ENTIRE FUNCTIONS WITH SPIRAL LIMITS

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It is (perhaps) well-known that given $0=t_0<t_1<\ldots<t_n=2\pi$ and complex numbers $c_1, \ldots, c_n$, there exists an entire function $f$ such that $\lim_{r \to \infty} f(re^{i\theta})=c_j$ for each $t_j=(t_{j-1}, t_j)$ and each $j=1, 2, \ldots, n$. (See G. Pólya [7] and Exercises IV 185—IV 186 of G. Pólya and G. Szegő [8].) In a similar vein K. Grandjot [5] proved the existence of a non-zero entire function $f$ such that $f(z)\to 0$ as $z\to \infty$ along any algebraic curve (cf. also H. Bohr [2]). The review Mathematical Reviews 52 # 8433 gives a brief historical account of this subject; it should be supplemented by Paragraphs 21 and 43 of the encyclopedia article [1] of L. Bieberbach. For some related results, we mention the anecdotal paper [10] by W. J. Schneider. In this note, we shall construct an entire function with "spiral" limits, where the limits are finitely many preassigned polynomials. Our method uses the well-known technique of shoving poles to infinity.

For each $p>0$, let $S_p$ denote the class of all continuously differentiable real-valued functions $\theta$ on $(0, \infty)$ such that

$$\int_1^\infty |\theta'(t)| \cdot t^{-p} \, dt < \infty.$$ (i)

Thus every function of the form $P(t) \cos Q(t) + R(t) \sin S(t)$ belongs to $S_p$ for some $p>1$, where $P, Q, R, S$ are polynomials with real coefficients. By a spiral region we mean an open set in the complex plane $C$ of the form

$$\Omega = \{re^{it} : r > 0 \text{ and } \theta_1(r) < t < \theta_2(r)\},$$ (ii)

where $\theta_1, \theta_2 \in S_p$ for some $p>0$ and $\theta_1(t) < \theta_2(t) \leq \theta_1(t) + 2\pi$ for all $t>0$. As is customary, we shall often identify a curve with its image set.

Theorem. Let $\Omega_1, \ldots, \Omega_k$ be pairwise disjoint spiral regions, $E_j \subset \Omega_j$ unbounded closed subsets of $C$, $P_j(z)$ polynomials in $z \in C$ for $j=1, 2, \ldots, k$, and $N$ a natural number. Then there exists an entire function $g$ such that

$$g(z) = P_j(z) + o(|z|^{-N}) \text{ as } z \in E_j \text{ tends to } \infty$$

for each $j=1, 2, \ldots, k$.

Corollary. Let $P$ be a polynomial, and $N$ a natural number. Then there exists a non-polynomial entire function $h$ such that $h(z) = P(z) + o(|z|^{-N})$ as $z \to \infty$ along any algebraic curve.

To prove these results, we need three lemmas.

**Lemma 1.** Suppose that \( \theta_1, \theta_2 \in S_p \) for some \( p > 1 \), and that \( \theta_1 < \theta_2 < \theta_3 \) on \( (0, \infty) \), where \( \theta_3 = \theta_1 + 2\pi \). For \( j = 1, 2 \), let

\[
\Omega_j = \{ |re^{it}| > 0 \quad \text{and} \quad \theta_j(r) < t < \theta_{j+1}(r) \},
\]

and let \( g_j \) be a holomorphic function on \( \Omega_j \) such that

\[
z \in \Omega_j \quad \text{and} \quad |z| > C \Rightarrow |g_j'(z)| < C/|z|^{2N}
\]

for some \( C \) and \( N > p + 1 \). Then there exist \( c_j \in C \) such that

\[
g_j(z) = c_j + o(|z|^{-N}) \quad \text{as} \quad z \in \Omega_j \quad \text{tends to} \quad \infty \quad (j = 1, 2).
\]

**Proof.** For \( j = 1, 2 \) and \( t \geq 0 \), define

\[
\tau_j(t) = 2^{-1}\{\theta_j(t) + \theta_{j+1}(t)\} \quad \text{and} \quad \gamma_j(t) = te^{itj(t)}.
\]

We fix \( j \), and write \( \tau = \tau_j \), \( \gamma = \gamma_j \), etc. Since \( N - 1 > p > 1 \), we have

\[
\int_1^\infty (1 + t|\tau'(t)|)t^{-N} dt \leq \int_1^\infty (1 + |\tau'_j(t)| + |\tau'_{j+1}(t)|)t^{-N} dt < \infty
\]

by (i).

Notice that \( \gamma((0, \infty)) \subset \Omega_j \), \( |\gamma(t)| = t \), and \( \gamma'(t) = (1 + i\tau'(t))e^{itj(t)} \) for all \( t > 0 \) by (1). It follows from (b) that \( s > r > C \) implies

\[
|g(\gamma(s)) - g(\gamma(r))| = \left| \int_r^s g'(\gamma(t))\gamma'(t) dt \right| \leq \int_r^s (1 + t|\gamma'(t)|) dt
\]

\[
= C e^{-N} \int_r^\infty (1 + t|\tau'(t)|)t^{-N} dt.
\]

From (2) and (3) we infer that \( g(\gamma(s)) \) converges to some complex number \( c = c_j \) as \( s \to \infty \), and that

\[
|c - g(\gamma(r))| = o(r^{-N}) \quad \text{as} \quad r \to \infty.
\]

Now suppose that \( z = re^{it} \in \Omega_j \), where \( r > C \) and that \( \theta_j(r) < s < \theta_{j+1}(r) \). Then we have

\[
|g(\gamma(r)) - g(z)| \leq \int_s^r (g'(re^{it})ire^{it} dt < 2\pi C/r^{2N-1}
\]

by (1) and (b), so

\[
|c - g(z)| \leq |c - g(\gamma(r))| + 2\pi C/r^{2N-1}.
\]

This inequality, combined with (4), yields the desired conclusion.

**Lemma 2.** Let \( \theta_j \), \( p \), and \( \Omega_j \) \((j = 1, 2)\) be as in Lemma 1. Fix a positive real number \( r_0 \) and a natural number \( N > p + 1 \). Let

\[
\alpha = r_0 \exp[i\theta_1(r_0)] \quad \text{and} \quad \beta = r_0 \exp[i\theta_2(r_0)],
\]
and let \( g(z) \) be a holomorphic antiderivative of \( [(z-x)(z-\beta)]^{-N} \) in the simply connected region

\[
\Omega_0 = \mathbb{C} \setminus \bigcup_{j=1}^{2} \{ r \exp[i\theta_j(r)]: r \equiv r_0 \}.
\]

Then there exist complex numbers \( c_1, c_2 \) such that

\[
g(z) = c_j + o(|z|^{-N}) \quad \text{as} \quad z \in \Omega_j \text{ tends to } \infty
\]

for \( j=1, 2 \). Moreover, \( c_1 \neq c_2 \).

**Proof.** The existence of \( c_j \) satisfying (c) is an immediate consequence of Lemma 1. So we only need to check that \( c_1 \neq c_2 \).

Let \( \gamma_1 \) and \( \gamma_2 \) be the two infinite curves defined as in the proof of Lemma 1. Notice that both \( \gamma_1 \) and \( \gamma_2 \) lie in \( \Omega_0 \), that \( \gamma_2 = -\gamma_1 \) and that

\[
g(v) - g(u) = \int_{\Gamma} (z-x)^{-N}(z-\beta)^{-N} \, dz \quad (u, v \in \Omega_0),
\]

where \( \Gamma = \Gamma(u, v) \) is any smooth curve in \( \Omega_0 \) from \( u \) to \( v \). Now pick any \( r > r_0 \), and consider the closed curve \( \gamma \) consisting of the following three pieces: \( \gamma_2(r-t) \) for \( 0 \leq t \leq r \), \( \gamma_1(t-r) \) for \( r \leq t \leq 2r \), and the semicircle \( C_r(t) = \gamma_1(r) \exp[i(t-2r)] \) for \( 2r \leq t \leq 2r + \pi \).

It is easy to check that \( x \) and \( \beta \) lie "outside" and "inside" of \( \gamma \), respectively.

It follows from (1) and Cauchy's residue theorem that

\[
g(\gamma_1(r)) - g(\gamma_2(r)) + \int_{C_r} (z-x)^{-N}(z-\beta)^{-N} \, dz = 2\pi i \text{Res}(\beta),
\]

where

\[
\text{Res}(\beta) = \frac{[2(N-1)!]}{[(N-1)!]^2} (-1)^{N-1}(\beta-x)^{1-2N} \neq 0.
\]

But it is routine to show that \( \lim_{r \to \infty} \int_{C_r} = 0 \). Letting \( r \to \infty \) in (2), we therefore conclude from (c) and (3) that \( c_1 - c_2 + 0 \neq 0 \), as desired.

Now we write \( P^*(w) = P(1/w) \) for a polynomial \( P \) and \( w \neq 0 \).

**Lemma 3.** Let \( E \) be a closed subset of \( \mathbb{C} \), \( K \) a compact connected subset of \( \mathbb{C} \setminus E \), and \( u, v \in K \). If \( N \) is a nonnegative integer, \( \varepsilon > 0 \), and \( R_1 \) is a polynomial, then there exists a polynomial \( R_2 \) such that

\[
|R_1^*(z-u) - R_2^*(z-v)| < \varepsilon/(2+|z|^N) \quad \forall z \in E.
\]

**Proof.** For \( N = 0 \), this is a consequence of Runge's theorem. (Indeed, it can be proved by an elementary method.) See, for example, Chapter IV, Paragraph 1 of S. Saks and A. Zygmund [9].

So assume that \( N \geq 1 \) and that the result is true with \( N \) replaced by \( N-1 \). Apply this inductive hypothesis to \( R_1(z) = (u-v)z \) to find a polynomial \( Q \) such that

\[
|(u-v)(z-u)^{-1} Q^*(z-v)| < \varepsilon/(2+|z|^{N-1}) \quad \forall z \in E.
\]
Divide both sides of this inequality by \(|z-v|\) to obtain

\[
|\frac{1}{z-u} - \{\frac{1}{z-v} + (z-v)^{-1}Q^*(z-v)\}| < \frac{\epsilon}{|z-v|(2 + |z|)^{N-1}} \quad \forall z \in E.
\]

But \((2 + |z|)/|z-v|\) is bounded on \(E\) and \(\epsilon > 0\) is arbitrary. Thus we conclude that there exists a polynomial \(R\) such that

\[
|\frac{1}{z-u} - R^*(z-v)| < \epsilon(2 + |z|)^{N} \quad \forall z \in E,
\]

which establishes the desired result for \(R_1(z)\). Since \((z-u)^{-1}\) is bounded on \(E\), (3) shows that \(R^*(z-v)\) is bounded on \(E\). Therefore the general case follows from this special case combined with the elementary formula \(A^n - B^n = (A - B)(A^{n-1} + \ldots + B^n)\). This completes the induction and hence the proof.

**Proof of the Theorem.** First consider the case \(k=1\). In this case we may assume that \(\Omega_1\) is the complement of a curve \(\Gamma\) of the form \(\Gamma(t) = te^{i\theta(t)}\) for \(t \geq 0\), where \(\theta\) is in \(S_p\) for some \(p > 0\). Then the hypothesis on the closed unbounded set \(E_1\) is that it be disjoint from \(\Gamma\). For such an \(E_1\), it is easy to construct an infinitely differentiable function \(\delta\) on \((0, \infty)\) such that \(0 < \delta(t) < 2\pi\) and \(|\delta(t)| < 1\) for all \(t > 0\) and such that

\[E_1 \subset \{e^{it}: r > 0\text{ and } \theta(r) < t < \theta(r) + 2\pi - \delta(r)\}.
\]

Therefore the case \(k=1\) can be reduced to the case \(k=2\). Also the desired result for \(k \geq 3\) follows from \(k\) applications of the result for \(k=2\) as follows: for each pair of complementary spiral regions \(\Omega_j\) and \(C \setminus \Omega_j\) and respective polynomials \(P_j\) and \(0\), the result for \(k=2\) supplies us with an appropriate entire function \(g_j\) and for the desired function \(g\) we take \(g_1 + g_2 + \ldots + g_k\). Thus it will be sufficient to deal with the case \(k=2\).

So assume that \(k=2\) and also, without loss of generality, that \(\Omega_1\) and \(\Omega_2\) are defined by (a) in Lemma 1. Let \(E_j\) be a closed unbounded set contained in \(\Omega_j\) for \(j=1, 2\). Choose and fix an infinitely differentiable function \(\delta\) on \((0, \infty)\), with bounded derivative, such that

\[
0 < \delta < 4^{-1}\min \{\theta_2 - \theta_1, \theta_3 - \theta_2\} \text{ on } (0, \infty), \quad \text{and for } j = 1, 2
\]

\[
E_j \subset \{e^{it}: r > 0 \text{ and } \theta_j^+(r) < t < \theta_{j+1}^-(r)\} \quad \text{def } \Omega_j^*,
\]

where

\[
\theta_j^+ = \theta_j + \delta \quad \text{and} \quad \theta_j^- = \theta_j - \delta \quad (j = 1, 2, 3).
\]

Now let \(\epsilon > 0\) and a natural number \(q > p + 1\) be given. Put

\[
U_n = \{e^{it}: r > n - 1/2, |t - \theta_1(t)| < \delta(r)\},
\]

\[
V_n = \{e^{it}: r > n - 1/2, |t - \theta_2(t)| < \delta(r)\},
\]

\[
\alpha_n = ne^{i\theta_1(n)} \quad \text{and} \quad \beta_n = ne^{i\theta_2(n)} \quad \text{for } n = 1, 2, \ldots.
\]
We shall construct two sequences of polynomials $Q_n$ and $R_n$ as follows. The rational function $f_1(z) = \frac{1}{(z-a_1)(z-b_1)^q}$ admits a representation of the form

$$f_1(z) = Q_1^*(z-a_1) + R_1^*(z-b_1)$$

for some polynomials $Q_1$ and $R_1$. Suppose that polynomials $Q_n$ and $R_n$ have been chosen for some $n \geq 1$. Notice that $a_n$ and $b_{n+1}$ are contained in an arc which is disjoint from the closed set $C \setminus U_n$. It follows from Lemma 3 that there exists a polynomial $Q_{n+1}^*$ such that

$$|Q_{n+1}^*(z-a_{n+1}) - Q_n^*(z-a_n)| < \varepsilon/(2 + |z|)^{2q_n} \quad \forall z \in C \setminus U_n.$$  

Similarly there exists a polynomial $R_{n+1}^*$ such that

$$|R_{n+1}^*(z-b_{n+1}) - R_n^*(z-b_n)| < \varepsilon/(2 + |z|)^{2q_n} \quad \forall z \in C \setminus V_n.$$ 

This completes our induction.

Now set $f_n(z) = Q_n^*(z-a_n) + R_n^*(z-b_n)$ for $n = 1, 2, \ldots$. Then (8) and (9) yield

$$|f_{n+1}(z) - f_n(z)| < 2\varepsilon/(2 + |z|)^{2q_n} \quad \forall z \in C \setminus (U_n \cup V_n).$$

It follows from (4), (5) and (10) that the rational functions $f_n$ converge to an entire function $f$ uniformly on each compact set. Moreover, we have

$$|f(z) - f_1(z)| < \sum_{n=1}^{\infty} \frac{2\varepsilon}{(2 + |z|)^{2q_n}} \equiv \frac{4\varepsilon}{(2 + |z|)^{2q}} \quad \forall z \in C \setminus (U_1 \cup V_1).$$

Let $g$ be the antiderivative of $f$ with $g(0) = 0$, and let $g_1$ be the antiderivative of $f_1$ with $g_1(0) = 0$ in

$$C \setminus \bigcup_{j=1}^{2} \{ t \exp[i\theta_j(t)] : t \equiv 1 \}.$$ 

By Lemma 2, there exist distinct complex numbers $c_1$ and $c_2$ such that

$$g_1(z) = c_j + o(|z|^{-q}) \quad \text{as} \quad z \in \Omega_j \quad \text{tends to} \quad \infty$$

for $j = 1, 2$. By the definition of $f_1$ and (11), we can find $C > 1$ so large that $|f(z)| < C/|z|^{2q}$ for all $z$ in $C \setminus (U_1 \cup V_1)$ with $|z| > C$. It follows from (two applications of) Lemma 1 and (1)—(5) that there exist two complex numbers $b_1$ and $b_2$ such that

$$g(z) = b_j + o(|z|^{-q}) \quad \text{as} \quad z \in \Omega_j^* \quad \text{tends to} \quad \infty$$

for $j = 1, 2$. We claim that $b_1 \neq b_2$, provided that $\varepsilon > 0$ is small enough.

Indeed, let $\tau_j$ and $\gamma_j$ be as in the proof of Lemma 1:

$$\tau_j(t) = \{ \theta_j(t) + \theta_{j+1}(t) \}/2 \quad \text{and} \quad \gamma_j(t) = te^{\tau_j(t)} \quad \text{for} \quad t \equiv 0.$$ 

According to (1)—(5) we have

$$\gamma_j(0, \infty) \subset \Omega_j^* \subset C \setminus (U_1 \cup V_1), \quad j = 1, 2.$$
Since \( g(0) = g_1(0) = 0 \), it follows from (11) and (14) that \( r > 0 \) implies

\[
|g(y_j(t)) - g_1(y_j(t))| = \left| \int_0^r \{ f(y_j(t)) - f_1(y_j(t)) \} y_j(t) \, dt \right|
\]

\[
< 4\varepsilon \int_0^\infty \frac{1 + t(\theta_j'(t) + |\theta_j'(t)|)}{(2 + |t|)^{2\alpha}} \, dt = 4\varepsilon B, \quad \text{say.}
\]

Notice that \( B \) is a finite constant which is independent of \( \varepsilon \). Letting \( r \to \infty \) in (15) we obtain from (12) and (13) that \( \|b_j - c_j\| \leq 4\varepsilon B \) for \( j = 1, 2 \). Hence \( \|b_1 - b_2\| \leq |c_1 - c_2| - 8\varepsilon B > 0 \), provided that \( \varepsilon > 0 \) is small enough, which confirms our claim. Upon setting \( h = \alpha + \beta \) for appropriate coefficients \( \alpha \) and \( \beta \), we therefore obtain an entire function \( h \) such that

\[
h(z) = \begin{cases} 
  a + o(|z|^{-\theta}) & \text{as } z \to 1, \\
  b + o(|z|^{-\theta}) & \text{as } z \to \infty,
\end{cases}
\]

where \( a \) and \( b \) are arbitrary, but preassigned, complex numbers.

Finally let \( P_1 \) and \( P_2 \) be two given polynomials. Write

\[
P_1(z) = \sum_{k=0}^M a_k z^k \quad \text{and} \quad P_2(z) = \sum_{k=0}^M b_k z^k.
\]

Choose a natural number \( q > M + N + p \) and entire functions \( h_k \) which behave as in (16) with \( a = a_k \) and \( b = b_k \) \((k = 0, 1, \ldots, M)\). Put \( F(z) = \sum_{k=0}^M z^k h_k(z) \). It is evident that \( F \) has the required properties.

**Proof of the Corollary.** Let \( f \) be any continuously differentiable, positive real-valued function on \((0, \infty)\) for which there exists \( p > 0 \) such that

(1) \[ x + f'(x) f(x) > 0 \quad \text{for all } x \equiv p, \]

and

(2) \[ \int_p^\infty |f'(x) x - f(x)| \cdot (x^2 + f(x)²)^{-p/2 - 2} \, dx < \infty. \]

The graph of such a function is essentially a curve of the kind which bounds spiral regions. To see this, define two functions \( t \) and \( \theta \) of \( x \) by setting \( t(x) = (x^2 + f(x)²)^{1/2} \) and \( \theta(x) = \arctan \left( f(x)/x \right) \). Then \( dt/dx = (x + f'(x) f(x))/t > 0 \) for all \( x \equiv p \) by (1), so \( t \) has a continuously differentiable inverse on \([b, \infty)\), where \( b = t(p) \). Accordingly, \( \theta \) may be regarded as a function of \( t \equiv b \). A simple calculation shows that

\[
\int_b^\infty \left| \frac{d\theta}{dt} \right| t^{-p} \, dt = \int_p^\infty \left| \frac{d\theta}{dx} \right| t^{-p} \, dx
\]

which is finite by (2). Thus (after an extension to \((0, b)\)) the function \( \theta \) belongs to \( S_p \).

It is evident that \( e^x \) satisfies (1) and (2) for all \( p > 0 \). Put

(3) \[ H = \{(x, y): x \equiv 1 \quad \text{and} \quad 2^{-1} e^x \equiv y \equiv 2e^x\}. \]

By the result established in the preceding paragraph, \( C \setminus H \) is essentially contained in a spiral region \( \Omega \). (More precisely, \( C \setminus H \) is contained in the union of \( \Omega \) and a
bounded set.) Given a natural number \( N \), our theorem yields an entire function \( g \) such that \( g(z) = o(|z|^{-N}) \) as \( z \in \Omega \) tends to \( \infty \). Indeed, such a \( g \) can be chosen so that \( g(z) = 1 + o(|z|^{-N}) \) as \( z \to \infty \) along the curve \( y = e^x \) for \( x > 0 \).

Now let \( \Gamma \) be an arbitrary algebraic curve. Thus, by definition, there exist finitely many polynomials \( Q_0, Q_1, \ldots, Q_n \) in \( x \), with \( Q_n \neq 0 \), such that each point \( (x, y) \) of \( \Gamma \) satisfies
\[
Q_0(x) + Q_1(x)y + \ldots + Q_n(x)y^n = 0.
\]
It is easily seen from (3) that if \( (x, y) \) is in \( H \) and \( x \) or \( y \) is large enough, then \( (x, y) \) does not satisfy (4). In other words, only a bounded portion of \( \Gamma \) lies outside \( \Omega \); hence \( g(z) = o(|z|^{-N}) \) as \( z \to \infty \) along \( \Gamma \). It follows that, given a polynomial \( P \), the entire function \( g + P \) has the required properties.

References


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Received 31 January 1984