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# ON A THEOREM OF ABIKOFF

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This note contains a new proof and an extension of a theorem of Abikoff [1] on (complex) boundaries of Teichmüller spaces. First we recall some definitions and results, cf. [3] for references.

Let $G$ be a Fuchsian group operating on the upper half-plane $U$, i.e., a discrete subgroup of $\mathrm{PSL}(2, \boldsymbol{R})$. Let $B(L, G)$ be the complex Banach space of holomorphic functions $\varphi(z)$ defined in the lower half-plane $L$, with norm

$$\|\varphi\| = \sup |y^2\varphi(z)| < \infty \quad (z = x+iy \in L),$$

and satisfying the functional equation of quadratic differentials

$$\varphi(g(z))g'(z)^2 = \varphi(z), \quad g\in G.$$

For every $\varphi \in B(L, G)$ the Schwarzian differential equation

$$\{W, z\} = \left(\frac{W''(z)}{W'(z)}\right)' - \frac{1}{2}\left(\frac{W''(z)}{W'(z)}\right)^2 = \varphi(z) \tag{1}$$

has meromorphic solutions in $L$; if $W$ is one, all others are of the form $\alpha W$ where $\alpha \in \mathrm{PSL}(2, \boldsymbol{C})$. It is convenient to denote by $W_\varphi$ the solution of (1) normalized by the requirement that

$$W_\varphi(t-i) = \frac{1}{t} + O(t), \quad t \to 0. \tag{2}$$

Every $\varphi \in B(L, G)$ induces a homomorphism $\chi_\varphi$ of $G$ into $\mathrm{PSL}(2, \boldsymbol{C})$ defined by the rule

$$W_\varphi \circ g(z) = \chi_\varphi(g) \circ W_\varphi(z) \quad (g\in G,\ z\in L).$$

(The group $G_\varphi = \chi_\varphi(G)$ is called the monodromy group of $\varphi$.)

The Teichmüller space $T(G)$ of $G$ can be defined as the set of those $\varphi \in B(L, G)$ for which $W_\varphi$ is the restriction to $L$ of a quasiconformal self-map $\hat{W}_\varphi$ of $\hat{\boldsymbol{C}}$, with

$$\hat{W}_\varphi \circ g \circ \hat{W}_\varphi^{-1}(z) = \chi_\varphi(g)(z) \quad (g\in G,\ z\in \hat{\boldsymbol{C}})$$

(so that $G_\varphi$ is a quasi-Fuchsian group).

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It is known that $T(G)$ is a domain (of holomorphy) in $B(L, G)$, containing the ball $\|\varphi\|<1/2$ and contained in the ball $\|\varphi\|<3/2$.

(We recall that $T(G)$ can be identified with the Teichmüller space $T(S_G)$ of the Riemann surface $S_G=U_G/G$ where $U_G$ is the complement in $U$ of all fixed points of elliptic elements of $U$. The dimension of $B(L, G)$ is finite if and only if $S_G$ is obtained from a closed surface of genus $p$ by removing $n$ distinct points, with $2p-2+n>0$; in this case $\dim T(G)=\dim B(L, G)=3p-3+n$.)

From now we assume that $\dim B(L, G)>0$, i.e., that $G$ is not a triangle group, and denote by $\partial T(G)$ the boundary of $T(G)$ in $B(L, G)$. It is known and easy to show (cf. [3], p. 592) that for all points $\varphi\in\partial T(G)$ the map $W_{\varphi_0}$ is injective (so that $G_{\varphi_0}$ is discrete and has a non-empty region of discontinuity).

Theorem. *Every neighborhood of every point $\varphi_0\in\partial T(G)$ contains points in the complement of $T(G)\cup\partial T(G)$.*

For $\dim T(G)<\infty$ Abikoff [1] proved (by an ingenious use of plurisubharmonic functions) that $\varphi_0\in\partial T(G)$ has the property stated in the theorem whenever the limit set of $G_{\varphi_0}$ has measure 0. He quotes an assertion by Thurston (the proof of which is not yet published) which implies that this is always so. Abikoff also notes that for $G=1$, the trivial group, the theorem follows at once from Gehring's characterization of the universal Teichmüller space $T(1)$ as the interior in $B(L, 1)$ of the set of Schwarzian derivatives of univalent functions [5].

The proof below (for all $G$) is based on an improvement of the "improved $\lambda$-lemma" of Sullivan—Thurston [6], which appears as Theorem 3 in Bers—Royden [4].

Assume that theorem is false for some $G$ and some $\varphi_0\in\partial T(G)$. Then there is an $\varepsilon>0$ such that for $\|\varphi-\varphi_0\|<\varepsilon$ the point $\varphi\in B(L, G)$ belongs to $T(G)\cup\partial T(G)$, so that $W_\varphi$ is injective.

Set

$$E = W_{\varphi_0}(L).$$

Choose a $\varphi_1\in T(G)$ with $\|\varphi_1-\varphi_0\|<\varepsilon/4$ and set

$$\psi_\lambda = (1-4\lambda)\varphi_0+4\lambda\varphi_1 \quad (|\lambda| < 1)$$

and

$$f(\lambda, z) = W_{\psi_\lambda}\circ W_{\varphi_0}^{-1}(z) \quad (|\lambda| < 1,\ z\in E).$$

Let $\Delta_a$ denote the disc $|\lambda|<a$ in $\boldsymbol{C}$.

The map $f\colon \Delta_1\times E\to\hat{\boldsymbol{C}}$ is an injection of $E$ for every fixed $\lambda$, since

$$\|\psi_\lambda-\varphi_0\| < \varepsilon \quad \text{for} \quad |\lambda| < 1,$$

reduces to the identity for $\lambda=0$, since $\psi_0=\varphi_0$, and is holomorphic in $\lambda$ for every fixed $z\in E$ (with $f(\lambda, \infty)=\infty$ for all $\lambda$), since the solution of the Schwarzian differential equation (1) with $\varphi=\psi_\lambda$, satisfying the initial condition (2), depends holomorphically on $\lambda$. (To see this, recall that $W_\varphi$ admits the representation $W_\varphi=\eta_1/\eta_2$

where $\eta_1$ and $\eta_2$ are solutions of the linear differential equation $2\eta''+\varphi\eta=0$ with $\eta_1=\eta_2'=1$ and $\eta_1'=\eta_2=0$ at $z=-i$.)

Applying Theorem 3 of [4] we conclude that there exists a unique map $\hat{f}$: $\Delta_{1/3}\times \hat{C}\to\hat{C}$ with the following properties:

(j) For every fixed $\lambda$, $|\lambda|<1/3$, $\hat{f}(\lambda, \cdot)$ is a quasiconformal self-mapping of $\hat{C}$, depending holomorphically on $\lambda$ and reducing to the identity for $\lambda=0$,

(jj) in every component $\omega$ of the complement $H$ of the closure $\hat{E}$ of $E$ the Beltrami coefficient of $f(\lambda, \cdot)$ is "harmonic" and depends holomorphically on $\lambda$,

(jjj) we have that

$$\hat{f}(\lambda, z)=f(\lambda, z) \quad \text{for} \quad |\lambda|<1/3 \quad \text{and} \quad z\in E.$$

Condition (jj) means that

$$\mu_\omega(\lambda, z)=\left(\frac{\partial\hat{f}(\lambda, z)}{\partial\bar{z}}\bigg/\frac{\partial\hat{f}(\lambda, z)}{\partial z}\right)\bigg|\omega=\varrho_\omega(z)^{-2}\overline{F_\omega(\lambda, z)} \tag{3}$$

where $\varrho_\omega(z)|dz|$ is the Poincaré metric in $\omega$ and $F_\omega(\lambda, z)$ a function holomorphic in $z\in\omega$, antiholomorphic in $\lambda\in\Delta_{1/3}$. (If $H=\varphi$, (jj) is vacuous and the argument simplifies by itself.)

Now let $g\in G$ and set $g_\lambda=\chi_{\psi_\lambda}(g)$. Then

$$g_\lambda(z)=W_{\psi_\lambda}\circ g\circ W_{\psi_\lambda}^{-1}(z) \quad \text{for} \quad |\lambda|<1 \quad \text{and} \quad z\in L$$

since $W_{\psi_\lambda}$ is injective. Also $g_0=\chi_{\varphi_0}(g)$ since $\psi_0=\varphi_0$.

Consider the map $h$: $\Delta_{1/3}\times\hat{C}\to\hat{C}$ defined as

$$h(\lambda, z)=g_\lambda^{-1}\circ\hat{f}(\lambda, g_0(z)). \tag{4}$$

It is holomorphic in $\lambda$, $|\lambda|<1/3$, for each fixed $z$, and is a quasiconformal self-mapping of $\hat{C}$ for each fixed $\lambda$. Also,

$$h(\lambda, z)=f(\lambda, z) \quad \text{for} \quad |\lambda|<1/3, \quad z\in E. \tag{5}$$

Indeed, $g_0(E)=E$ and for $z\in E$ one has

$$\begin{aligned}\hat{f}(\lambda, g_0(z))&=f(\lambda, g_0(z))=W_{\psi_\lambda}\circ W_{\varphi_0}^{-1}\circ W_{\varphi_0}\circ g\circ W_{\varphi_0}^{-1}(z)\\&=W_{\psi_\lambda}\circ g\circ W_{\varphi_0}^{-1}(z)=g_\lambda\circ W_{\psi_\lambda}\circ W_{\varphi_0}^{-1}(z)=g_\lambda\circ f(\lambda, z)\end{aligned}$$

whence (5) follows.

If $\omega$ is a component of the complement $H$ of $E$, so is $g_0(\omega)$, and a direct calculation based on (3) and (4) shows that for $z\in\omega$ the Beltrami coefficient of $h(\lambda, z)$ equals

$$\begin{aligned}\left(\frac{\partial h(\lambda, z)}{\partial\bar{z}}\bigg/\frac{\partial h(\lambda, z)}{\partial z}\right)\bigg|\omega&=\mu_{g(\omega)}(\lambda, g_0(z))\frac{|g_0'(z)|^2}{g_0'(z)^2}\\&=[\varrho_{g(\omega)}(g_0(z))|g'(z_0)]^{-2}\overline{F_{g(\omega)}(\lambda, g_0(z))g_0'(z)^2}\\&=\varrho_\omega(z)^{-2}\overline{F_{g(\omega)}(\lambda, g_0(z))g_0'(z)^2}.\end{aligned} \tag{6}$$

Since the function under the conjugation bar is holomorphic in $z$ and antiholomorphic in $\lambda$, and $\omega$ was arbitrary, we conclude that $h(\lambda, z)$ has a "harmonic" Beltrami coefficient in every component of the complement of $\hat{E}$. By the uniqueness part of Theorem 3 in [4] we obtain that

$$h = \hat{f},$$

and this shows that the restriction $\mu_H$ of the Beltrami coefficient of $\hat{f}$ to the complement $H$ of $\hat{E}$,

$$\mu_H(\lambda, z) = \left(\frac{\partial \hat{f}(\lambda, z)}{\partial \bar{z}} \Big/ \frac{\partial \hat{f}(\lambda, z)}{\partial z}\right)\Big| H,$$

satisfies the relation

$$\mu_H(\lambda, g_0(z)) \frac{|g_0'(z)|^2}{(g_0'(z))^2} = \mu_H(\lambda, z). \tag{7}$$

Since $g \in G$ was arbitrary, this relation holds for all such $g$.

It follows from (7) that, for every $g \in G$ and every $\lambda \in \Delta_{1/3}$, the Beltrami coefficient of $\hat{f}(\lambda, \cdot) \circ g_0 | H$ equals to $\mu_H(\lambda, \cdot)$ so that there exists a holomorphic function $\gamma(z)$, $z \in H$, depending on $g$ and $\lambda$, such that

$$\hat{f}(\lambda, \cdot) \circ g_0 | H = \gamma \circ \hat{f}(\lambda, \cdot) H$$

or

$$\gamma = \hat{f}(\lambda, \cdot) \circ g_0 \circ \hat{f}(\lambda, \cdot)^{-1} | H.$$

This relation shows that $\gamma$ has a continuous extension to the boundary $\partial H$ of $H$, which is also the boundary $\partial E$ of $E$. But for $z \in E$

$$\hat{f}(\lambda, \cdot) \circ g_0 \circ \hat{f}(\lambda, \cdot)^{-1}(z) = f(\lambda, \cdot) \circ g_0 \circ f(\lambda, \cdot)^{-1}(z)$$
$$= W_{\psi_\lambda} \circ W_{\varphi_0}^{-1} \circ W_{\varphi_0} \circ g \circ W_{\varphi_0}^{-1} \circ W_{\varphi_0} \circ W_{\psi_\lambda}^{-1}(z) = g_\lambda(z).$$

Hence the boundary values of $\gamma$ in $H$ coincide with the values of $g_\lambda$ on $\partial H$. Since $\gamma$ and $g_\lambda$ are both holomorphic in $H$, $\gamma = g_\lambda$. We established that

$$\hat{f}(\lambda, \cdot) \circ g \circ \hat{f}(\lambda, \cdot)^{-1} = g_\lambda.$$

Since $\psi_{1/4} = \varphi_1 \in T(G)$, the map $W_{\psi_{1/4}}$ has a quasiconformal extension $\hat{W}_{\psi_{1/4}}$ to $\hat{C}$ such that

$$\hat{W}_{\psi_{1/4}} \circ g \circ \hat{W}_{\psi_{1/4}}^{-1} = \chi_{\psi_{1/4}}(g) \quad (g \in G).$$

One checks at once that the map

$$\hat{W}_{\varphi_0} = \hat{f}(1/4, \cdot)^{-1} \circ \hat{W}_{\psi_{1/4}}$$

is a quasiconformal extension of $W_{\varphi_0}$ to $\hat{C}$ such that

$$\hat{W}_{\varphi_0} \circ g \circ \hat{W}_{\varphi_0}^{-1} = \chi_{\varphi_0}(g) \quad (g \in G).$$

This shows that $\varphi_0 \in T(G)$. Contradiction.

*Postscript.* Abikoff observed that the proof above could be somewhat simplified by noting that (under the assumption made about $\varphi_0$, $\varepsilon$ and $\varphi_1$) the domain $E=$

$W_{\varphi_0}(L)$ is a Jordan domain, being the image of the Jordan domain $W_{\varphi_1}(L)$ under the quasiconformal self-map $f(1/4, \circ)^{-1}$ of $\boldsymbol{C}$.

We note that this conclusion could be obtained already from the original "improved $\lambda$-lemma" by Sullivan and Thurston [6] without using the uniqueness statement in [4], and that it establishes the theorem for the case $\dim T(G) < \infty$. The uniqueness statement seems to be essential for treating the case $\dim T(G) = \infty$.

*P.P.S.* After this note was written I learned about Žuravlev's paper [7]. The main result of this paper asserts that $T(G)$ is the component (containing the origin) of the interior (in $B(L, G)$) of the set of Schwarzian derivatives (belonging to $B(L, G)$) of univalent functions. This assertion implies the theorem discussed here.

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