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SOME ESTIMATES OF HARMONIC MAJORANTS

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1. Introduction

The problem of the harmonic majorization of a given subharmonic function in an unbounded domain in R^d involves the Dirichlet problem with the given boundary values and hence the use of harmonic measure.

In the plane case there is the following connection with the theory of Hardy spaces. Let F be a univalent analytic function from the unit disc onto a domain D in the z-plane and let $0<p<\infty$. Then $F\in H^p \Leftrightarrow |z|^p$ has a harmonic majorant in D (see e.g. [4, p. 28]). We note that the implication $\Leftarrow$ holds without F being univalent. This classical result has been associated with the theory of Brownian exit times by Burkholder, [2], [3].

Let τ denote the exit time from D (an open connected set in R^d) of a Brownian motion, starting at zero time from a point x in D. Burkholder proved (for $0<p<\infty$) that the expectation $E_x\tau^{p/2}$ is finite if and only if $|x|^p$ has a least harmonic majorant $u(x)$ in D.

In fact (see [2, p. 191 (3.3)]),

$$k_{p,d}|x|^p \leqq k_{p,d}u(x) \leqq d^{p/2}E_x\tau^{p/2}+|x|^p \leqq K_{p,d}u(x),$$

with constants $k_{p,d}$ and $K_{p,d}$ depending only on p and d.

Thus, one can ask the question whether the order of magnitude of $u(x)$ and hence that of $E_x\tau^{p/2}$ can be greater than that of $|x|^p$. One purpose of this paper is to consider an estimate of the type

$$u(x) \leqq C|x|^p \log|x|, \quad |x| \geqq \gamma > 1,$$

which can be shown to be sharp in suitable domains; see Section 3. The majorization of some other subharmonic functions is also treated; see Section 4. A preliminary version of the results was given in [8].

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2. Notation

Let R_d (where $d \geqq 2$) denote a d-dimensional Euclidean space with $x=(x_1, x_2, \ldots, x_d)$ and $|x|=(\sum_{k=1}^{d} x_k^2)^{1/2}$. However, points in the complex plane are also denoted by z.

Assuming that a given point x_0 belongs to the domain D, let D_r denote the component of $D \cap \{x\colon |x|<r\}$ containing x_0. The following harmonic measures are considered:

$$\omega_r(x) = \begin{cases} 1 & \text{on } \partial D_r \cap \{x\colon |x| = r\} \cap D \\ 0 & \text{on the rest of } \partial D_r \end{cases} \qquad |x| \leqq r,$$

$$v_r(x) = \begin{cases} 1 & \text{on } \partial D \cap \{x\colon |x| > r\} \\ 0 & \text{on } \partial D \cap \{x\colon |x| \leqq r\}. \end{cases}$$

3. Harmonic majorization of $|x|^p$

In the complex plane the following result was given by Tsuji [9, p. 118].

Theorem 1 (Tsuji). *If*

$$\omega_r(z) \leqq c_0 r^{-p_1} \tag{1}$$

for some $p_1 > p > 0$, $r \geqq r_1 > 0$, c_0 *depending on* z, *then there exists a least harmonic majorant* $u(z)$ *of* $|z|^p$ *in* D. *If further*

$$\omega_r(z) \leqq c(r/|z|)^{-p_1} \tag{2}$$

for some $p_1 > p$, $r \geqq r_0|z| > 0$, c, p_1, r_0 *being constants, then for some constant* C

$$|z|^p \leqq u(z) \leqq C|z|^p \quad \text{in} \quad D.$$

In order to estimate $\omega_r(z)$ Tsuji used a well-known inequality of the type

$$\omega_r(z) \leqq c \exp\left(-\pi \int_{|z|}^{r} \theta^{-1}(r) r^{-1} dr\right), \tag{3}$$

where $r\theta(r)$ denotes the maximum length of the arcs occurring on $\partial D_r \cap \{z\colon |z|=r\} \cap D$ (provided this set is $\neq \{z\colon |z|=r\}$, otherwise let $\theta(r)=\infty$). This type of inequality corresponds to Ahlfors' First Distortion Inequality (for simply-connected domains) and can also be proved by Carleman's classical method (irrespective of connectivity). See e.g. [7] and [9, p. 112]. Hence the condition (2) above is implied by [9, p. 118]:

$$\varliminf_{r/|z| \to \infty} \pi (\log (r/|z|))^{-1} \int_{|z|}^{r} \theta^{-1}(r) r^{-1} dr > p. \tag{4}$$

In order to generalize inequalities of the types (3) and (4) to R^d, $d \geqq 3$, one uses Carleman's method with integrals $\pi \int \theta^{-1}(r) r^{-1} dr$ generalized to integrals $\int \alpha(r) r^{-1} dr$, where $\alpha(r)$ is the so-called characteristic constant connected with the

Laplace—Beltrami equation. Thus one obtains estimates from above of the following type:

(5) $$\omega_r(x) \leqq c \exp\left(-\int_{|x|}^{r/2} \alpha(r) r^{-1} dr\right), \quad x \in D, \quad |x| < r/2,$$

(see [6, in particular pp. 136—138], cf. also [7]).

For the sake of simplicity one can consider domains D such that $D \cap \{x\colon |x|=r\}$ is a spherical cap $\{x\colon |x|=r\} \cap \{x\colon 0 \leqq \varphi < \varphi(r)\}$, where φ, $0 \leqq \varphi \leqq \pi$, is defined by $|x| \cos\varphi = x_1$. Actually, ω_r is increased by a spherical symmetrization [1, p. 267], while $\alpha(r)$ is decreased; cf. [6, p. 139].

Tsuji's result depends on the expression (assuming that $0 \in D$)

(6) $$u(x) = p \int_0^\infty v_r(x) r^{p-1} dr$$

for the least harmonic majorant $u(x)$ of $|x|^p$, if it exists. Cf. [2, Theorem 3.1, Remark 3.1, p. 191]. Let $\omega_r(x)=1$ when $r<|x|$. Of course, $v_r(x) \leqq \omega_r(x)$. However, it follows from Burkholder's probabilistic results [2, Theorem 2.2, p. 189] (again assuming that $0 \in D$) that

(7) $$\int_0^\infty \omega_r(x) r^{p-1} dr \leqq C \int_0^\infty v_r(x) r^{p-1} dr$$

with a constant depending on p and d (provided that the complement of D is not too small). Hence, for some constant c,

(8) $$c \int_0^\infty \omega_r(x) r^{p-1} dr \leqq u(x) \leqq p \int_0^\infty \omega_r(x) r^{p-1} dr.$$

The difficult part of (7) to prove concerns two-dimensional domains D with irregular boundaries. If ∂D meets every circle $\{x\colon |x|=r\}$, symmetrization and simple estimates of harmonic measure are sufficient for the proof.

By altering Tsuji's conditions on $\omega_r(x)$ one obtains a somewhat greater estimate from above of $u(x)$.

Theorem 2. *Assume that $0<p<\infty$ and $a>1$. If, for a constant γ and a constant c_0 depending on x,*

(9) $$\omega_r(x) \leqq c_0 r^{-p} (\log r)^{-a}, \quad r \geqq \gamma > 1,$$

then there exists a least harmonic majorant $u(x)$ of $|x|^p$ in D. If, further, for some constants c and γ,

(10) $$\omega_r(x) \leqq c(|x|/r)^p ((\log|x|)/\log r)^a, \quad r \geqq r_0|x| > |x| \geqq \gamma > 1,$$

then for some constant C

(11) $$|x|^p \leqq u(x) \leqq C|x|^p \log|x|, \quad x \in D, \quad |x| \geqq \gamma > 1.$$

Proof. Assume that $0 \in D$. The existence part follows from (8) and (9). The estimate from above in (11) follows from (8) and (10):

$$u(x) \leqq p\int_0^\infty \omega_r(x)\, r^{p-1}\, dr \leqq p\int_0^{r_0|x|} r^{p-1}\, dr$$

$$+cp|x|^p(\log|x|)^a \int_{r_0|x|}^\infty r^{-1}(\log r)^{-a}\, dr$$

$$\leqq (r_0|x|)^p + C_1|x|^p \log|x| \leqq C|x|^p \log|x|, \quad |x| \geqq \gamma > 1, \quad x \in D.$$

Remark 1. The condition (10) is implied by the following condition in the complex plane, cf. (3) and (4):

$$\varliminf_{r/|z|\to\infty} \Big(\pi\int_{|z|}^r \theta^{-1}(r)\, r^{-1}\, dr - p\log(r/|z|)\Big)/\log(\log r/\log|z|) > 1.$$

Remark 2. If a condition opposite to (9) with $a \leqq 1$ is satisfied, it follows by (8) that $|x|^p$ does not have a least harmonic majorant in D.

Remark 3. In suitable domains it is possible to satisfy the condition (10) as well as a similar estimate *from below* of $\omega_r(x)$; see Example 1. Thus, by (8) the least harmonic majorant $u(x)$ of $|x|^p$ can have the order of magnitude $|x|^p \log|x|$; cf. Burkholder's result about Brownian exit times, mentioned in Section 1. In general it is more difficult to obtain good estimates from below of harmonic measures than from above, especially in R^d, $d \geqq 3$. Ahlfors' Second Distortion Inequality for simply connected strip domains in the complex plane is based on assumptions about the variation of the width of the domain. In R^d, $d \geqq 3$, few results from below are available; see [7, p. 24] and [5].

Example 1. Let $|x| \cos\varphi = x_1$, $0 \leqq \varphi \leqq \pi$, and let (for a given $\delta > 0$) $T_{\delta,d}(\pi/2)$ denote the domain $\{x\colon |x| > 1,\ 0 \leqq \varphi < \pi/2 - \operatorname{arc\,tan}\big(1/(\delta\log|x|)\big)\}$ in R^d. Then the conditions (9) and (10) of Theorem 2 are satisfied in $T_{\delta,d}(\pi/2)$ for $p=1$ if and only if $\delta < 2\sigma_{d-1}/\sigma_d$, where σ_d is the area of the unit sphere in R^d, $d \geqq 2$; $\sigma_1 = 2$. Moreover, for such δ, by an inequality opposite to (10), on the x_1-axis,

$$u(x) \geqq c|x| \log|x|, \quad |x| \geqq \gamma > 1, \quad x = (x_1, 0, 0, \ldots, 0).$$

In general let $T_{\delta,d}(\varphi_0)$ denote a similar domain $\{x\colon |x|>1,\ 0 \leqq \varphi < \varphi(|x|)\}$, where $\varphi(|x|) = \varphi_0 - 1/(\delta\log|x|) + O\big(1/(\log|x|)^2\big)$ for large $|x|$ (for details see [5]). Then the conditions (9) and (10) of Theorem 2 are satisfied in $T_{\delta,d}(\varphi_0)$ for $p = \alpha_0$ if and only if $\delta < |\alpha_0'|$, α_0 and $|\alpha_0'|$ being defined by (12), (13), (14), p. 121. For such δ, by an inequality opposite to (10), on the x_1-axis

$$u(x) \geqq c|x|^{\alpha_0} \log|x|, \quad |x| \geqq \gamma > 1, \quad x = (x_1, 0, 0, \ldots, 0).$$

Here, the characteristic constant α_0 for the Laplace—Beltrami operator on a spherical cap $\{x\colon |x| = r,\ 0 \leqq \varphi < \varphi_0\}$ is defined by the first eigenvalue of the Legendre equation:

$$f''(\varphi)+(d-2)\cot\varphi\, f'(\varphi)+\lambda_0 f(\varphi)=0,\quad f(0)=1,\quad f'(0)=0,$$

$$f(\varphi_0)=0,\quad f(\varphi)>0,\quad 0\leqq\varphi<\varphi_0, \tag{12}$$

$$\alpha_0(\alpha_0+d-2)=\lambda_0,\quad \alpha_0>0, \tag{13}$$

$$\alpha_0'=\frac{d\alpha_0}{d\varphi_0}(\varphi_0). \tag{14}$$

In particular for $\varphi_0=\pi/2$, the constant α_0 equals 1 and $\alpha_0'=-2\sigma_{d-1}/\sigma_d$ (see [5]).

Proof. a) The estimates (9) and (10) in $T_{\delta,d}(\varphi_0)$ are obtained by Carleman's method; see (5). Here for large r

$$\alpha(r)=\alpha_0-\alpha_0'\frac{1}{\delta\log r}+O\left(\frac{1}{(\log r)^2}\right),$$

and hence in (9) and (10) $p=\alpha_0$ and $a=-\alpha_0'/\delta$, which is >1 if and only if $\delta<|\alpha_0'|$.

b) See Remark 2. An estimate opposite to (9) for $a=-\alpha_0'/\delta\leqq 1$, that is, $\delta\geqq|\alpha_0'|$ and $p=\alpha_0$, implies the non-existence of the least harmonic majorant of $|x|^{\alpha_0}$. Also an estimate opposite to (10) for $p=\alpha_0$ and $\delta<|\alpha_0'|$, on the x_1-axis, implies that $u(x)\geqq c|x|^{\alpha_0}\log|x|$ for sufficiently large values of $|x|$, $x=(x_1,0,0,\dots,0)$. Such estimates of harmonic measure are given in [5, Theorem 2 and Theorem 3].

In fact, by use of Harnack's inequality, we even obtain a pointwise inequality $v_r(x)\geqq c\omega_r(x)$, $x=(x_1,0,\dots,0)$, $x\geqq k>1$, $x\in T_{\delta,d}(\varphi_0)$.

Remark 4. Iterated logarithms. By suitable modifications of the conditions (9) and (10) one obtains

$$|x|^p\leqq u(x)\leqq C|x|^p\,\Pi_{k=1}^n\log_k|x|$$

for $|x|$ sufficiently large. For example, instead of (10) the condition

$$\omega_r(x)\leqq c\left(\frac{r}{|x|}\right)^{-p}\left(\Pi_{k=1}^{n-1}\frac{\log_k r}{\log_k|x|}\right)^{-1}\left(\frac{\log_n r}{\log_n|x|}\right)^{-a}$$

can be assumed to be satisfied for some $a>1$ and $r\geqq r_0|x|>|x|$ sufficiently large.

4. Harmonic majorization of other subharmonic functions

In Section 3 there is a natural connection between the derivative of the subharmonic function r^p, where $r=|z|$ in the complex plane and $r=|x|$ in R^d, and the estimates of harmonic measures. Some generalizations to general subharmonic functions $\Phi(r)$ can be given. In Section 1, Burkholder's inequalities between the least harmonic majorants of $|x|^p$ and expectations $E_x\tau^{p/2}$ are referred to. For his generalizations to subharmonic functions $\Phi(|x|)$ see [2, Theorem 3.5 and Theorem 3.6, p. 197].

For notation see Section 2. Some assumptions about Φ are introduced:

$$(15)\quad \begin{cases} \Phi \text{ continuously differentiable, non-decreasing with } \Phi(0)=0,\ \Phi(\infty)=\infty, \\ \Phi(2t)\leqq c\Phi(t), \quad t>0, \\ \Phi(|x|) \text{ subharmonic in } D\subset R^d. \end{cases}$$

Tsuji's result in Theorem 1 can immediately be generalized in the following way.

Theorem 3. *Let Φ satisfy conditions* (15). *Assume that the estimate $\omega_r(x_0)\leqq g(r)$, $r\geqq |x_0|>0$, holds for some fixed $x_0\in D_r$ and that*

$$(16)\qquad \int_{|x_0|}^{\infty} \Phi'(r)\,g(r)\,dr < \infty.$$

Then $\Phi(|x|)$ has a least harmonic majorant $u(x)$ in D. If, further, for some constants r_0 and C_1,

$$(17)\qquad \omega_r(x) \leqq g\left(\frac{r}{|x|}\right) \quad \textit{for} \quad r\geqq r_0|x|>0,$$

and

$$(18)\qquad |x|\,\Phi'(|x|\,t) \leqq C_1\Phi'(t)\,\Phi(|x|), \quad t\geqq r_0,$$

then

$$(19)\qquad \Phi(|x|) \leqq u(x) \leqq C\Phi(|x|), \quad x\in D.$$

Remark 5. For convex Φ satisfying (15) the inequality (18) is guaranteed by

$$(18a)\qquad \Phi(t|x|) \leqq C_2\Phi(|x|)\Phi(t).$$

Indeed, if Φ is convex, it follows from (15) that $\Phi(|x|)\leqq |x|\,\Phi'(|x|)\leqq (c-1)\,\Phi(|x|)$. Thus it follows from (18a) that Φ' satisfies the same type of condition as (18a). Hence $|x|\,\Phi'(t|x|)\leqq C_3|x|\,\Phi'(t)\,\Phi'(|x|)\leqq C_4\Phi'(t)\,\Phi(|x|)$, which yields (18).

(If, for instance, $\Phi(|x|)=\log|x|$ for large $|x|$, this deduction of (18) from (18a) does not work; however, (18) and (18a) are satisfied.)

Proof of Theorem 3. Cf. the proof of Theorem 2. In particular, the estimate from above follows from

$$(20)\qquad u(x) = \int_0^\infty \Phi'(r)\,v_r(x)\,dr \leqq \int_0^{r_0|x|} \Phi'(r)\,dr + \int_{r_0|x|}^{\infty} \Phi'(r)\,g\left(\frac{r}{|x|}\right)dr$$
$$\leqq \Phi(r_0|x|) + \int_{r_0}^{\infty} \Phi'(t\,|x|)\,g(t)\,|x|\,dt.$$

The term $\Phi(r_0|x|)$ is taken care of if we use (15). The desired estimate of the last integral in (20) follows from (16) and (18).

Remark 6. The assumptions of Theorem 3 are satisfied for large $|x|$ if $\Phi(|x|)=|x|^p(\log|x|)^q$, $p>0$, $q>0$, and $g(r)=Cr^{-p_0}$, $p_0>p$. With the same $g(r)$ but $q<0$, (18) and (18a) are not satisfied, but the condition $p_0>p$ is sufficient to guarantee (19) for large $|x|$.

Remark 7. *Generalization of Theorem* 2. Let Φ satisfy the conditions (15). Assume that the estimate $\omega_r(x_0) \leqq g(r)h(\log r)$, $r \geqq |x_0| > 1$, holds for some fixed $x_0 \in D_r$ and that the functions Φ', g, h and l are such that

$$\int_{|x_0|}^{\infty} \Phi'(r)\, g(r)\, h(\log r)\, dr \leqq \int_{|x_0|}^{\infty} r^{-1} l(\log r)\, dr < \infty.$$

Then $\Phi(|x|)$ has a least harmonic majorant $u(x)$ in D.

If, further, for some constants $r_0 > 1$ and $\gamma > 1$,

$$\omega_r(x) \leqq g\left(\frac{r}{|x|}\right) h\left(\frac{\log r}{\log |x|}\right), \quad \text{for} \quad r \geqq r_0 |x|, \ |x| \geqq |\gamma|,$$

and

$$|x|\, \Phi'(|x|\, t)\, g(t)\, h\left(\frac{\log(|x|\, t)}{\log |x|}\right) \leqq C_1 \Phi(|x|)\, t^{-1} l\left(\frac{\log(|x|\, t)}{\log |x|}\right), \quad \text{for} \quad t \geqq r_0, \quad |x| \geqq |\gamma|,$$

then

$$\Phi(|x|) \leqq u(x) \leqq C\Phi(|x|) \log |x|, \quad |x| \geqq |\gamma|, \quad x \in D.$$

Proof. Cf. the proof of Theorem 2, in which $h(r) = l(r) = r^{-a}$, $a > 1$, $\Phi(r) = r^p$, and that of Theorem 3. Here the substitution $(\log(|x|t))/\log |x| = \log y$ is appropriate to round off the proof.

As an example one can take $\Phi(r) = r^p (\log r)^q$, $p > 0$, for large r and $g(r) = r^{-p}$, $h(r) = r^{-q-a}$ and $l(r) = r^{-a}$ with $a > 1$. That the magnitude of the estimate of u from above can be attained for $q > 0$ can be seen by comparison with Example 1, with δ now depending also on q.

We note that the condition on $\omega_r(x)$ in this remark, as in Theorem 2, contains $(\log |x|)/\log r$ rather than $\log(|x|/r)$ as in Theorem 3, and that this particular form occurs in known estimates (cf. Example 1). As a constructed example containing $\log(|x|/r)$ let us take $\Phi(|x|) = |x|^p/\log |x|$ for large $|x|$ and $g(r) = r^{-p}/\log r$ for large r in the setting of Theorem 3. Then (18) is no longer satisfied and an evaluation of the last integral in (20) yields the estimate $u(x) \leqq C\Phi(|x|) \log\log |x|$ for large $|x|$.

Thus, on the whole, various results of the interplay between Φ' and the estimates of ω_r are possible.

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