ON FUNCTIONS WITH A FINITE OR LOCALLY BOUNDED DIRICHLET INTEGRAL

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1. Introduction

The modulus of a curve family is an effective tool in the geometric function theory and in the theory of quasiconformal and quasiregular mappings (cf. [1], [13], [18], [19]). The aim of this paper is to show that the modulus method applies to the study of real-valued functions as well. We shall give an approach to the theory of Dirichlet finite functions, which is based on the well-known connection between the Dirichlet integral of an ACL function $u$ and the modulus of the family of curves joining given level sets of $u$. In this approach the modulus method has a role similar to that of the length-area principle in [7] and [17]. We shall extend some results, which were previously known in the case of quasiconformal mappings, to the case of Dirichlet finite functions (cf. [21], [24]).

A function $u: R_n^+ \rightarrow R$ is said to have a finite Dirichlet integral (or to be Dirichlet finite) if $u$ is ACL and if

$$\int_{R_n^+} |\nabla u|^n \, dm < \infty.$$  \hspace{1cm} (1.1)

In this paper all functions are required to be continuous. We shall consider the following weaker condition: $u$ is ACL and there are numbers $M, B \in (0, \infty)$ such that

$$\int_{D(x,M)} |\nabla u|^n \, dm \leq B$$  \hspace{1cm} (1.2)

for all $x \in R_n^+$, where $D(x,M)$ is the hyperbolic ball in $R_n^+$ with the centre $x$ and radius $M$. If (1.2) holds, $u$ is said to have a locally bounded Dirichlet integral.

In the preliminary Section 2 we prove, using the modulus method, that functions of $R_n^+$ which are monotone (in the sense of Lebesgue) and have a locally bounded Dirichlet integral are uniformly continuous with respect to the hyperbolic metric of $R_n^+$. In Section 3 we prove that a monotone function satisfying (1.1) and having a limit $\alpha$ at 0 through a set $E \subset R_n^+$, has in fact an angular limit $\alpha$ at 0 provided that the lower capacity density of $E$ at 0 is positive, $\text{cap dens}(E, 0) > 0$. Similar results for some other classes of functions were given in [21] and [22]. An example is given to show that the condition $\text{cap dens}(E, 0) > 0$ is not sufficient here. As an application of the results of Section 3 we can prove that a Dirichlet finite quasiregular mapping

2. Preliminary results

We shall follow, as a rule, the notation and terminology of Väisälä's book [18], which the reader is referred to for some definitions etc. Some notation will be introduced at first.

2.1. For $x \in \mathbb{R}^n, n \geq 2$, and $r > 0$, let $B^n(x, r) = \{z \in \mathbb{R}^n: |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$. If $x \in \mathbb{R}^n$ and $b > a > 0$, then we write $R(x, b, a) = B^n(x, b) \setminus B^n(x, a)$ and $R(b, a) = R(0, b, a)$. The standard coordinate unit vectors are $e_1, \ldots, e_n$. If $A \subset \mathbb{R}^n$, let $A_+ = \{x = (x_1, \ldots, x_n) \in A: x_n > 0\}$. The hyperbolic metric $\varrho$ in $\mathbb{R}^n_+$ is defined by the element of length $d\varrho = |dx|/x_n$. If $x \in \mathbb{R}^n_+$ and $M > 0$, we write $D(x, M) = \{z \in \mathbb{R}^n_+: \varrho(z, x) < M\}$. A well-known fact is that the hyperbolic balls are balls in the euclidean geometry as well, for instance

$$D(te_n, M) = B^n((t \cosh M)e_n, t \sinh M), \quad t > 0.$$  

For $x, y \in \mathbb{R}^n_+$ the following formula holds [2, (3.3.4) p. 35]:

$$\cosh \varrho(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n}.$$  

Sometimes we shall regard $B^n$ as a hyperbolic space as well and use the same symbols as in the case of $\mathbb{R}^n_+$. The hyperbolic metric $\varrho$ is then defined by $d\varrho = 2|dx|/(1 - |x|^2)$. The counterpart of (2.2) for $B^n$ is

$$D(x, M) = B^n(y, r) \begin{cases} y = \frac{x(1 - \tanh^2(M/2))}{1 - |x|^2 \tanh^2(M/2)} \\ r = \frac{(1 - |x|^2) \tanh(M/2)}{1 - |x|^2 \tanh^2(M/2)} \end{cases}.$$  

2.5. Monotone functions. Let $G \subset \mathbb{R}^n$ be an open set. A continuous function $u: G \to \mathbb{R}$ is said to be monotone (in the sense of Lebesque) if

$$\max_D u(x) = \max_{\partial D} u(x), \quad \min_D u(x) = \min_{\partial D} u(x),$$

whenever $D$ is a domain with compact closure $\bar{D} \subset G$. 

\[ f: \mathbb{R}^n_+ \to \mathbb{R}^n, \quad f = (f_1, \ldots, f_n), \] will have an angular limit $(x_1, \ldots, x_n)$ at 0 if each coordinate function $f_j$ has a limit $x_j$ through a set $E_j$ with cap $\text{dens}(E_j, 0) > 0$. A related result for bounded analytic functions was proved by F. W. Gehring and A. J. Lohwater [4]. In Section 4 we shall give an example illustrating the behaviour of a monotone function satisfying (1.2). The topic of Section 5 is the behaviour of a monotone function at an isolated singularity. In the final section, Section 6, we prove a variant of the Iversen—Tsuji theorem for monotone Dirichlet finite functions.
2.6. Remark. It follows from the above definition that if $t \in \mathbb{R}$, then each component $A \neq \emptyset$ of the set \{z \in G: u(z) > t\} is not relatively compact, i.e., $\overline{A} \cap (\partial G \cup \{\infty\}) \neq \emptyset$. A similar statement holds if $>$ is replaced by $\leq$, $\leq$, or $\leq$. Hence monotone functions obey a sort of maximum principle.

2.7. The modulus of a curve family. Let $\Gamma$ be a family of curves in $\mathbb{R}^n$. The modulus of $\Gamma$ is defined by

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{\mathbb{R}^n} \varrho \, dm,$$

where $F(\Gamma)$ is the family of all non-negative Borel functions $\varrho: \mathbb{R}^n \to [0, \infty]$ with $\int_{\mathbb{R}^n} \varrho \, ds \leq 1$ for all locally rectifiable $\gamma \in \Gamma$. For the properties of the modulus the reader is referred to Väisälä's book [18]. If $E, F, G \subseteq \mathbb{R}^n$, then $\Delta(E, F; G)$ is the family of all curves $\gamma \subseteq G$ joining $E$ to $F$ in the following sense: $\gamma \cap E \neq \emptyset \neq \gamma \cap F$.

2.8. Condenser and its capacity. A pair $(A, C)$ is said to be a condenser if $A \subseteq \mathbb{R}^n$ is open and $C \subseteq A$ is compact. The capacity of $E=(A, C)$ is defined by

$$\text{cap } E = \text{cap } (A, C) = \inf_{u} \int_{\mathbb{R}^n} |\nabla u|^n \, dm,$$

where $u$ runs through the set of all ACL$^n$ functions with compact support in $A$ and with $u(x) \equiv 1$ for $x \in C$. An alternative definition is

$$\text{cap } (A, C) = M(A(C, \partial A; \mathbb{R}^n)) = M(A(C, \partial A; A)).$$

Ziemer [26] proved that (2.9) and (2.10) agree for bounded $A$, from which the same conclusion for unbounded $A$ follows easily. A compact set $E$ in $\mathbb{R}^n$ is said to be of capacity zero if $\text{cap } B^n(t), E) = 0$ for some $t > 0$ such that $E \subseteq B^n(t)$.

The following lower bound for the Dirichlet integral of an ACL$^n$ function will be applied several times in what follows. In fact, it is the easy part in the proof of (2.10). The proof is standard (cf. [1, p. 65], [9, p. 577], [26, Lemma 3.1]).

2.11. Lemma. Let $u: G \to \mathbb{R}$ be an ACL$^n$ function, $-\infty < a < b < \infty$, and let $A, B \subseteq G$ be such that $u(x) \leq a$ for $x \in A$ and $u(x) \geq b$ for $x \in B$. Then

$$M(A(A, B;\mathbb{R}^n)) \leq (b-a)^{-n} \int_{G} |\nabla u|^n \, dm.$$
2.13. Lemma. Let \( u: B^n \to \mathbb{R} \) be a monotone ACL\( n \) function satisfying (1.2). Then
\[
|u(x) - u(y)|^n \leq C \left( \frac{1}{r} \right)^{-1} (1 - r)^{1 - n},
\]
where \( r = \tanh \left( \frac{q(x, y)}{4} \right) \) and \( C \) is a positive constant depending only on \( n, M, \) and \( B \) in (1.2). In particular, \( u: (B^n, q) \to (\mathbb{R}, |\cdot|) \) is uniformly continuous.

Proof. Clearly we may assume \( u(x) < u(y) \). Since the right side depends only on \( x \) and \( y \) through the Möbius invariant quantity \( q(x, y) \), we may assume \( x = re_1 = -y, \ r = \tanh \left( \frac{q(x, y)}{4} \right) \) (cf. (2.4)). Let
\[
E = \{ z \in B^n: u(z) \equiv u(x) \},
F = \{ z \in B^n: u(z) \equiv u(y) \},
\]
\[
\Gamma_r = B^n(\sqrt{r}).
\]
Then by Remark 2.6 and [18, 10.12]
\[
M(\Gamma_r) \equiv c_n \log \frac{1}{\sqrt{r}}.
\]
Lemma 2.11 yields
\[
M(\Gamma_r) \equiv |u(x) - u(y)|^{-n} \int_{B^n(\sqrt{r})} |\nabla u|^n \, dm.
\]
In view of (1.2) the integral can be estimated from above in terms of \( B \) and the following number
\[
k = \inf \left\{ p: B^n(\sqrt{r}) \subset \bigcup_{j=1}^p D(x_j, M) \right\}.
\]
An upper bound for \( k \) can be found by a method involving estimation of the hyperbolic volume of \( \bigcup \{ D(x, M): |x| \equiv \sqrt{r} \} \). (For details see [25, Section 9]). This method yields the estimate
\[
\int_{B^n(\sqrt{r})} |\nabla u|^n \, dm \leq d (1 - \sqrt{r})^{1 - n},
\]
where \( d \) is a positive number depending only on \( n, M, \) and \( B \). The desired estimate with \( C = 2^nd/c_n \) follows from this and the preceding estimates.

The uniform continuity in 2.13 can also be proved with the help of an oscillation inequality of Gehring [3, p. 355]. (See also Lelong—Ferrand [7, p. 7]). For \( n \)-dimensional version of the oscillation inequality see Mostow [12]. The oscillation inequality is applied in the proof of [18, 10.12], which was exploited above.

2.14. Remark. For large values of \( q(x, y) \) the above upper bound is not sharp. Indeed, one can replace the factor \( (\log (1/r))^{-1} (1 - r)^{1 - n} \) in 2.13 by \( (1 + \log ((1 + r)/(1 - r)))^{-1} \cdot \text{const.} \), which yields a better estimate for large values of
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Such an estimate can be deduced from the above version of Lemma 2.13, in particular, from the fact that \( u \) is uniformly continuous.

A continuous function \( v: \mathbb{R}^n_+ \to \mathbb{R} \cup \{0\} \), \( R_+ = \{x \in \mathbb{R}: x_1 > 0\} \), is said to satisfy a Harnack inequality if there exist numbers \( \lambda \in (0, 1) \) and \( C_\lambda \geq 1 \) such that (cf. [23])

\[
(2.15) \quad \max_{B^n(x, \lambda r)} v(z) \leq C_\lambda \min_{B^n(x, \lambda r)} v(z)
\]

whenever \( B^n(x, r) \subset R^n_+ \). The next result should be compared with [5, p. 200].

2.16. Corollary. If \( u: \mathbb{R}^n_+ \to \mathbb{R} \) is a monotone function satisfying (1.2), then \( e^u \) satisfies (2.15) for every \( \lambda \in (0, 1) \) with

\[
\log C_\lambda = C_\lambda^{1/n} \left( \frac{1}{\lambda} \right)^{-1/n} \left( \frac{1 + \lambda}{1 - \lambda} \right)^{(n-1)/(2n)},
\]

where \( C_\lambda \) is the number in 2.13.

Proof. Fix \( \lambda \in (0, 1) \). Choose \( B^n(x, r) \subset R^n_+ \). Then \( r \equiv r(x) \), where \( x = (x_1, \ldots, x_n) \), and \( \varrho(B^n(x, \lambda r)) \equiv \log ((1 + \lambda)/(1 - \lambda)) \) by (2.2) or (2.3). For \( z, y \in B^n(x, \lambda r) \) we get by 2.13 and by some elementary inequalities that

\[
|u(z) - u(y)| \leq C_\lambda^{1/n} \left( \frac{1}{\lambda} \right)^{-1/n} \left( \frac{1 + \lambda}{1 - \lambda} \right)^{(n-1)/(2n)},
\]

from which (2.15) for \( e^u \) follows.

We shall now give examples of functions satisfying (1.2) in \( B^n \).

2.17. The function \( u_F \). Let \( F \) be a relatively closed subset of \( B^n \). For \( x \in B^n \) set (cf. [23, 3.6])

\[
u_F(x) = \exp(-\varrho(x, F)).
\]

As shown in [23], the function \( u_F \) has some extremal properties for appropriate choices of \( F \). For \( x, y \in B^n \) and \( \varrho(x, F) \equiv \varrho(y, F) \) we get

\[
|u_F(x) - u_F(y)| \equiv |e^{-\varrho(x, F)} - e^{-\varrho(y, F)}| = e^{-\varrho(x, F)} |1 - e^{-a}|
\]

\[
\leq e^{-\varrho(x, F)} a \equiv e^{-\varrho(x, F)} \varrho(x, y) \equiv \varrho(x, y),
\]

where \( a = \varrho(y, F) - \varrho(x, F) \). Similarly \( |u_F(x) - u_F(y)| \equiv \varrho(x, y) \) for \( \varrho(x, F) \equiv \varrho(y, F) \) as well. Next we apply [22, 2.11] to get

\[
\varrho(x, y) \leq \log \left( 1 + \frac{|x-y|}{d(x) - |x-y|} \right) \leq \frac{|x-y|}{d(x) - |x-y|},
\]

for \( y \in B^n(x, d(x)) \) where \( d(x) = d(x, \partial B^n) \). Therefore

\[
\limsup_{y \to x} \frac{|u_F(x) - u_F(y)|}{|x-y|} \leq \limsup_{y \to x} \frac{1}{d(x) - |x-y|} = \frac{1}{d(x)}.
\]
It follows that $u_F$ is locally Lipschitz continuous and hence $ACL^n$. Fix $M > 0$. Then we get by (2.4) that

$$\int_{D(x, M)} |\nabla u_F(x)|^n \, dm(x) \leq \int_{D(x, M)} d(x)^{-n} \, dm(x) \leq c_1(n, M).$$

In conclusion, $u_F$ satisfies (1.2). Note, however, that $u_F$ is usually not monotone.

2.18. Remark. It follows from [5, p. 23, formula (2.32)] and (2.4) that all bounded harmonic functions satisfy (1.2).

3. Behaviour at an individual boundary point

For $\varphi \in (0, \pi/2)$ let $C(\varphi) = \{y \in R^+_n : (y|e_n) = |y| \cos \varphi\}$, where $(z|u)$ is the inner product $\sum z_i u_i$. A function $u : R^+_n \to R$ is said to have an angular limit $\alpha$ at $0$ if, for each $\varphi \in (0, \pi/2)$, $\lim_{x \to 0, x \in C(\varphi)} u(x) = \alpha$. A function $u$ is said to have an asymptotic value $\alpha$ at $0$ if there exists a continuous curve $\gamma : [0, 1) \to R^+_n$ with $u(\gamma(t)) \to \alpha$ and $\gamma(t) \to 0$ as $t \to 1$.

3.1. Lemma. Let $u : R^+_n \to R$ be a monotone $ACL^n$ function satisfying (1.2) and let $E \subset R^+_n$ be a measurable set such that

$$\lim_{r \to 0} m((R^+_n \setminus E) \cap B^n(r)) r^{-n} = 0.$$

If $u(x) \to \alpha$ as $x \to 0$ and $x \in E$, then $u$ has an angular limit $\alpha$ at $0$.

Proof. The proof follows from 2.13 and [23, 6.13].

3.2. Remarks. (1) It is not difficult to show that the assumption in 3.1 is equivalent to the condition that $u$ has an approximate limit $\alpha$ at $0$ [23, 6.3]. For the definition of an approximate limit see [23, 6.1]. For a related result see J. Lelong—Ferrand [7, p. 16] and [21, 5.9].

(2) The monotone $ACL$ function $u : R^+_n \to R$, $u(x, y) = \arctan(y/x)$, satisfies (1.2) (but not (1.1)) and has infinitely many distinct asymptotic values at $0$ but no angular limit at $0$. Hence an approximate limit in the hypotheses of 3.1 cannot be replaced by an asymptotic value.

For $E \subset R^n$, $x \in R^n$, and $t > r > 0$ set

$$M_t(E, r, x) = M(A(S^{n-1}(x, t), E \cap \overline{B^n}(x, r); R^n)),
M(E, r, x) = M_{2r}(E, r, x).$$

The lower and upper capacity densities of $E$ at $x$ are defined by

$$\text{cap dens}^-(E, x) = \lim_{r \to 0} \inf_{r > 0} M(E, r, x),$$
$$\text{cap dens}^+(E, x) = \lim_{r \to 0} \sup_{r > 0} M(E, r, x).$$
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3.3. Lemma. Let \( u: (\mathbb{R}^n_+, \| \cdot \|) \to (\mathbb{R}, | \cdot |) \) be uniformly continuous, let \( b_k \in \mathbb{R}^n_+, \ b_k \to 0 \), and let \( u(b_k) \to \beta \neq \infty \). For every \( \varepsilon > 0 \) there exists \( M \in (0, \infty) \) and \( p \geq 1 \) such that
\[
|u(x) - \beta| < \varepsilon, \quad x \in E_M = \bigcup_{k \geq p} D(b_k, M).
\]
Moreover, if there exists \( \varphi \in (0, \pi/2) \) such that \( b_k \in C(\varphi) \) for all \( k \), then there exists a positive number \( c \) depending only on \( n, \varphi \), and \( M \) such that
\[
\text{cap}_{\text{dens}}(E_M, 0) \geq c > 0.
\]

Proof. The first part follows from the definition of uniform continuity. For the proof of the second part we assume \( b_k \in C(\varphi), \ k = 1, 2, \ldots \). It follows from (2.2) that \( F_M \subset E_M \),
\[
F_M = \bigcup_{k \geq p} B^\varphi(b_k, b_k (1 - e^{-M})),
\]
where \( b_n \equiv |b_k| \cos \varphi \) is the \( n \)-th coordinate of \( b_k \). Hence the proof follows from [18, 10.12]. For more details see [21, 2.5 (2)].

3.4. Theorem. Let \( u: (\mathbb{R}^n_+, \| \cdot \|) \to (\mathbb{R}, | \cdot |) \) be a uniformly continuous and Dirichlet finite function and let \( E \subset \mathbb{R}^n_+ \) be a set with \( \text{cap}_{\text{dens}}(E, 0) > 0 \). If \( u(x) \to \alpha \) as \( x \to 0 \), \( x \in E \), then \( u \) has an angular limit \( \alpha \) at 0.

Proof. Fix \( \varphi \in (0, \pi/2) \). Suppose, on the contrary, that there exists a sequence \( (b_k) \) in \( C(\varphi) \) with \( b_k \to 0 \) and \( u(b_k) \to \beta \neq \alpha \). We shall assume that \( -\infty < \alpha < \beta < \infty \); in other cases the proof is similar. Let \( B_k \) be the \( b_k \)-component of the set \( B = \{ z \in \mathbb{R}^n_+: u(z) > (\alpha + 2\beta)/3 \} \) and let \( A = \{ z \in \mathbb{R}^n_+: u(z) < (2\alpha + \beta)/3 \} \). Since \( u \) is uniformly continuous, there exist by Lemma 3.3 numbers \( M > 0 \) and \( p \in \mathbb{N} \) such that \( D(b_k, M) \subset B_k \) for \( k \geq p \) and
\[
\text{cap}_{\text{dens}}(B, 0) \equiv c > 0.
\]
Since \( \text{cap}_{\text{dens}}(E, 0) > 0 \), it follows from [21, 4.3] and [20, 3.8] that
\[
M(\Delta(A, B; R^o_n)) \equiv M(\Delta(A, B; \mathbb{R}^o_n))/2 = \infty.
\]
A contradiction follows from (1.1) and 2.11.

3.5. Remark. The condition \( \text{cap}_{\text{dens}}(E, 0) > 0 \) is satisfied for instance if \( E \) is a curve terminating at 0. This fact follows from [18, 10.12]; for details and other sufficient conditions for \( \text{cap}_{\text{dens}}(E, 0) > 0 \) see [21]. On the other hand, there are compact sets \( E \) of zero Hausdorff dimension with \( \text{cap}_{\text{dens}}(E, 0) > 0 \) (cf. [21, 2.5 (3)]).

We shall now construct an example showing that the condition \( \text{cap}_{\text{dens}}(E, 0) > 0 \) would not suffice in 3.4.
3.6. Example. There exists a monotone $ACL^n$ function $u: R^n_+ \rightarrow R$ with a finite Dirichlet integral such that for a sequence $r_k \rightarrow 0$ with $0 < r_{k+1} < r_k$

(a) \[ u \left( \bigcup_{k=1}^{\infty} S_{r_k}^{n-1} (r_{2k-1}) \right) = 1, \]

(b) \[ u \left( \bigcup_{k=1}^{\infty} S_{r_k}^{n-1} (r_{2k}) \right) = 0. \]

Let $r_1 = 1$. Select $r_{k+1} \in (0, r_k/2)$, $k = 1, 2, \ldots$, such that

\[ \left( \log \frac{r_k}{r_{k+1}} \right)^{1-n} < 2^{-k}. \]

Define $u: R^n_+ \rightarrow R$ by $u(x) = 1$ for $x \in R^n_+ \setminus B^n$ and

\[ u(x) = \frac{\log |x| - \log r_{2k}}{\log (r_{2k-1}/r_{2k})}; \quad r_{2k-1} > |x| \equiv r_{2k}, \quad x \in R^n_+, \]

\[ u(x) = \frac{\log r_{2k} - \log |x|}{\log (r_{2k}/r_{2k+1})}; \quad r_{2k} > |x| \equiv r_{2k+1}, \quad x \in R^n_+, \]

for $k = 1, 2, \ldots$. It follows from (3.7) that (1.1) holds (cf. [18, 7.5]). Clearly $u$ is monotone and satisfies (a) and (b). In addition, if $\text{cap} \left\{ \text{dens} (B, 0) > 0 \right\}$ and $u(z) \rightarrow 0$, as $z \rightarrow 0$, $z \in B$, and $u$ fails to have an asymptotic value and hence an angular limit at 0. Therefore the condition $\text{cap} \left\{ \text{dens} (E, 0) > 0 \right\}$ in Theorem 3.4 cannot be replaced by \( \text{cap} \left\{ \text{dens} (E, 0) > 0 \right\} \).

3.8. Theorem. Let $u: R^n_+ \rightarrow R$ be a monotone and Dirichlet finite function, let \( (b_k) \subset R^n_+ \) be a sequence with \( q(b_k, b_h) \equiv 4M \) for \( k \neq h \), and let \( a_k \in D(b_k, M) \). For each \( \varepsilon > 0 \) let

\[ P_{\varepsilon} = \{ k \in N : \ |u(a_k) - u(b_k)| \equiv \varepsilon \}. \]

Then

\[ \text{card} \ P_{\varepsilon} \leq A \varepsilon^{-n} \int_{R^n_+} |\nabla u|^n \, dm, \]

where $A$ is a positive number depending only on $n$ and $M$.

Proof. Fix $\varepsilon > 0$. Let

\[ A_k = \{ z \in R^n_+ : \ |u(z) - u(a_k)| < \varepsilon/3 \}, \]

\[ B_k = \{ z \in R^n_+ : \ |u(z) - u(b_k)| < \varepsilon/3 \}, \]

\[ \Gamma_k = A(A_k, B_k; D(b_k, 2M)); \quad k \in P_{\varepsilon}. \]

From 2.6 it follows that the $a_k$-component of $A_k$ meets $\partial D(b_k, 2M)$, and that so does the $b_k$-component of $B_k$, when $k \in P_{\varepsilon}$. It follows from the conformal invariance of the modulus [18, 8.1], from $q(a_k, b_k) \equiv M$, and from (2.4) that

\[ M(\Gamma_k) \geq c_n \log \frac{\tanh M}{\tanh (M/2)} \geq c_n (\log 2) e^{-M} \]
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holds for \( k \in P \). Let \( A = \bigcup_{k \in P} A_k \), \( B = \bigcup_{k \in P} B_k \) and \( \Gamma = A(A, B; R^n) \). Since \( q(b_k, b_h) \geq 4M \) for \( k \neq h \), it follows that \( \{ \Gamma_k \}; k \in P \) are separate subfamilies of \( \Gamma \). Hence we get by [18, 6.7]

\[
\sum_{k \in P} M(\Gamma_k) \equiv M(\Gamma) \leq \left( \frac{3}{\ell} \right)^n \int_{R^n} |\nabla u|^n dm.
\]

The desired upper bound for \( \text{card } P \) follows from this and the preceding inequality with \( A = (3^n e^M)/(c_n \log 2) \).

3.9. Remark. A similar result holds for a uniformly continuous Dirichlet finite function \( u: (R^n, \rho) \to (R, | |) \) as well, but with a more complicated dependence on the Dirichlet integral and the modulus of continuity of the function.

3.10. Theorem. Let \( u: (R^n, \rho) \to (R, | |) \) be a uniformly continuous and Dirichlet finite function, let \( (b_k) \in R^n \), \( b_k \to 0 \), \( u(b_k) \to \beta \) and let \( M \in (0, \infty) \). Then \( u(x) \to \beta \) as \( x \to 0 \), \( x \in \bigcup D(b_k, M) \).

Proof. In the case of monotone functions the proof follows from 3.8. The general case follows from 3.9.

The next theorem has its roots in [24], where a similar result was proved for quasi-conformal mappings. We shall omit the proof since it parallels the proofs of Theorems 3.8 and that of [24, 4.9]. For the statement of the theorem the following condition is needed. Let \( (a_k) \) and \( (b_k) \) be sequences in \( R^n \) tending to 0 and let \( J_k = J[a_k, b_k] \) be the closed geodesic segment in the hyperbolic geometry joining \( a_k \) with \( b_k \). Thus \( J_k \) is the arc between \( a_k \) and \( b_k \) on a circle through \( a_k \) and \( b_k \), which is orthogonal to \( \partial R^n \). Suppose that there exists a positive number \( M \) such that

\[
q(J_k, J_h) \geq M > 0 \text{ for } k \neq h.
\]

As in [24, 4.9], condition (3.11) is needed to guarantee that some curve families are separate.

3.12. Theorem. Let \( (a_k) \) and \( (b_k) \) be sequences in \( R^n \) tending to 0 and satisfying (3.11), let \( u: (R^n, \rho) \to (R, | |) \) be uniformly continuous and Dirichlet finite, and let \( u(a_k) \to \alpha \), \( u(b_k) \to \beta \). If \( \sum q(a_k, b_k)^{1-n} = \infty \), then \( \alpha = \beta \).

Proof. The proof is similar to the proof of 3.8 and [24, 4.9]. The details are left to the reader.

3.13. Remarks. (1) Theorem 3.4 was proved by V. M. Mikljukov [10] in the case when the set \( E \) is a curve and the mapping is vector-valued and of class \( BL \). One variant of Theorem 3.10 for these mappings was given by G. D. Suvorov [17, p. 122].

(2) It should be observed that Theorem 3.12 follows from Theorem 3.10 in the case \( \lim \inf q(a_k, b_k) < \infty \).
(3) We shall next show that condition (3.11) cannot be removed from the hypotheses of Theorem 3.12 even in the case of continuous mappings (and, a fortiori, in the case of uniformly continuous Dirichlet finite functions). Let \( u: \mathbb{R}_+^n \rightarrow \mathbb{R} \) be a continuous function such that \( u(\mathbf{a}_k) \rightarrow 0, u(\mathbf{b}_k) \rightarrow 1 \), where \( \mathbf{a}_k, \mathbf{b}_k \rightarrow 0 \). We shall construct two new sequences \((a_k)\) and \((b_k)\) with \( \{a_j\} = \{\mathbf{a}_k\} \) and \( \{b_j\} = \{\mathbf{b}_k\} \) for which (3.11) fails to hold and for which \( \sum \varrho(a_k, b_k)^{1-n} = \infty \). To this end, let \( p_1 = 1, \) \( p_{j+1} > p_j \) be an increasing sequence of integers such that

\[
(p_{k+1} - p_k) \varrho(\mathbf{a}_k, \mathbf{b}_k)^{1-n} > 1/k
\]

for all \( k = 1, 2, \ldots \) Set

\[
a_j = \mathbf{a}_i \quad \text{and} \quad b_j = \mathbf{b}_i \quad \text{if} \quad p_i \leq j < p_{i+1}.
\]

Then \( a_k, b_k \rightarrow 0 \) and \( \sum \varrho(a_k, b_k)^{1-n} = \infty \) hold while \( u(a_k) \rightarrow 0, u(b_k) \rightarrow 1 \).

We show that Theorem 3.12 is sharp in a sense.

3.14. Theorem. Let \((b_k)\) be a sequence in \( \mathbb{R}_+^n \) with \( |b_{k+1}| < |b_k|, b_k \rightarrow 0 \), let \( a_k = |b_k| e_n \), and suppose that (3.11) holds. If \( \sum \varrho(a_k, b_k)^{1-n} < \infty \), then there exists a monotone Dirichlet finite function \( u: \mathbb{R}_+^n \rightarrow \mathbb{R} \) having an angular limit 0 at 0 and satisfying \( u(b_k) \rightarrow 1 \).

Proof. Since \( a_n = |b_n| e_n \), we obtain by (3.11), (2.2), and (2.3) that

\[
\varrho(J_k, J_{k+1}) = \log \frac{|b_k|}{|b_{k+1}|} \geq M > 0.
\]

It follows that the annuli \( R(|b_k|, |b_k|/\lambda) \) are disjoint when \( \lambda = \exp M/\pi \). Let \( w_k \in \partial \mathbb{R}_+^n \) be a unit vector such that \( b_k - b_{kn} e_n = c w_k \), where \( b_{kn} \) is the \( n \)-th coordinate of \( b_k \) and \( c \) is a positive number such that \( |b_k|^2 = c^2 + b_{kn}^2 \). The balls \( \mathcal{B}_k = \mathbb{B}^n(|b_k| w_k, r_k) \), \( r_k = |b_k|(1-1/\lambda) \) are then disjoint. Let \( t_k = |b_k - b_k| w_k \). It follows from (2.3) that

\[
t_k < 2 |b_k| \exp (-\varrho(a_k, b_k))
\]

(for more details see [23, (2.4)]). Since \( \sum \varrho(a_k, b_k)^{1-n} < \infty \), it follows that \( \varrho(a_k, b_k) \rightarrow \infty \) as \( k \rightarrow \infty \). By relabelling and passing to a subsequence if necessary we may hence assume, in view of the above estimate for \( t_k \), that \( t_k < r_k \) for all \( k \). Choose now a monotone \( \text{ACL}^n \) function \( u_k \) such that (cf. 3.6)

\[
u_k|_{\mathbb{R}_+^n \setminus \mathcal{B}_k} = 0, \quad u_k|_{\mathbb{R}_+^n \cap \overline{\mathcal{B}_k}(|b_k| w_k, t_k)} = 1,
\]

\[
d_k = \int_{\mathbb{R}_+^n} |\nabla u_k|^n \, dm = \frac{\alpha_{n-1}}{2} \left( \log \frac{r_k}{t_k} \right)^{1-n}.
\]

There exist numbers \( k_0 \) and \( c(n, M) \) such that for \( k \geq k_0 \)

\[
d_k \geq c(n, M) \varrho(a_k, b_k)^{1-n}.
\]

Set \( u = \sum_{k=k_0} u_k \). Then \( u \) is monotone, \( \text{ACL}^n \), \( u(b_k) \rightarrow 1 \), \( u(t e_k) = 0 \), \( t > 0 \) and \( u \) has a finite Dirichlet integral, as desired.
We shall next give some applications of the preceding results to the theory of quasiregular mappings. A continuous $ACL^n$ mapping $f: R^n_+ \to R^n$ is called quasiregular (qr) if there exists a constant $K \in [1, \infty)$ such that
\[
\sup_{|h|=1} |f'(x) h|^n \leq K J_f(x)
\]
a.e. in $R^n_+$, where $J_f$ is the Jacobian determinant of $f$. A sense-preserving homeomorphism is quasiregular if and only if it is quasiconformal (qc). For the basic parts of the theory of qc and qr mappings the reader is referred to [15], [18], [19]. For the following result see Rešetnjak's book [15, p. 118].

3.15. Lemma. The coordinate functions $f_1, \ldots, f_n$ of a qr mapping $f: R^n_+ \to R^n$, $f=(f_1, \ldots, f_n)$, are monotone.

3.16. Theorem. Let $f: R^n_+ \to R^n$ be a qr mapping with
\[
\int_{R^n_+} |\nabla f_j|^n dm < \infty, \quad j = 1, \ldots, n.
\]
If $f_j(x) \to \alpha_j$ as $x \to 0$, $x \in E_j$ and $\text{cap} \text{dens}(E_j, 0) > 0$, then $f_j$ has an angular limit $\alpha_j$ at 0, $j = 1, 2, \ldots, n$.

Proof. The proof follows from 3.15, 2.13, and 3.4.

3.17. Remarks. For bounded analytic functions a result similar to 3.16 holds without a condition about finite Dirichlet integral (Gehring—Lohwater [4]). In the case of bounded qr mappings $f: R^n_+ \to R^n$ such a condition is, however, necessary if $n \geq 3$. This fact follows from an example due to Rickman [16].

4. On the behaviour at a typical boundary point

In this section we shall study the behaviour of a Dirichlet finite function at a "typical" boundary point. We shall employ the following result of Rešetnjak [14].

4.1. Lemma. Let $u: R^n_+ \to R$ be an $ACL^n$ function with a finite Dirichlet integral. Then there exists a set $E \subset \partial R^n_+$ such that every compact set $F$ in $E$ is of zero $n$-capacity and such that $u$ has an essential value at every point of $\partial R^n_+ \setminus E$, i.e., for every $x \in \partial R^n_+ \setminus E$ there exists a number $\alpha$ with
\[
\lim_{r \to 0} r^{-n} \int_{B^+_r(x, r)} |f(y) - \alpha| dm = 0.
\]

4.2. Theorem. Let $u: R^n_+ \to R$ be a monotone $ACL^n$ function with a finite Dirichlet integral. Then $u$ has an angular limit at every point of $\partial R^n_+ \setminus E$, where $E$ is as in 4.1.
Proof. Since \( u \) has an essential value at the points of \( \partial R^*_n \setminus E \), it has an approximate limit as well by [23, 6.7 (1)]. By 3.1 and 3.2 (1) it has an angular limit, too.

We next show that 4.2 fails to hold for monotone functions satisfying (1.2) but not (1.1).

4.3. Example. There exists a bounded monotone ACL\(^2\) function \( u: R^*_n \to R \) satisfying condition (1.2), having an asymptotic value at each point of a dense subset of \( \partial R^*_n \), but having no angular limits.

Divide the square \( Q=[0,1] \times (0,1) \subset R^2 \) into four equal squares by joining the midpoints of opposite sides with (euclidean) segments. Repeat the division in those resulting squares which have one side on the x-axis. By continuing this process we get a division of \( Q \) into closed squares \( Q_{i,j} \) of constant hyperbolic size, where \( Q_{i,j} \) has euclidean side-length \( 2^{-i} \). Join the center of \( Q_{i,j} \) by (euclidean) segments to the centres of those two adjacent squares in \( \{Q_{i,j+1}: j=1,2,...,2^i+1\} \), each of which has a side lying on a side of \( Q_{i,j} \), for each \( i \) and \( j \). As a result we get two distinct “treelike” infinite polygonal curves approaching the x-axis. The union of these curves will be denoted by \( T \).

Define \( u(x)=0 \) if \( x \) is located on a side \( A \) of a square \( Q_{i,j} \) and \( A \cap T=\emptyset \) and \( u(y)=1 \) if \( y \in T \). In \( (\text{int } Q) \setminus T \) define \( u \) in such a way that \( u: Q \to R \cup [0,1] \) will be monotone, have all partial derivatives, continuous in \( \text{int } Q \) and (4.4)

\[
|\nabla u(z)| \leq 2^{k+3} \quad \text{for} \quad z \in (\text{int } Q_{i,j}) \setminus T,
\]

\( j=1,2,...,2^k \). Extend the domain of definition of \( u \) to \( R^*_n \) as follows. If \( \text{Im } z > 1 \), set \( u(z)=0 \). If \( p \in \mathbb{Z} \) and \( z \in Q + \{(p,0)\} \), then \( z-(p,0) \in Q \); set \( u(z)=u(z-(p,0)) \). Then \( u \) is defined in \( R^*_n \), has an asymptotic value 1 at the points of \( T \cup \partial R^*_n \setminus \{\infty\} \) through the set \( T \) and is monotone, and it follows from (4.4) that (1.2) holds. Moreover, it is clear that \( u \) has no angular limits.

5. On isolated singularities and Phragmén—Lindelöf-type behaviour

A function with a finite Dirichlet integral need not have a limit at an isolated singularity. To see this fact we may consider the function in Example 3.5 and extend it by reflection in \( \partial R^*_n \) to a map \( v: R^n \setminus \{0\} \to R \) with a finite Dirichlet integral and with no limit at 0. This function is not, however, monotone although \( v|_{R^n} \) indeed is monotone.

5.1. Theorem. Let \( u: R^n \setminus \{0\} \to R \) be a monotone ACL\(^n\) function. If \( u \) has no limit at 0, then

\[
\liminf_{t \to 0} \int_{R(0,t)} |\nabla u|^n \, dm / \log \frac{1}{t} > 0.
\]
Proof. Suppose that there are sequences \( \{a_k\}, \{b_k\} \) in \( B^n \setminus \{0\} \) with \( a_k, b_k \to 0 \) and \( u(a_k) \to \alpha, u(b_k) \to \beta \neq \alpha \). We may assume \( -\infty < \alpha < \beta < \infty \). Let \( A_k \) be the \( a_k \)-component of the set
\[
A = \{ z \in \mathbb{R}^n \setminus \{0\} : u(z) \equiv (2\alpha + \beta)/3 \},
\]
and \( B_k \) the \( b_k \)-component of the set
\[
B = \{ z \in \mathbb{R}^n \setminus \{0\} : u(z) \equiv (2\beta + \alpha)/3 \}.
\]
By 2.6, \( \tilde{A}_k \cap \{0, \infty\} \neq \emptyset \neq \tilde{B}_k \cap \{0, \infty\} \) for all large \( k \). There is a sequence \( (j_k) \) such that either \( 0 \in \tilde{A}_{j_k} \) for all \( j_k \) or \( \infty \in \tilde{A}_{j_k} \) for all \( j_k \). Consider the first case, the proof being similar in the second case. For \( t \in (0, 1) \) set
\[
\Gamma_t = \Lambda(B, A_{j_k}; R(1, t)).
\]
Suppose that \( 0 \in \tilde{B}_k \) for some \( k \) such that \( |b_k| < |a_{j_k}| \). Then we get by [18, 10.12]
\[
M(\Gamma_t) \equiv c_n \log \frac{|b_k|}{t} = c_n \log |b_k| + c_n \log \frac{1}{t}; \quad t = |b_k|.
\]
Otherwise \( \infty \in \tilde{B}_k \) for all \( k \) such that \( |b_k| = |a_{j_k}| \) and thus \( S^{-1}(r) \cap B \neq \emptyset \) for all \( r \in (0, |a_{j_k}|) \), because \( b_k \to 0 \) (cf. 2.6). Hence (5.2) holds in this case for all \( t \in (0, |a_{j_k}|) \) by [18, 10.12]. Lemma 2.11 yields
\[
M(\Gamma_t) \equiv \left( \frac{3}{\beta - \alpha} \right)^n \int_{\mathbb{R}^n} \nabla u |\nabla u| dm,
\]
where \( C = R(1, t) \). This estimate together with (5.2) gives the desired lower bound.

5.3. Corollary. Let \( u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a bounded monotone ACL\(^n \) function and let \( \alpha = \lim \inf_{x \to 0} u(x), \quad \beta = \lim \sup_{x \to 0} u(x) \). Then
\[
\lim \inf_{t \to 0} \left( \int_{R(0, t)} |\nabla u| \, dm \right) |\log \frac{1}{t} | \equiv c_n (\beta - \alpha)^n,
\]
where \( c_n \) is the positive constant in the proof of 5.1.

A counterpart of condition (1.2) for the ACL\(^n \) function \( u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) is the following one. There are constants \( \mu \in (0, 1) \) and \( A > 0 \) such that
\[
\int_{B_x} |\nabla u| \, dm \equiv A, \quad B_x = \tilde{B}^\mu(x, \mu|x|)
\]
for all \( x \in \mathbb{R}^n \setminus \{0\} \). From a standard covering argument (cf. [25]) and from (5.4) it follows that
\[
\int_{R(0, \mu|x|)} |\nabla u| \, dm \equiv d(n, A, \mu)
\]
for \( t \in (0, 1) \), where \( d(n, A, \mu) \) depends only on \( n, A \) and \( \mu \). Furthermore, it follows
from (5.5) that for \( t \in (0, 1/2) \)

\[
\int_{R^{(0,0)}} |\nabla u|^n \, dm \leq c(n, A, \mu) \log \frac{1}{t}.
\]

A direct calculation shows that the monotone ACL² function \( v(x, y) = y^2/(x^2+y^2) \), \((x, y) \in R^2 \setminus \{0\}\) satisfies (5.4) and (5.6), but \( v \) fails to have a limit at 0. This example should be compared with 5.1.

According to Theorem 5.1 a monotone function with a finite Dirichlet integral has a limit at an isolated singularity. A natural question is whether a similar result holds for a countable sequence of isolated singularities.

5.7. Example. There is a monotone ACLⁿ function \( u: R^n \setminus \{2^{-k}e_1: k = 1, 2, \ldots\}\setminus \{0\} \to R \) with

\[
\lim_{x \to 2^{-k}e_1} u(x) = 1, \quad \lim_{t \to 0^+} u(-te_1) = 0
\]

\( k = 1, 2, \ldots \) with a finite Dirichlet integral. The existence of such a function \( u \) can be seen by a direct construction. Clearly \( u \) has no limit at 0.

The next result is a Phragmén—Lindelöf type theorem.

5.8. Theorem. Let \( G \subset R^n \) be a domain such that \( M(R^n \setminus G, r, 0) \geq \delta > 0 \) for all \( r \equiv r_0 \), and let \( u: G \to R \) be a monotone ACLⁿ function. If

\[
\lim_{x \to y} \sup u(x) = 1
\]

for all \( y \in \partial G \setminus \{\infty\} \), then either \( u(x) \equiv 1 \) for all \( x \in G \) or

\[
\lim_{t \to 0^+} \inf \int_{G \cap B^n(t)} |\nabla u|^n \, dm / \log t > 0.
\]

Proof: Suppose that \( u(x_0) = c > 1 \) for some \( x_0 \in G \). Let \( E = \{x \in G: u(x) < (2+c)/3\} \). Then \( \partial G \subset E \) by the assumption. Let \( F \) be the \( x_0 \)-component of \( \{z \in G: u(z) > (1+2c)/3\} \). Then \( \infty \in F \) by 2.6. Let

\[
\Gamma_t = \Delta(E, F; G \cap B^n(t)), \quad t \equiv r_0,
\]

\[
\bar{\Gamma}_t = \Delta(E, F; B^n(t)), \quad t \equiv r_0.
\]

By the geometry of the situation it follows that \( M(\Gamma_t) = M(\bar{\Gamma}_t) \) (cf. [18, 11.3] and (2.10)). From [20, 3.5] we obtain

\[
M(\bar{\Gamma}_t) \equiv c(n, \delta) \log t
\]

for large values of \( t \). The proof follows from Lemma 2.11.
6. Some properties of boundary values

Next we shall compare the limit values of a monotone Dirichlet finite function on the closure of its domain of definition to the limit values on the boundary.

6.1. Theorem. Let \( u: \mathbb{R}^n_+ \to \mathbb{R} \) be a monotone Dirichlet finite function and let \( E \subset \partial \mathbb{R}^n_+ \) be a compact set of capacity zero with \( 0 \in E \). Then

\[
\limsup_{x \to 0} u(x) = \limsup_{x \to 0} (\limsup_{y \to x} u(y)).
\]

**Proof.** Since \( \text{cap } E = 0 \), it follows that \( 0 \in (\partial \mathbb{R}^n_+ \setminus E) \) ([15, p. 72]) and hence the right side of the above equality makes sense. Denote the left and right sides by \( d \) and \( 6 \), respectively. Clearly \( d = 6 \). Hence it remains to be shown that \( \tilde{a} \geq \tilde{b} \) is impossible. Choose \( a \) and \( b \) such that \( \tilde{b} < b < a < \tilde{a} \). Let \( r > 0 \) be such that

\[
\limsup_{y \to x} u(y) < b
\]

for all \( x \in (\partial \mathbb{R}^n_+ \setminus E) \cap B^n(r) \). Choose a sequence \( (a_k) \) in \( B^n_+(r) \) with \( u(a_k) \to a \) and \( |a_k| < r/k \). Let \( A_k \) be the \( a_k \)-component of the set \( \{z \in \mathbb{R}^n_+: u(z) > a\} \). It follows from 2.6 that \( A_k \cap (\partial \mathbb{R}^n_+ \cup \{-\infty\}) \neq \emptyset \) for all \( k \). From (6.2) it follows that \( A_k \cap (\partial \mathbb{R}^n_+ \setminus B^n(r)) \neq \emptyset \) for all \( k \). Let \( B = \{z \in \mathbb{R}^n_+: u(z) < b\} \) and \( \Gamma_k = A_k \Delta B ; \mathbb{R}^n_+ \). It follows from 2.11 that

\[
M(\Gamma_k) \equiv (a - b)^{-n} \int_{\mathbb{R}^n_+} |\nabla u|^n \, dm < \infty.
\]

If \( A_k \cap (E \cap B^n(r)) \neq \emptyset \), then \( M(\Gamma_k) = \infty \) because \( A_k \) is a connected set and \( \text{cap } E = 0 \) (cf. [18, 10.12]). Otherwise \( A_k \cap (\partial \mathbb{R}^n_+ \setminus B^n(r)) \neq \emptyset \), and since \( \text{cap } E = 0 \) and \( A_k \) is connected, we get by [18, 10.12] that

\[
M(\Gamma_k) \equiv c_n \log k.
\]

In either case we obtain a contradiction with (6.3) when \( k \to \infty \).

6.4. Remark. By inspecting the above proof we see that the condition \( \text{cap } E = 0 \) can be weakened. In fact, it suffices to assume that \( E \subset \partial \mathbb{R}^n_+ \) is a compact set which has no interior points (in the topology of \( \partial \mathbb{R}^n_+ \)) and which satisfies \( M(y, \partial \mathbb{R}^n_+ \setminus E) = \infty \) for all \( y \in E \) in the sense of [9].

6.5. A bound for a Dirichlet finite function. Let \( u: \mathbb{R}^n_+ \to \mathbb{R} \) be a monotone Dirichlet finite function, let \( E \subset \mathbb{R}^n_+ \), and let \( u \) have a continuous extension, denoted by \( u \), to the points \( E \cap \partial \mathbb{R}^n_+ \) such that \( u(x) = b \) for \( x \in E \). Define

\[
\sigma(x, E) = \inf_{C} M(A(C, E; \mathbb{R}^n_+)),
\]

where the infimum is taken over all continua \( C \) with \( x \in C \) and \( C \cap (\partial \mathbb{R}^n_+ \cup \{-\infty\}) \neq \emptyset \).
It follows then that

\[(6.7) \quad u(x) \equiv b + \left( \int_{R^n} |\nabla u|^n \, dm \right)^{1/n} \]

for all \( x \in R^n \). This estimate follows directly from Lemma 2.11 and Remark 2.6. In fact, this idea has been applied several times in this paper. The inequality (6.7) suggests that the quantity \( \sigma(x, E) \) is of some interest in the theory of Dirichlet finite functions.

References

On functions with a finite or locally bounded Dirichlet integral


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