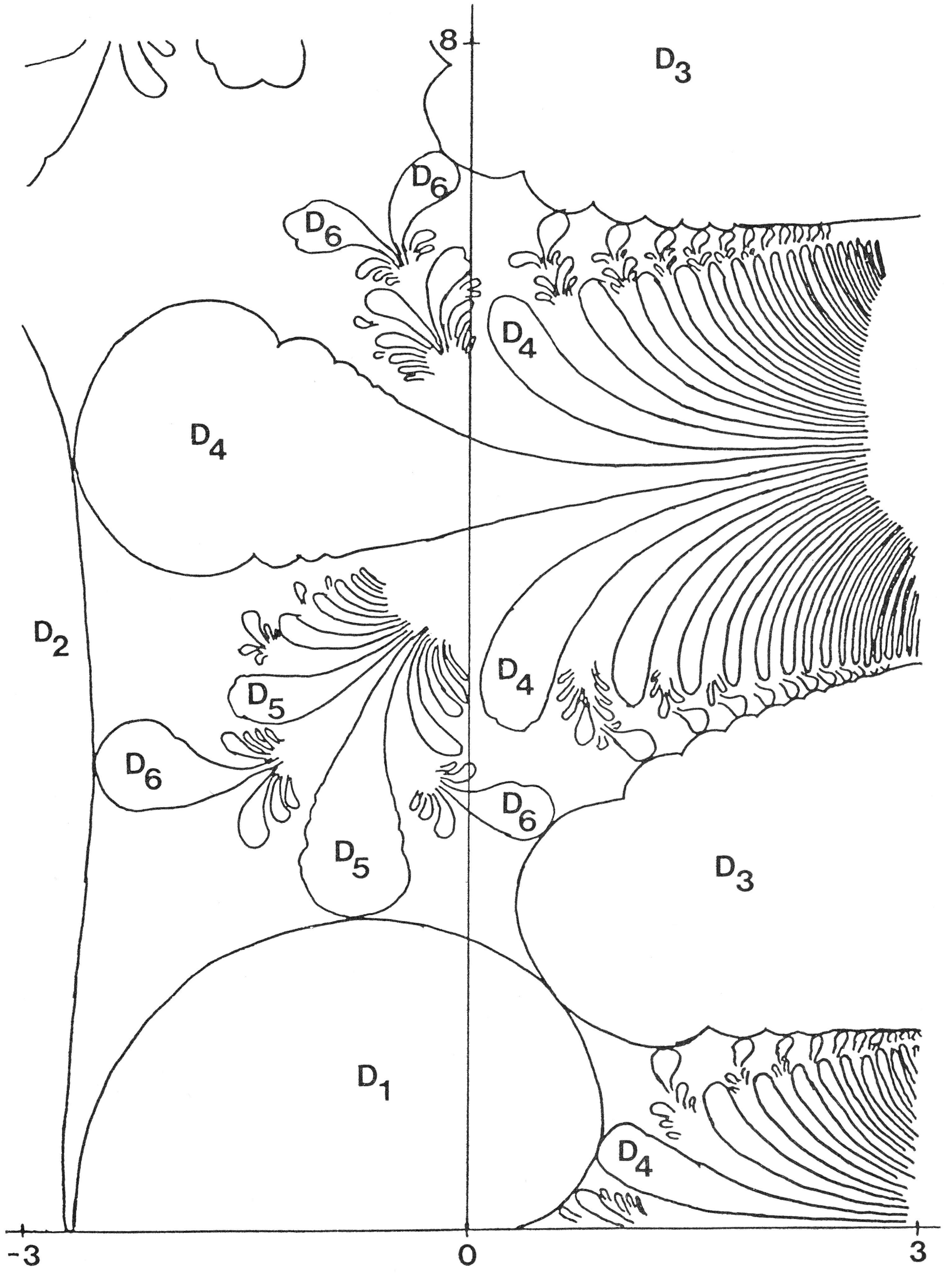

Figure 2

Now observe that for a in $\bar{D}_2$ we have $|\xi|=|e^t|\geqq e^{-1}$ and thus $1\geqq|st|=|a^2\xi\eta|\geqq|\eta a^2|/e$, $|\eta|<e/|a|^2$ and $|t|=|a\eta|<e/|a|$. For large a in D_2 then

$$|se^s| = |te^t| < \frac{1}{e}.$$

Now from the series inversion of $te^t=w$ (see e.g. [16, p. 141]) one obtains

$$e^{-t} = -\sum_0^\infty (-1)^{n-1}\frac{(1+n)^{n-1}}{n!}w^n, \quad |w| < \frac{1}{e}.$$

Since $te^t=se^s$ and $a=se^{-t}$ we obtain

$$a = \sum_0^\infty (-1)^n \frac{(n+1)^{n-1}}{n!} s(se^s)^n, \quad |se^s| < \frac{1}{e},$$

and

$$st = s(te^t)e^{-t} = (se^s)a = \sum_0^\infty (-1)^n \frac{(n+1)^{n-1}}{n!} s(se^s)^{n+1}, \quad |se^s| < \frac{1}{e}. \tag{12}$$

Notice that the map $a\to s\to a=se^{-t}$ is one-to-one and conformal between D_2 and a region E of the s-plane which lies in the set E_1, where $|se^s|<1$, $|s|>1$, which is part of the left half-plane. We may define $st(a)=\varphi(s)$, s in E. For all large a in $\bar{D}_2$ we have $|se^s|<1/e$ and $\varphi(s)$ is given by (12). On the boundary of E one has $|\varphi(s)|=1$.

By (12) $|\varphi(s)/s^2e^s|>1/2$ if $|se^s|=\delta$ where δ is some constant such that $0<\delta<1/e$. Now the unbounded component E_2 of $|se^s|<\delta$ belongs to E_1. Moreover (12) gives an analytic extension of φ to E_2 which contains all large points of E. The boundary of E_2 is a simple arc $s(\tau)=x(\tau)+iy(\tau)$, $-\infty<\tau<\infty$, such that $y(\tau)\to\pm\infty$ as $\tau\to\pm\infty$. For large $y(\tau)$ we have $|\varphi(s(\tau))|>(\delta/2)|s(\tau)|>1$ while $\varphi(s)\to 0$ as $s=x+iy$, y constant, $x\to-\infty$, since $\varphi=O(s^2e^s)$. Thus for any large τ there is a point $x_1(\tau)+iy(\tau)$ in E_2 which lies on a curve $|\varphi|=1$. Thus there is a level curve $|\varphi|=1$ on which $\operatorname{Im} s\to+\infty$ and one on which $\operatorname{Im} s\to-\infty$. Further $\varphi(s)\to 0$ as $\operatorname{Re} s\to-\infty$.

Since $\varphi(s)=O(|s|)$ as $s\to\infty$ in E_2 the Phragmèn—Lindelöf principle implies that if $|\varphi(s)|\leqq K$ on a simple curve Γ in E_2 which approaches ∞ at both its ends, then $|\varphi(s)|\leqq K$ holds in the component of the complement of Γ which lies in E_2. If E_2 contains two distinct level curves $|\varphi|=1$ then there is a region $\{s;\ |\varphi(s)|>1\}$ which is necessarily unbounded, which lies in E_2 for large s and in which $|\varphi(s)|\to\infty$ as $s\to\infty$. This is impossible by the preceding remarks. Thus there is a unique level curve γ, on which $|\varphi(s)|=1$, in $\bar{E}$, whose ends have been described above. E is the domain bounded on the right by γ. The conformal mapping back to D gives finite a for finite s so that the image of γ is a single boundary component. Thus the uniqueness of γ shows that there is a unique D_2 whose boundary is a single curve.

On the boundary of E we have from (12)

$$|\varphi(s)| = |s^2e^s - s^3e^{2s}\ldots| = 1.$$

As $s\to\infty$ on ∂E, $se^s\to 0$ so that we have $|s^2e^s|(1+o(1))=1$, which for large $s=X+iY$ gives $X^2+Y^2=e^{-X}(1+o(1))$. As $s\to\infty$ we thus have $X\to-\infty$, $X^2e^X\to 0$ and so $Y^2=e^{-X}(1+o(1))$. The map between a and s satisfies $a=s-s(se^s)+\ldots\sim s$ as $s\to\infty$ on ∂E so that the equation of ∂D_2 has a similar form.

7. D_p, $p>2$

The distribution of the components D_p, $p>2$, is extremely complicated and we shall be able to give only a partial description.

Denote by a strip a region which is bounded by a single Jordan arc receding to ∞ in both directions and lying entirely in a set of the form $\{x+iy;\ \alpha<y<\beta,\ x>\delta\}$ where α,β,δ are real.

Any family of disjoint strips has a natural ordering, one strip being "less than" another if it lies below the other in some right half-plane.

The figures suggest that the D_p have the following structure.

(a) Each D_p, $p>2$, is a strip.

(b) For each $p>2$ the components D_p form themselves into families (which we call p-families) in such a way that there is an order-preserving bijection between each p-family and the integers. Moreover, between each adjacent pair of components in a given p-family there is to be found a single $(p+1)$-family.

Some confirmation of these properties is given by the following result.

Theorem 5. *There exists a family of strips Δ_p, $p>2$, which form p-families as described by* (b) *above, such that the restriction of each strip Δ_p to some right half-plane lies in a single component D_p.*

We first give a criterion which guarantees that the complex number a lies in one of the components D_p, $p>2$.

Lemma 7.1. *Suppose that $|a|>1/e$. Set $f(z)=e^{az}$ and $g(z)=e^{|a|z}$; suppose that, for some $p>2$,*

$$e|f^p(0)|<\left(|a|\,g^{p-1}\left(\frac{1}{|a|}\right)\right)^{-p}.$$

Then f^p has an attractive fixed point.

Proof. Let $0<\varrho\leq 1/|a|$. Then, for $|z|\leq\varrho$,

$$\left|\frac{f^p(z)}{f^p(0)}\right|\leq\exp\left|\int_0^z\frac{(f^p)'(\xi)}{f^p(\xi)}\,d\xi\right|\leq\exp\left(\varrho\max_{|\xi|\leq 1/|a|}\left|\frac{(f^p)'(\xi)}{f^p(\xi)}\right|\right)$$

$$\leq\exp\left(\varrho|a|^p\max_{|\xi|\leq 1/|a|}|f^{p-1}(\xi)\ldots f(\xi)|\right)\leq\exp\left(\varrho(|a|\,g^{p-1}(1/|a|))^p\right)$$

since $\max_{|\xi|\leqq 1/|a|} |f^j(\xi)| \leqq g^j(1/|a|)$, and $|a|>1/e$ implies $g(x)>x$, $g^j(1/|a|)\leqq g^{p-1}(1/|a|)$ for $j\leqq p-1$. Choosing ϱ, as we may, so that

$$e|f^p(0)| < \varrho < \left(|a|\, g^{p-1}(1/|a|)\right)^{-p},$$

we deduce that, for $|z|\leqq \varrho$,

$$|f^p(z)| < e|f^p(0)| < \varrho.$$

Hence f^p maps $\{|z|\leqq \varrho\}$ into itself properly, and so has an attractive fixed point.

To apply the lemma set $a=x+iy$ and, for $k=1, 2, \dots, p$, define $\lambda_k=\lambda_k(a)$ and $\theta_k=\theta_k(a)$ by $\lambda_1=\theta_1=0$ and

$$\lambda_{k+1} = e^{\lambda_k}(x\cos\theta_k - y\sin\theta_k), \quad k = 1, 2, \dots, p-1,$$
$$\theta_{k+1} = e^{\lambda_k}(x\sin\theta_k + y\cos\theta_k), \quad k = 1, 2, \dots, p-1.$$

Thus $f^k(0)=e^{\lambda_k+i\theta_k}$, $k=1, \dots, p$.

Next put, for $p>2$,

$$\Omega_p = \left\{a = x+iy;\ x > 4|y|,\ \cos\theta_k(a) > \tfrac{1}{2},\ k = 1, 2, \dots, p-2,\ \cos\theta_{p-1}(a) < -\tfrac{1}{2}\right\}.$$

For a in Ω_p we have $|\tan\theta_k|<\sqrt{3}$, $k=1, 2, \dots, p-1$, and so

$$x\cos\theta_k - y\sin\theta_k > \tfrac{1}{2}x\cos\theta_k > \tfrac{1}{4}x, \quad k = 1, 2, \dots, p-2,$$

and

$$x\cos\theta_{p-1} - y\sin\theta_{p-1} < \tfrac{1}{2}x\cos\theta_{p-1} < -\tfrac{1}{4}x.$$

Hence

$$\lambda_{k+1} > \tfrac{1}{4}xe^{\lambda_k}, \quad k = 1, 2, \dots, p-2,$$

and

$$\lambda_p < -\tfrac{1}{4}xe^{\lambda_{p-1}}.$$

Because $\lambda_2=x$ we obtain

$$|f^p(0)| = \exp(\lambda_p) < \exp\left(-\tfrac{1}{4}xe^{1/4\,xe^{\cdot^{\cdot^{1/4\,xe^x}}}}\right),$$

where the expression on the right contains the term x exactly $(p-1)$ times. Thus for $x>4$ we have

$$|f^p(0)| < 1/e^{p-1}(x), \quad \text{where} \quad e(x) = \exp x.$$

We wish to show $f^p(0)$ satisfies the inequality of Lemma 7.1. Since for all sufficiently large a in Ω_p we have $x>|a|/2$ and $g^{p-1}(1/|a|)=g^{p-2}(e)$ the desired result will follow if we can show that

$$e^{p-1}\left(\frac{|a|}{2}\right) > e\left(|a|\, g^{p-2}(e)\right)^p$$

holds for all sufficiently large a. This is easily done by induction. Thus we have shown that there is a constant K_p such that all points of

$$\Omega_p \cap \{\operatorname{Re} a > K_p\}$$

lie in components D_p.

We now show that the sets Ω_p, $p>2$, have the kind of structure described in (b) above. First put

$$A=\left\{a;\ \frac{2\pi}{3}<\arg a<\frac{4\pi}{3}\right\},$$

$$B=\left\{a;\ -\frac{\pi}{3}<\arg a<\frac{\pi}{3}\right\}$$

and

$$C=\{a=x+iy;\ x>4|y|\}.$$

Thus

$$\Omega_p=\{a;\ a\in C,\quad f^k(0)\in B,\quad k=2,\dots,p-2,\quad f^{p-1}(0)\in A\}.$$

The set

$$\Omega_3=\{a\in C;\ e^a\in A\}$$

is evidently a sequence of strips, which we label S_n, $n\in \boldsymbol{Z}$, by writing S_n for the component of Ω_3 such that $2n\pi<y=\theta_2(a)<(2n+2)\pi$. These form the required family of strips referred to as Δ_3 in Theorem 5.

Between the S_n are the components of

$$\left\{a=x+iy;\ x>4|y|,\ \cos\theta_2(a)>\tfrac{1}{2}\right\}=\{a\in C;\ e^a\in B\}.$$

These are also strips which we label T_n, $n\in\boldsymbol{Z}$, one between each adjacent pair of S_n, S_{n+1}.

Now apply the following lemma repeatedly.

Lemma 7.2. *Suppose that* $T\subset C$ *is a strip and that* φ *maps* T *conformally onto* $B\cap\{|z|\geqq R\}$ *for some* $R>0$. *If* $\psi(a)=a\varphi(a)$, $a\in\overline{T}$, *then* T *contains a strip* T' *such that* ψ *maps* T' *conformally onto* $C\cap\{|x|\geqq R'\}$ *for some* $R'>0$.

Proof. Since

$$|\arg a|\leqq\tan^{-1}\left(\tfrac{1}{4}\right)<\pi/8,\quad a\in\overline{C}-\{0\}$$

and

$$|\arg a|=\pi/3,\quad a\in\partial B-\{0\},$$

the boundary of $\psi(T)$ does not intersect $C\cap\{|z|\geqq R'\}$ when R' is large enough. Therefore this set is covered exactly once by ψ and we can take

$$T'=\psi^{-1}(C\cap\{|z|>R'\}).$$

Applying Lemma 7.2 to each of the strips T_n, $n\in\boldsymbol{Z}$, defined above we obtain a strip T_n', lying in T_n, which is mapped by $\psi(a)=ae^a$ conformally onto $C\cap\{|z|\geqq R_n\}$ for some $R_n>0$. We can then define, for $m,n\in\boldsymbol{Z}$,

$$S_{nm}=\left\{a\in T_n';\ ae^a\in C\cap\{|z|\geqq R_n\}\cap S_m\right\}$$

and

$$T_{nm}=\left\{a\in T_n';\ ae^a\in C\cap\{|z|\geqq R_n\}\cap T_m\right\}.$$

Evidently, for $a\in S_{nm}$ we have $a\in C$, $e^a\in B$ and $e^{ae^a}\in A$. Thus the strips S_{nm} form themselves into the 4-families of strips referred to as Δ_4. On the other hand for

$a\in T_{nm}$ we have $a\in C$, $f^k(0)\in B$, $k=2, 3$, and by another application of Lemma 7.2 we can find 5-families Δ_5 inside the strips T_{nm}. In this way the proof of Theorem 5 may be completed by induction.

We can push this approach a little further and show for example that there are components D_3 having width π at ∞ and components D_4 asymptotic to the lines $\operatorname{Im} a=\pm 2k\pi$ (from below in the cases in the upper half-plane). Also it can be shown that the lines $\operatorname{Im} a=0$, $\pm 2\pi$, $\pm 4\pi$ meet no component D_3, but we omit the details.

The figures suggest that the union of the components D_p, $p\geqq 1$, is dense in the complex plane, but we have not been able to prove this.

8. Wandering domains

Lemma 8.1. *If* $f(z)=e^{az}$, $a\neq 0$, *then any component of* $C(f)$ *is simply-connected.*

Proof. Consider a component $D\neq\emptyset$ of $C(f)$. By Section 2, property IX we need consider only the case when $f^n\to\infty$ in D. Since all f^n omit the value 0, the reciprocals $1/f^n$ are entire and converge to 0 uniformly in D. It then follows that D is simplyconnected.

Definition 8.2. *A component* U *of* $C(f)$ *is a wandering domain of* f *if* $f^m(U)\cap f^n(U)=\emptyset$ *for all non-negative integers* m, n *such that* $m\neq n$. (Here $f^0(z)\equiv z$.)

We shall prove

Theorem 6. *If* $f(z)=e^{az}$, $a\neq 0$, *then* f *has no wandering domain.*

Remark. If U is a wandering domain then there is at most one integer $j\geqq 0$ such that $0\in f^j(U)$. We may replace U by $f^{j+1}(U)$ and assume $0\notin f^j(U)$ for all $j\geqq 0$. For any $k\geqq 0$ there is then a branch of f^{-1} which is analytic in $f^{k+1}(U)$ and (by Section 2, property II) maps $f^{k+1}(U)$ univalently onto $f^k(U)$. Thus f is a one-to-one conformal map between $f^k(U)$ and $f^{k+1}(U)$.

We shall prove Theorem 6 by following Sullivan's method [22, 23] which uses quasiconformal maps.

Definition 8.3. *A function* $u(x, y)$ *is* ACL *(absolutely continuous on lines) in a plane domain* D *if, for every closed rectangle* R *in* D *with sides parallel to the* x *and* y *axes,* $u(x, y)$ *is absolutely continuous on almost every horizontal and almost every vertical line in* R.

Definition 8.4. ([1], [18]). *A topological map* φ *of the plane domain* D *into* $\hat{C}$ *is called quasiconformal if*

(i) φ *is* ACL,

(ii) *there is a constant k such that* $0 \leqq k < 1$ *and* $|\varphi_{\bar z}| \leqq k|\varphi_z|$ *holds almost everywhere, where*

$$\varphi_z = \tfrac{1}{2}(\varphi_x - i\varphi_y), \quad \varphi_{\bar z} = \tfrac{1}{2}(\varphi_x + i\varphi_y).$$

The quantity $\mu(z) = \varphi_{\bar z}/\varphi_z$, which exists almost everywhere, is called the complex dilatation of the map.

In the above definition the map φ is conformal if and only if $\mu(z)=0$ almost everywhere. We shall often use the following formulae. (See e.g. [18, p. 191].)

Lemma 8.5. *If* $\varphi: G \to H$, $\psi: H \to K$ *are quasiconformal then* $\psi\varphi = \psi(\varphi)$ *is also, and the dilatations satisfy*

(i) *if* ψ *is conformal* $\mu_{\psi\varphi}(z) = \mu_\varphi(z)$ *a.e.*

(ii) *if* φ *is conformal* $\mu_{\psi\varphi}(z) = \mu_\psi(\varphi(z))\left(\overline{\varphi'(z)}/\varphi'(z)\right)$ *a.e.*

If φ, ψ *are onto then we can reverse the implication in* (i):

(iii) *if* $\mu_{\psi\varphi} = \mu_\varphi$ *a.e. in G then* ψ *is conformal.*

Lemma 8.6. *Suppose that f is a one-to-one conformal map from a domain D to a domain* D_1, *and that* φ *is a quasiconformal map defined on D and* D_1 *and whose complex dilatation* $\mu = \mu_\varphi$ *satisfies*

$$\mu(f(z)) = \mu(z) f'(z)/\overline{f'(z)} \quad \text{a.e. in } D. \tag{13}$$

Then $\varphi f \varphi^{-1}$ *is conformal in* $\varphi(D)$.

For by 8.5 (ii) $\mu_{\varphi f}(z) = \mu_\varphi(f)\overline{(f)'}/f' = \mu_\varphi(z)$ a.e. in D, that is $\mu_{(\varphi f \varphi^{-1})\varphi} = \mu_\varphi$ a.e. in D and the result follows from Lemma 8.5.

The existence of a large family of quasiconformal maps is guaranteed by an existence theorem for solutions of the Beltrami differential equation $\varphi_{\bar z} = \mu\varphi_z$.

Lemma 8.7 (e.g. [1], [18]). *Given any measurable function* μ *on the plane such that* $\|\mu\|_\infty = \text{ess sup } |\mu| < 1$, *there exists a unique sense-preserving quasiconformal homeomorphism* $\varphi = \varphi^\mu$ *of* $\hat{C}$ *onto* $\hat{C}$ *such that* $\varphi_{\bar z} = \mu\varphi_z$ *(a.e.) and* φ *fixes* 0, 1, ∞.

Clearly we can use a different normalisation by replacing φ by $L \circ \varphi$, where L is an arbitrary Moebius transformation. For a fixed normalisation, for example that chosen in 8.7, L. Ahlfors and L. Bers [2] proved that if μ depends continuously and differentiably on parameters, the same is true of φ^μ. Their result includes the following.

Lemma 8.8. *Write* $t = (t_1, \ldots, t_n)$ *and* $s = (s_1, \ldots, s_n)$. *Suppose that for all t in some open set* $\Delta \subset R^n$ *one has for the* μ *of Lemma* 8.7 *that* $\mu(t, z) \in L_\infty$ *as a function of z with* $\|\mu\|_\infty < 1$ *and that (suppressing z)*

$$\mu(t+s) = \mu(t) + \textstyle\sum_{i=1}^n a_i(t) s_i + |s| \alpha(t, s) \tag{14}$$

with $\|\alpha(t,s)\|_\infty \leqq c$, *c constant, and* $\alpha(t, s) \to 0$ *a.e. in z as* $s \to 0$. *Suppose that* $\|a_i(t+s)\|_\infty$ *are bounded and that* $a_i(t+s) \to a_i(t)$ *a.e. for* $s \to 0$.

Then $\varphi^{\mu(t)}$ *is in* $C^1(\Delta)$ *as a function of t for fixed z.*

Remark. Ahlfors and Bers prove much more, namely that φ^{μ} has an expansion

$$\varphi^{\mu(t+s)} = \varphi^{\mu(t)} + \sum_{i=1}^{n} \theta^{\mu(t), a_i(t)} s_i + |s| \gamma(t, s),$$

with $\|\gamma(t, s)\|_B \to 0$ for $s \to 0$, where for a certain integer $p > 2$ and an arbitrary (fixed) $R > 0$ one has

$$\|w\|_B = \sup_{|z_i| \leq R} \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{1-2/p}} + \Big(\iint_{|z| \leq R} |w_z|^p \, dx \, dy\Big)^{1/p}.$$

They also show that as $t' \to t$ under these assumptions then $\|\theta^{\mu(t'), a_i(t')} - \theta^{\mu(t), a_i(t)}\|_B \to 0$. Since γ and $\theta^{\mu, a}$ vanish for $z = 0$ it follows that for fixed z, $\gamma(t, s)(z) \to 0$ as $s \to 0$ and that $\theta^{\mu, a_i}(z)$ is continuous in t.

First stage of the proof of Theorem 6. Preliminaries

Suppose that f has the wandering domain U. By an earlier remark we may assume that U is simply-connected and f maps every $f^k(U)$, $k \geq 0$, $1-1$ conformally onto $f^{k+1}(U)$.

Define the equivalence relation $\sim$ on $\mathbf{C}$: $x \sim y$ if and only if there exist positive integers m and n such that $f^m(x) = f^n(y)$. If $x \sim y$ and $y \sim z$ then there are positive integers m, n, k, p such that $f^m(x) = f^n(y)$, $f^k(y) = f^p(z)$ and so $f^{mk}(x) = f^{np}(z)$, which gives $x \sim z$.

A class of equivalence $[x]$ meets U in at most one point, for if $x, y \in U$, $f^m(x) = f^n(y)$, then $m = n$ by the definition of wandering domain and then $x = y$ since f^m is a homeomorphism.

Lemma 8.9. *Given a measurable function μ on U such that $|\mu| \leq k < 1$ on U, there is an extension of μ to a function in $L^\infty(\mathbf{C})$ which is "f-invariant" in the sense that*

$$\mu(f(z)) = \mu(z) f'(z) / \overline{f'(z)} \quad \text{a.e. in } \mathbf{C}. \tag{14}$$

Further $\|\mu\|_\infty = \sup_U |\mu|$.

Proof. Given μ on U set $\mu(z) = 0$ if $[z] \cap U = \emptyset$. If $[z] \cap U = \{x\}$, $x \in U$, then there exist positive integers m, n such that $f^m(x) = f^n(z)$. In this case define

$$\mu(x)(f^m)'(x) / \overline{(f^m)'(x)} = \mu(z)(f^n)'(z) / \overline{(f^n)'(z)}. \tag{15}$$

The derivatives involved never vanish. The equation (15) gives a well-defined and indeed unique extension of μ to the set $[U]$ of classes which contain a representative in U. $[U]$ is a countable union of components of $C(f)$ and thus open. The function μ thus extended is measurable on each component $f^{\pm n}(U)$ and thus measurable in the plane. The function μ defined in this way satisfies all the conditions asserted in the lemma.

Lemma 8.10. *If the measurable function* μ *satisfies* $\|\mu\|_\infty<1$ *and the condition* (14) *of Lemma* 8.9 *denote by* $\varphi=\varphi^\mu$ *the quasiconformal homeomorphism of* $\hat{C}$ *which satisfies* $\varphi_{\bar z}=\mu\varphi_z$ *a.e. and which fixes* 0, 1 *and* ∞. *Then for* $f(z)=e^{az}$ *the function* $f_\mu=\varphi f\varphi^{-1}$ *is an entire function of the form* e^{bz} *where* $b=b_\mu$ *is a constant.*

Proof. By construction f_μ maps $\boldsymbol{C}$ to $\boldsymbol{C}$, $f_\mu(z)\neq 0$ and f^μ is a local homeomorphism. By Lemma 8.6., f_μ is locally conformal and thus is an entire function whose derivative never vanishes.

Now by for example [18, p. 74] there are positive constants C and K such that for all large $|z|$ we have

$$|\varphi(z)|<C|z|^K,\quad |\varphi^{-1}(z)|<C|z|^K.$$

For large $|z|=r$ this leads to $|f\varphi^{-1}(z)|<\exp(|a|Cr^K)$ and $|f_\mu(z)|<C\exp(Ar^K)$ for $A=|a|CK$. Thus f_μ has finite order and so $f_\mu=e^g$ where g is a polynomial. Since $f'_\mu\neq 0$, g is linear, $f_\mu(z)=e^{bz+d}$. But $\varphi(1)=1$ implies $f_\mu(0)=e^d=1$ and we have the form claimed above.

Second stage. The main quasiconformal construction

Following Sullivan we construct a family of dilatations μ which satisfy (14) and depend in a C^1 manner on $N\geqq 3$ real parameters t_j. The associated b_μ of Lemma 8.10 is a single complex parameter (equivalent to two real ones) which depends C^1 on the t_j. This implies the existence of a non-constant continuous arc in the t-space along which b_μ is constant. A contradiction is derived from this in the final part of the proof.

The construction begins in the unit disc and is carried over by conformal mapping first to the wandering domain and then by extension to the plane.

Notation. Henceforth ψ denotes a fixed conformal map from the unit disc D onto the simply-connected wandering domain U, while $\tilde a$, $\tilde b$ and $\tilde c$ are three distinct points of ∂D.

N is a fixed integer such that $N\geqq 3$. Set

$$T=\{(t_1,\dots,t_N);\ t_i\in\boldsymbol{R},\ \ |t_i|<1,\ 1\leqq i\leqq N\}.$$

On the arc $(\tilde a,\tilde b)$ pick values $e^{i\theta_v}$, $v=1,2,\dots,2N$ such that $\arg\tilde a<\theta_1<\theta_2<\dots$ $\dots<\theta_{2N}<\arg\tilde b$.

Let δ_j be a real continuously differentiable function, $1\leqq j\leqq N$, such that

$$\text{(16)}\qquad \begin{cases}\delta_j=0,\quad \theta\notin I_j=[\theta_{2j-1},\theta_{2j}];\quad \delta_j>0,\quad \theta\in I_j;\\ \delta_j(\theta)+\theta<\theta_{2j},\ \theta-\delta_j(\theta)>\theta_{2j-1}\ \text{ for }\ \theta\in I_j;\\ |\delta_j'(\theta)|<\dfrac{1}{2N}.\end{cases}$$

For any $t \in T$ the function $\theta + \sum_1^N t_j \delta_j(\theta)$ is monotone and gives a map of $[0, 2\pi]$ to itself. This extends to

$$\varphi(t, re^{i\theta}) = r \exp\left(i\left(\theta + \sum_1^N t_j \delta_j(\theta)\right)\right), \tag{17}$$

which is a homeomorphism of the disc D onto itself. Different choices of t give different maps φ, all of which fix $\tilde{a}$, $\tilde{b}$, $\tilde{c}$. In particular $t \neq 0$ implies $\varphi \neq \mathrm{Id}$.

Each $\varphi(t, z)$ is quasiconformal in D, for $\varphi \in C^1(D)$ and

$$\begin{aligned} \mu_\varphi = \varphi_{\bar{z}}/\varphi_z &= e^{2i\theta}\left(\varphi_r + \frac{i}{r}\varphi_\theta\right)\Big/\left(\varphi_r - \frac{i}{r}\varphi_\theta\right) \\ &= -\left(e^{2i\theta} \sum_1^N t_j \delta_j'(\theta)\right)/\left(2 + \sum_1^N t_j \delta_j'(\theta)\right). \end{aligned} \tag{18}$$

Thus in D, μ_φ satisfies $|\mu_\varphi| < 1/3$ by (16). Further $\mu_\varphi(z)$ is continuous in z.

Observe that $\chi = \psi\varphi\psi^{-1}$, with φ given by (17), is quasiconformal in U and maps U onto itself. The dilatation of χ is, by Lemma 8.5., given by

$$\mu_\chi = \mu_\varphi(\psi^{-1})\overline{(\psi^{-1})'}/(\psi^{-1})', \tag{19}$$

which is continuous and thus measurable in z in U. Further $|\mu_\chi| < 1/3$ in U.

Use the construction of Lemma 8.9 to extend μ_χ to an f-invariant complex dilatation in the plane, which we still denote by the same symbol, and which satisfies $|\mu_\chi| < 1/3$ in the plane.

Denote by $\Phi = \Phi_t$ the quasiconformal map of C to itself which has complex dilatation χ and fixes $0, 1, \infty$. By Lemma 8.10

$$\Phi_t f \Phi_t^{-1} = e^{bz}, \tag{20}$$

where b is a constant which depends on t.

We now check that μ_χ satisfies the differentiability conditions of Lemma 8.8. For z such that $[z]$ meets U there is some z_1 in U and positive integers m and n such that $f^n(z) = f^m(z_1)$ and by (15)

$$\mu_\chi(z) = \overline{((f^n)'(z)}(f^m)'(z_1)\mu_\chi(z_1))/((f^n)'(z)\overline{(f^m)'(z_1)}).$$

Using (18), (19) we have

$$\mu_\chi(z) = e^{i\gamma(z)}\left(\sum_{j=1}^N t_j \beta_j\right)/\left(2 + \sum_{j=1}^N t_j \beta_j\right), \tag{21}$$

where $\gamma(z)$ is real and $\beta_j = \delta_j'\,(\arg \psi^{-1}(z_1))$ is constant in t and satisfies $|\beta_j| < 1/2N$ by (16). For other values of z we have $\mu_\chi = 0$ and can retain the formula (21) if we set $\gamma = \beta_j = 0$.

Calculation gives

$$a_i(t) = e^{i\gamma} 2\beta_i / \left(2 + \sum_1^N \beta_i t_i\right)^2$$

and

$$\alpha(s, t) = -e^{i\gamma}\left(\sum_1^N \beta_i s_i\right)^2 \left(2 + \sum_1^N \beta_i(s_i + t_i)\right)^{-1} \left(2 + \sum_1^N \beta_i t_i\right)^{-2}$$

which satisfy the requirements of Lemma 8.8. Thus we have proved the first part of the following

Lemma 8.11. *The quasi-conformal maps Φ_t constructed above depend on t in a C^1 manner for $t \in T$ and fixed z. Further $\Phi_t f \Phi_t^{-1} = e^{bz}$, where b is a constant in z and is in $C^1(T)$.*

To prove the last part of the lemma take a fixed point z_1 of f: $e^{az_1} = z_1$ and observe that $\Phi_t(z_1)$ is a fixed point of $e^{bz} = \Phi_t f \Phi_t^{-1}$. Thus

$$b = \big(1/\Phi_t(z_1)\big) \log \Phi_t(z_1). \tag{22}$$

Now $t=0$ makes $\chi=0$ and thus Φ_0 is analytic and with the given normalisation must be the identity map. Then for $t=0$ we have $b=a$, $\Phi_0(z_1)=z_1$ and we can choose the value of $\log z_1$ to make $a=(1/z_1)\log z_1$. But, as t varies in T, $\Phi_t(z_1)$ is continuous in t and never zero so that the right hand side of (22) is locally well-defined and continuously differentiable in t. Thus $b \in C^1(T)$.

Lemma 8.12. *Take $N=3$. Then there is a non-constant arc α in T such that $b(t)$ is constant on α.*

Proof. Put $b = X_1 + iX_2$. If rank $(\partial X_i/\partial t_j)$, $1 \leqq i \leqq 2$, $1 \leqq j \leqq 3$, has its maximum value of 2 at $t = \tau \in T$, we can assume that say $\partial(X_1, X_2)/\partial(t_1, t_2) \neq 0$ at τ and hence in a neighbourhood of τ. The inverse function theorem applied to the map $X_1 = X_1(t)$, $X_2 = X_2(t)$, $Z = t_3$ near τ gives C^1 solutions $t_1 = F_1(X_1, X_2, Z)$, $t_2 = F_2(X_1, X_2, Z)$, $t_3 = F_3(X_1, X_2, Z) \equiv Z$ near $X_1(\tau)$, $X_2(\tau)$, τ_3. Holding X_1, X_2 constant we obtain the arc α: $X_1 = \text{const}$, $X_2 = \text{const}$, $t_3 = Z$ which is in T for $|Z - \tau_3|$ small.

If the maximal rank of $(\partial X_i/\partial t_j)$ is 1, occurring at $t = \tau$, we can assume $\partial X_1/\partial t_1 \neq 0$ near $t = \tau$ and apply the inverse function theorem to $X_1 = X_1(t)$, $Y = t_2$, $Z = t_3$ near τ to obtain C^1 solutions $t = F(X_1, Y, Z)$ near $X_1(\tau), \tau_2, \tau_3$, such that $X_1\big(F(X_1, Y, Z)\big) = X_1$, $Y\big(F(X_1, Y, Z)\big) = Y$, $Z\big(F(X_1, Y, Z)\big) = Z$. Then for t on the arc α: $t = F(X_1, \tau_2, Z)$ where $X_1 = X_1(\tau), X_2 = X_2(\tau)$ and $|Z - \tau_3|$ is small we have $t \in T$ and α is the required arc.

If rank $(\partial X_i/\partial t_j)$ is always zero then any arc will do.

Conclusion of the proof

The last stage is to derive a contradiction from the result of Lemma 8.12. Suppose that $t = t(\sigma)$, $0 \leqq \sigma \leqq \sigma_0$, is the equation of the arc α in that lemma. Write $\Omega_\sigma = \Phi_{t(0)}^{-1} \Phi_{t(\sigma)}$. Since $t(0)$ and $t(\sigma)$ give the same b in (20) it follows that $\Omega_\sigma f = f \Omega_\sigma$. From this we deduce

Lemma 8.13. *Ω_σ leaves every point of $F(f)$ fixed; Ω_σ maps U onto U, for $0 \leqq \sigma \leqq \sigma_0$.*

Proof. Suppose that z_0 is a fixed point of some order p of f. Then Ω_σ commutes with f^p and so $\Omega_\sigma(z_0)=\Omega_\sigma(f^p(z_0))=f^p(\Omega_\sigma(z_0))$. Thus $\Omega_\sigma(z_0)$ is one of the discrete set of fixed points of order p of f, $\Omega_\sigma(z_0)$ is continuous in σ and $\Omega_0(z_0)=z_0$. Hence Ω_σ fixes every fixed point of every order of f. These points are dense in $F(f)$ by Section 2, property VI and the first statement follows. Consequently Ω_σ maps each component of $C(f)$ into a component of $C(f)$. Again it follows by continuity from the case $\sigma=0$ that U is mapped to U.

Lemma 8.14. *Ω_σ is a homeomorphism of U onto U which leaves each prime end fixed.*

For the definition of prime ends see e.g. [10]. The point of Lemma 8.14 is that different prime ends may have the same "impression", that is correspond to the same point sets on the boundary. Lemma 8.13 asserts only that Ω_σ is the identity as a point mapping on ∂U. The result of Lemma 8.14 is asserted in [23]. The following proof is due to T. A. Lyons.

Proof. Fix any σ with $0\leqq\sigma<\sigma_0$ and write $\Omega_\sigma=\Omega$, $\omega=\psi^{-1}\Omega_\sigma\psi$. Then ω is an orientation-preserving homeomorphism of the unit disc D to itself and the boundary points of D correspond 1—1 to the prime ends of U under the map ψ. The map ω extends to an orientation-preserving homeomorphism of $S=\partial D$ to itself. It is enough to prove that ω is the identity on S. If this is not the case we may choose $\theta_1\neq\theta_2$ in $[\theta, 2\pi]$ so that $\omega(e^{i\theta_1})=e^{i\theta_2}$ and ψ has radial limits α_1, α_2 at $e^{i\theta_1}, e^{i\theta_2}$.

Now Ω is the identity on ∂U and as $r\to 1^-$ we have $\psi\omega(re^{i\theta_1})=\Omega\psi(re^{i\theta_1})\to\Omega(\alpha_1)$ since Ω is continuous and $\Omega(\alpha_1)=\alpha_1$. But by assumption $\omega(re^{i\theta_1})\to\omega(e^{i\theta_1})=e^{i\theta_2}$ so that we must have $\alpha_1=\alpha_2$. The image of the radii $\{re^{i\theta_i};\ 0\leqq r<1\}$ under ψ, together with α_1, therefore constitutes a Jordan curve J in U.

We now choose a small positive δ such that $\theta_1+\delta<\theta_2$ and $\omega(\theta_1+\delta)\in(\theta_2, \theta_1+2\pi)$. Then for $\theta_1'\in(\theta_1, \theta_1+\delta)$ we have $\omega(\theta_1')\in(\theta_2, \theta_1+2\pi)$. For almost all θ_1' in $(\theta_1, \theta_1+\delta)$ the radial limit α' of ψ exists at $e^{i\theta_1'}$ and also at $e^{i\theta_2'}$, where $\theta_2'=\omega(\theta_1')$. Thus we obtain a second Jordan curve J' formed by the image under ψ of the radii from 0 to $e^{i\theta_1'}$, $e^{i\theta_2'}$ together with the value α'. J and J' cut at $\psi(0)$ and are otherwise disjoint except perhaps at α', α if $\alpha=\alpha'$. But the curves must cut more than once if at all, so we do have $\alpha=\alpha'$.

Thus $\lim_{r\to 1^-}\psi(re^{i\theta'})=\alpha$ for almost all θ' in $(\theta_1, \theta_1+\delta)$. But this is impossible since ψ is univalent. Thus the lemma is established.

By Lemma 8.14 all $\Phi_{t(\sigma)}$ map U onto the same domain $\hat{U}$ and induce the same map of the prime ends of U to the prime ends of $\hat{U}$. Denote by $\hat{\psi}$ some conformal map of D onto $\hat{U}$. Then $h_\sigma=\hat{\psi}^{-1}\Phi_{t(\sigma)}\ \psi$ is a quasiconformal map of D onto D.

The dilatation of h_σ is, by Lemma 8.5 (i) and noting that $\mu_\Phi=\mu_\chi$ in U, $\chi=\psi\varphi\psi^{-1}$, the same as the dilatation of $\psi\varphi\psi^{-1}\psi$, that is μ_φ, where $\varphi=\varphi(t(\sigma), z)$ is given by (17).

By Lemma 8.5 (iii) applied to $h_\sigma \circ \varphi^{-1}(t(\sigma))$ and $\varphi(t(\sigma))$ it follows that $h_\sigma = L_\sigma \varphi(t(\sigma))$, where L_σ is a conformal self-mapping of D, that is a Moebius transformation, which may of course depend on σ. By our remarks on prime ends all h_σ have the same boundary values for $0 \leqq \sigma < \sigma_0$, which shows that on ∂D $L_0\varphi(t(0)) = L_\sigma \varphi(t(\sigma))$, these expressions being continuous on $\bar{D}$. Since $\varphi(t(\sigma))$ is the identity map on the boundary arc $\tilde{b}\tilde{c}$ one has $L_\sigma = L_0$ for all σ. But this implies that $\varphi(t(\sigma)) = \varphi(t(0))$ on ∂D for an arc of values $t(\sigma)$ not all zero. This contradicts the construction of φ in (17). The proof is complete.

9. Limit functions and domains of normality

We shall give a few results about the iterative behaviour of the functions e^{az}. Some previous contributions in the case $a=1$ are [19] and [24].

Theorem 6 shows that any component D of $C(f)$, $f=e^{az}$, is preperiodic in the sense that there exist integers $k \geqq 0, p>0$, such that $D_0 = f^k(D)$ and $f^p(D_0) \subset D_0$. By an extension of the Wolff—Denjoy theorem one sees (cf. [8]) that only the following two possibilities can arise:

(i) (f^{np}), $n \in \mathbf{N}$, converges in D_0 to a constant limit λ. Either $\lambda = \infty$ or λ is a finite solution of $f^p(\lambda) = \lambda$. In the latter case we have either $\lambda \in D_0$ and λ is an attractive fixed point of order p, or $\lambda \in \partial D_0$ and in this case $(f^p)'(\lambda)$ must be a root of unity (see [14, Section 54] where Fatou proves for the rational case that we cannot have convergence of f^{np} to λ in such a domain D_0 if $(f^p)'(\lambda)$ has modulus 1 but is not a root of unity. The proof remains valid for entire functions.).

(ii) D_0 contains in its interior a "centrum", that is a fixed point λ such that $(f^p)'(\lambda) = \exp(2\pi i\theta)$, where θ is irrational. In this case there is a subsequence of f^{np} which converges to the identity in D_0.

Remark. Case (ii) will occur only if a is in the boundary of some D_p, since $(f^p)'(\lambda)$ varies analytically with a. Thus (i) and (ii) are mutually exclusive by the remark preceding Lemma 5.1.

These are the cases found by Sullivan [22] for the corresponding iteration of rational functions, except that our domains D, D_0 are simply-connected and thus the case of the Herman rings, which may arise for rational functions, cannot occur here.
As a further simplification the case $\lambda = \infty$ in (i) cannot in fact occur.

Theorem 7. *For $f(z) = e^{az}$, $a \neq 0$, there is no component D of $C(f)$ and sequence of integers n_i such that $n_i \to \infty$ and $f^{n_i}(z) \to \infty$ in D.*

Proof. It is enough to show that if $f^p(D) \subset D$ for some positive integer p, then f^{np} does not have limit ∞ in D. Suppose that on the contrary $f^{np} \to \infty$ in D as $n \to \infty$.

Then $f^{np-1}=f^{-1}(f^{np})$ also has limit ∞ in D and similarly for f^{np-i} for any fixed i. Thus the whole sequence f^m has limit ∞ in D as $m\to\infty$.

Consider any disc Δ of centre z_1 such that $\bar{\Delta}\subset D$. Since $(f^n)'(z_1)=a^n f^n(z_1)\cdot f^{n-1}(z_1)\dots f(z_1)$, while $f(z)\neq 0$ and $a\neq 0$, $f^m(z_1)\to\infty$ implies that $(f^n)'(z_1)\to\infty$ as $n\to\infty$. By Bloch's theorem there is an integer n_0 such that for $n>n_0$ the domain $f^n(\Delta)$ contains a disc of arbitrarily large radius and hence a segment γ such that $a\gamma$ is vertical and has length at least 2π. Then $f^{n+1}(\Delta)$ contains $f(\gamma)$ which is a circumference surrounding 0, and for large n this circumference has arbitrarily large radius since $f^{n+1}(\Delta)\to\infty$. But $f(\gamma)$ belongs to some simply-connected component D' of $C(f)$. Thus $C(f)$ contains the interior of $f(\gamma)$, which implies $F(f)=\emptyset$, against Section 2, property I.

Corollary 1. *If* $f^n(0)\to\infty$ *then* $F(f)=\mathbf{C}$.

This follows from Theorem 7 and Section 2, property XI. It occurs at least for real a such that $a>1/e$. This extends the result of Misiurewicz [19] for $a=1$.

Corollary 2. $C(e^{az})\neq\emptyset$ *implies that* $a\in\bar{D}_p$ *for some* p.

Thus in particular for any of the countable set of values a such that the sequence (1) has terminating convergence of some period q, that is for a such that $f^n(0)=f^{n+q}(0)$ holds for some positive integers n, q, Theorem 2 shows that $F(e^{az})=\mathbf{C}$. Further examples of these maximal Fatou—Julia sets are given by

Theorem 8. *The set* $S=\{a;\ F(e^{az})=\mathbf{C}\}$ *is dense in every* ∂D_p. *Indeed, each arc of a* ∂D_p *contains a non-countable subset of* S.

Proof. Observe that two different D_p, D_q can touch at most at one point of their boundaries, since at a dense subset of ∂D_p the region D_p is tangent to other unbounded disjoint $D_{p'}$. Thus the set A of points a which belong to two boundaries ∂D_p is countable.

Fix a region D_p. Denote by $M(a)$ the multiplier of the attractive cycle of fixed points of e^{az}. Then $M(a)$ is analytic in D_p and $|M(a)|=1$ on ∂D_p. Take an arc γ of ∂D_p, on which $M(a)$ is analytic and maps γ to an arc of the unit circumference. Take $a\in\gamma-A$ such that $M(a)=\exp(i\theta\pi)$, θ irrational. We note that e^{az} has just one cycle whose multiplier has modulus one, since a is not on two ∂D_p's, and that there are no attractive cycles. Thus if $C(e^{az})\neq\emptyset$, then there are no admissible constant limit functions by (i) and there is therefore a centrum of f^p with multiplier $e^{i\theta\pi}$. If we choose θ appropriately the centrum case can also be ruled out.

Lemma 9.1. Cremer [11]. *If*

(i) $g(z)=e^{i\varphi}z+\dots$ *is a non-linear entire function,*
(ii) $F(r)$ *is a positive function of* r *such that*

$$M(g,r)<F(r),\qquad r>r_0,$$

(iii) $\liminf_{n\to\infty} |e^{in\varphi}-1|^{1/\log F^n(r)}=0$, $r>r_0$, *where* F^n *is the n-th iterate of* F, *then* 0 *is a non-centrum of* g. *For fixed* F *there is in any neighbourhood of any angle* φ_0 *an uncountable subset of* φ *which satisfy* (iii).

We continue the proof of Theorem 8. There is a constant B such that $|a|\leqq B$, $|z(a)|\leqq B$ holds for $a\in\gamma$ and $z(a)$ a point of the p-cycle whose multiplier has modulus 1. Apply Cremer's lemma to $g(z)=f^p(z+z(a))-z(a)$. For $a\in\gamma$ we have for $r>r_0(a)$ that

$$M(g, r) < M(f^p, r+B)+B < M(f^p, 2r) < h^p(2r),$$

where $h(x)=e^{Bx}$. Take $F(r)=h^p(2r)$ and we see that for $\theta\pi=\varphi$ in a dense uncountable set and hence for a in a dense uncountable subset of γ we do not have a centrum. Thus $F(e^{az})=\mathbf{C}$. The theorem is proved.

Turning to the cases when $C(e^{az})\neq\emptyset$ the simplest situation occurs when $C(e^{az})$ has a single component, which is then a completely invariant domain.

Theorem 9. *$C(e^{az})$ has a single non-empty component if and only if $a\in D_1\cup\{e^{-1}\}$. In all other cases when $C(e^{az})$ is non-empty it has an infinity of components.*

Proof. If $a\in D_1\cup\{e^{-1}\}$ then $a=te^{-t}$ for $|t|<1$ or $t=1$. Thus the fixed point $\xi=e^t$ of multiplier t is in a component G of $C(f)$, where $f=e^{az}$ (or in the case $t=1$ we have $\xi\in\partial G$). Since $0\in G$ and $f(G)=G-\{0\}$ we may take z_1 near 0 in G and have for some branch of f^{-1} that $f^{-1}(z_1)\in G$. By continuation of f^{-1} around a circle centred at 0, which we may suppose to stay in G, it follows from the complete invariance of $C(f)$ that every branch of $f^{-1}(z_1)\in G$. By analytic continuation $f^{-1}(z)\in G$ for all $z\in G$, $z\neq 0$, and all branches f^{-1}. Thus G is completely invariant. But for any component G_1 of $C(f)$ we must have $f^k(G_1)\subset G$ for some k by the basic classification (i) and (ii). Thus $C(f)=G$ has a single component.

Conversely, if $C(f)$ has a single component G then G is completely invariant. Since $f(G)\subset G$ there must, by the above analysis of cases, be a first order fixed point η whose multiplier has modulus not exceeding 1. Since f cannot be a univalent map of G to G the centrum case is ruled out and $f^n(z)\to\eta$ in G as $n\to\infty$. Thus either $a\in D_1$ or the multiplier of η must be some p-th root of unity $p\geqq 1$. It remains to show that $p=1$.

Suppose that $p>1$. An examination of the local iteration of f near η shows that the p components $G_1, \dots, G_p$ of $C(f)$ whose boundaries contain η, as mentioned in Section 2, property X, are necessarily different. For (see e.g. [3]) there is an angle A of directions at η such that for any small segment σ which approaches η in A, if σ' is the rotation of σ through $2\pi/p$, then $f^n(z)\to\eta$ uniformly on σ and σ'. If we suppose that σ, σ' are in the same G_i we can join their ends in G_i to obtain a Jordan curve γ on which $f^n(z)\to\eta$ uniformly. It follows that the interior of γ belongs to G_i and so to $C(f)$. But η is a point of $F(f)$ which is invariant under $z\to f$ and thus under (approximate) rotations about η by $2\pi k/p$, for some integer k prime to p. This is incompatible with the existence of γ. Hence if $p>1$ there are at least p components of $C(f)$.

Finally, if there are finitely many, say N, components of $C(f)$, they are permuted among themselves by the map f. Hence each is invariant, and completely invariant under f^m where $m=N!$. By [5] the entire transcendental function f^m can have at most one completely invariant component of $C(f^m)=C(f)$ and so $N=1$.

Most other cases are covered by

Theorem 10. *For $a\in D_p$, $p>1$, or for $a\in\partial D_p$ such that the multiplier of the non-repulsive p-cycle for $f=e^{az}$ is 1, there is a cycle $G_1, \ldots, G_p$ of components of $C(f)$, such that $f(G_i)=G_{i+1}$, $G_{p+1}=G_1$, and f^{np} converges to a fixed point in each G_i. Further, every component $\hat{G}$ of $C(f)$ satisfies $f^N(\hat{G})\subset G_i$ for some N, i.*

All this is obvious. Further, since 0 belongs to one of the cycles G_i, it follows first that $G_{i-1}=f^{-1}(G_i)$ is unbounded and, by repetition of maps by f^{-1}, that every component of $C(f)$ is unbounded.

Theorems 8, 9 and 10 cover all except the centrum cases. If $f(z)=e^{az}$ has a centrum then 0 and hence all $f^n(0)\in F(f)$.

We sketch the proof in the case when, in addition, $a\in\partial D_1$, so that there is among the components of $C(f)$ a centrum domain G which contains a first-order fixed point ξ. G is mapped univalently onto itself by f. Further there is a sequence n_k such that $n_k\to\infty$ and $f^{n_k}(z)\to z$, $f^{-n_k}(z)\to z$ in G, where f^{-n_k} is the branch of the inverse of f^{n_k} such that $f^{-n_k}(\xi)=\xi$.

We show that every point η of ∂G belongs to the set L of Section 2, property XI. If not there is a neighbourhood K of η which does not meet L. Then by [6, Lemma 1] any set of branches of f^{-n} which are analytic in a domain are also normal there. We take a simply-connected subset K of H such that K meets G and K contains a point η of $F(f)$. The branches f^{-n_k} are analytic and normal in K and so have limit z there.

If $K'=\{z\colon |z-\eta|\leqq\varrho\}\subset K$ and $K''=\{z\colon |z-\eta|<\varrho/2\}$, then for large n_k we have $f^{-n_k}(K')\supset K''$, $f^{n_k}(K'')\subset K'$. But this contradicts Section 2, property VIII since $\eta\in F(f)$. Thus $\eta\in L$.

If $0\in C(f)$ then for some k, p we have $f^k(0)\in H$, where H is a component of $C(f)$ such that $f^p(H)\subset H$. If H is not a centrum domain then L has only p limit points, while if H is a centrum domain the limit points of L are all in $C(f)$. This contradicts $\partial G\subset L$.

While this article was in the press we learned of forthcoming work by R. L. Devaney and (independently) A. E. Eremenko and M. Yu. Lyubich which seems to have points in common with some of our results.

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