SEQUENTIAL FOURIER—FEYNMAN TRANSFORMS

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In a forthcoming Memoir of the American Mathematical Society [3], the authors give a simple sequential definition of the Feynman integral which is applicable to a rather large class of functionals. In Corollary 2 to Theorem 3.1 of [3] we showed that the elements of the Banach algebra $\tilde{S}$ defined in [3] (and below) are all sequentially Feynman integrable.

In the present paper we use the sequential Feynman integral to define a set of sequential Fourier—Feynman transforms. We also show that they form an abelian group of isometric transformations of $\tilde{S}$ onto $\tilde{S}$.

**Notation.** Let $C=C[a, b]$ be the space of continuous functions $x(t)$ on $[a, b]$ such that $x(a)=0$, and let $C'=\times_{j=1}^{\nu} C[a, b]$.

Let a subdivision $\sigma$ of $[a, b]$ be given:

$$\sigma: [a = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots < \tau_m = b].$$

Let $X=X(t)$ be a polygonal curve in $C'$ based on a subdivision $\sigma$ and the matrix of real numbers $\tilde{\xi} \equiv \{\xi_{j,k}\}$, and defined by

$$\tilde{X}(t) \equiv \tilde{X}(t, \sigma, \tilde{\xi}) = [X_1(t, \sigma, \tilde{\xi}), \ldots, X_\nu(t, \sigma, \tilde{\xi})]$$

where

$$X_j(t, \sigma, \tilde{\xi}) = \frac{\xi_{j,k-1}(\tau_k-t) + \xi_{j,k}(t-\tau_{k-1})}{\tau_k-\tau_{k-1}}$$

when

$$\tau_{k-1} \leq t \leq \tau_k; \quad k = 1, 2, \ldots, m; \quad \text{and} \quad \xi_{j,0} \equiv 0.$$

(We note that as $\tilde{\xi}$ ranges over all of $vm$ dimensional real space, the polygonal functions $\tilde{X}((\cdot), \sigma, \tilde{\xi})$ range over all polygonal approximations to the functions $C'[a, b]$ based on the subdivision $\sigma$. Specifically if $x$ is a particular element of $C'[a, b]$ and we set $\xi_{j,k} = x_j(\tau_k)$, the function $\tilde{X}((\cdot), \sigma, \tilde{\xi})$ is the polygonal approximation of $\tilde{x}$ based on the subdivision $\sigma$.) Where there is a sequence of subdivisions $\sigma_1, \sigma_2, \ldots$, then $\sigma, m$ and $\tau_k$ will be replaced by $\sigma_n, m_n$ and $\tau_{k,n}$.

**Definition.** Let $q \neq 0$ be a given real number and let $F(x)$ be a functional defined on a subset of $C'[a, b]$ containing all the polygonal elements of $C'[a, b]$.
Let \( \sigma_1, \sigma_2, \ldots \) be a sequence of subdivisions such that norm \( \sigma_n \to 0 \) and let \( \{ \lambda_n \} \) be a sequence of complex numbers with \( \Re \lambda_n \geq 0 \) such that \( \lambda_n \to -i\eta \). Then if the integral in the right hand side of (1.0) exists for all \( n \) and if the following limit exists and is independent of the choice of the sequences \( \{ \sigma_n \} \) and \( \{ \lambda_n \} \), we say that the sequential Feynman integral with parameter \( q \) exists and is given by

(1.0) \[ \int_{S_q} F(\bar{x}) \, d\bar{x} \equiv \lim_{n \to \infty} \gamma_{\sigma_n, \lambda_n} \int_{\mathcal{R}_{VM_n}} \exp \left\{ -\frac{\lambda_n}{2} \int_a^b \left\| \frac{d\bar{X}}{dt} (t, \sigma_n, \xi) \right\|^2 \right\} F(\bar{X}(\cdot), \sigma_n, \xi) \, d\xi, \]

where

\[ \gamma_{\sigma, \lambda} = \left( \frac{\lambda}{2\pi} \right)^{v/2} \prod_{k=1}^m (\tau_k - \tau_{k-1})^{-v/2}. \]

**Definition.** Let \( D[a, b] \) be the class of elements \( x \in C[a, b] \) such that \( x \) is absolutely continuous on \( [a, b] \) and \( x' \in L_a[a, b] \). Let \( D' = \times_1 D \).

**Definition.** Let \( \mathcal{M} \equiv \mathcal{M}(L_a[a, b]) \) be the class of complex measures of finite variation defined on \( B(L_2^v) \), the Borel measurable subsets of \( L_2[a, b] \). We set \( \| \mu \| = \text{var} \mu \). (In this paper, \( L_2 \) always means real \( L_2 \).)

**Definition.** The functional \( F \) defined on a subset of \( C^v \) that contains \( D' \) is said to be an element of \( \tilde{S} \equiv \tilde{S}(L_2^v) \) if there exists a measure \( \mu \in \mathcal{M} \) such that for \( \bar{x} \in D^v \)

(1.1) \[ F(\bar{x}) \equiv \int_{L_2^v} \exp \left\{ i \sum_{j=1}^{v} \int_a^b v_j(t) \left( \frac{dx_j(t)}{dt} \right) dt \right\} d\mu(\bar{v}). \]

We also define \( \| F \| \equiv \| \mu \| \).

**Lemma.** If \( F \in \tilde{S} \) and \( \bar{y} \in D^v \), then the translate of \( F \) by \( \bar{y} \) is in \( \tilde{S} \); i.e. \( F((\cdot) + \bar{y}) \in \tilde{S} \). Moreover if for \( \bar{x} \in D^v \), \( F(\bar{x}) \) is given by equation (1.1) where \( \mu \in \mathcal{M} \), it follows that

(1.2) \[ F(\bar{x} + \bar{y}) = \int_{L_2^v} \exp \left\{ i \sum_{j=1}^{v} \int_a^b v_j(t) \left( \frac{dx_j(t)}{dt} \right) dt \right\} d\sigma(\bar{v}) \]

where \( \sigma \in \mathcal{M} \) and for each Borel subset \( E \) of \( L_2^v \), \( \sigma(E) \) is given by

(1.3) \[ \sigma(E) = \int_E \exp \left\{ i \sum_{j=1}^{v} \int_a^b v_j(t) \left( \frac{dy_j(t)}{dt} \right) dt \right\} d\mu(\bar{v}). \]

**Proof of the Lemma.** If \( \sigma \) is given by equation (1.3) it is clearly in \( \mathcal{M} \) and \( \| \sigma \| \equiv \| \mu \| \). For \( \bar{x} \in D^v \), it follows from (1.1) that

\[ F(\bar{x} + \bar{y}) = \int_{L_2^v} \exp \left\{ i \sum_{j=1}^{v} \int_a^b v_j(t) \left( \frac{dx_j(t)}{dt} \right) dt \right\} \exp \left\{ i \sum_{j=1}^{v} \int_a^b v_j(t) \left( \frac{dy_j(t)}{dt} \right) dt \right\} d\mu(\bar{v}) = \int_{L_2^v} \exp \left\{ i \sum_{j=1}^{v} v_j(t) \left( \frac{dx_j(t)}{dt} \right) dt \right\} d\sigma(\bar{v}). \]
and hence $F((\cdot)+\bar{y})\in\mathcal{S}$ and the Lemma is proved. (Cf. also Corollary 2 of Theorem 4.1 of [3].)

**Definition.** If $p\neq 0$ and if for each $y\in D'[a,b]$ the sequential Feynman integral

$$(1.4) \quad (\Gamma_p F)(\bar{y}) = \int_{L^y_x} \exp\left\{ i \sum_{j=1}^y \int_a^b v_j(t) \frac{dy_j(t)}{dt} dt \right\} \exp\left\{ \frac{p}{2i} \sum_{j=1}^y \int_a^b [v_j(t)]^2 dt \right\} d\mu(\bar{y}).$$

exists, then $\Gamma_p F$ is called the sequential Fourier—Feynman transform of $F$. If $p=0$ we define $\Gamma_0$ to be the identity transformation, $\Gamma_0 F = F$.

**Theorem 1.** If $F\in\mathcal{S}$ and $p$ is real, then the sequential Fourier—Feynman transform of $F$ exists and $\Gamma_p F\in\mathcal{S}$. Moreover for $\bar{y}\in D'$ and $F$ given by equation (1.1),

$$(1.6) \quad (\Gamma_p F)(\bar{y}) = \int_{L^y_x} \exp\left\{ i \sum_{j=1}^y \int_a^b v_j(t) \frac{dy_j(t)}{dt} dt \right\} \exp\left\{ \frac{p}{2i} \sum_{j=1}^y \int_a^b [v_j(t)]^2 dt \right\} d\mu(\bar{y}).$$

**Proof of Theorem 1.** By the Lemma, $F((\cdot)+\bar{y})\in\mathcal{S}$, and hence by Corollary 2 of Theorem 3.1 of [3] when $p\neq 0$, the right hand member of (1.4) exists. Also in terms of the measure $\sigma$ given in equation (1.3), we have from (1.2) and Corollary 2 of Theorem 3.1 of [3] that

$$\int_{L^y_x} F(\vec{x} + \bar{y}) d\vec{x} = \int_{L^y_x} \left[ \int_{L^y_x} \exp\left\{ i \sum_{j=1}^y \int_a^b v_j(t) \frac{dy_j(t)}{dt} dt \right\} d\sigma(\bar{y}) \right] d\vec{x}$$

$$= \int_{L^y_x} \exp\left\{ \frac{p}{2i} \sum_{j=1}^y \int_a^b [v_j(t)]^2 dt \right\} d\sigma(\bar{y}).$$

Equation (1.6) follows by substituting for $\sigma$ using equation (1.3). Now let the measure $\tau$ be defined on the Borel subsets $E$ of $L^y_x$ by

$$(1.7) \quad \tau(E) = \int_E \exp\left\{ \frac{p}{2i} \sum_{j=1}^y \int_a^b [v_j(t)]^2 dt \right\} d\mu(\bar{y}).$$

Clearly $\tau \in \mathcal{M}$ and equation (1.6) can be written

$$(1.8) \quad (\Gamma_p F)(\bar{y}) = \int_{L^y_x} \exp\left\{ i \sum_{j=1}^y \int_a^b v_j(t) \frac{dy_j(t)}{dt} dt \right\} d\tau(\bar{y}).$$

Hence $\Gamma_p F\in\mathcal{S}$ and the theorem is proved for the case $p\neq 0$. When $p=0$, $\Gamma_0 F$ is the identity transformation and the theorem follows from (1.1).
Corollary to Theorem 1. In addition to the hypotheses of Theorem 1, assume that \( \Phi \) is a bounded measurable functional defined on \( L_2^w \), and let

\[
H(\bar{x}) = \int_{L_2^w} \exp \left\{ i \sum_{j=1}^r \int_a^b v_j(t) \frac{dx_j(t)}{dt} \ dt \right\} \Phi(\bar{v}) \ d\mu(\bar{v}).
\]

Then the functional \( H \in \hat{S} \) and

\[
(\Gamma_p H)(\bar{y}) = \int_{L_2^w} \exp \left\{ i \sum_{j=1}^r \int_a^b v_j(t) \frac{dy_j(t)}{dt} \ dt \right\} \exp \left\{ \frac{p}{2i} \sum_{j=1}^r \int_a^b [v_j(t)]^2 \ dt \right\} \Phi(\bar{v}) \ d\mu(\bar{v}).
\]

**Proof.** Let a measure \( \sigma \) be defined on each Borel set \( E \) of \( L_2^w \) by

\[
\sigma(E) = \int_E \Phi(\bar{v}) \ d\mu(\bar{v}).
\]

Clearly \( \sigma \in \mathcal{M} \) and for \( \bar{x} \in D^r \)

\[
H(\bar{x}) = \int_{L_2^w} \exp \left\{ i \sum_{j=1}^r \int_a^b v_j(t) \frac{dx_j(t)}{dt} \ dt \right\} \ d\sigma(\bar{v})
\]

so that \( H \in \hat{S} \). Applying the theorem to \( H \) and replacing \( d\sigma(\bar{v}) \) by \( \Phi(\bar{v}) \ d\mu(\bar{v}) \), we obtain (1.10) and the Corollary is proved.

**Theorem 2.** The set of sequential Fourier—Feynman transforms \( \Gamma_p \) for real \( p \) forms an abelian group of isometries of the Banach algebra \( \hat{S} \), with multiplication rule

\[
\Gamma_q \Gamma_p = \Gamma_{p+q} \quad \text{for} \quad p, q \text{ real}
\]

and identity

\[
\Gamma_0 = I
\]

and inverses

\[
(\Gamma_p)^{-1} = \Gamma_{-p} \quad \text{for} \quad p \text{ real}.
\]

**Proof of Theorem 2.** By equation (1.6),

\[
(\Gamma_p F)(\bar{y}) = \int_{L_2^w} \exp \left\{ i \sum_{j=1}^r \int_a^b v_j(t) \frac{dy_j(t)}{dt} \ dt \right\} \exp \left\{ \frac{p}{2i} \sum_{j=1}^r \int_a^b [v_j(t)]^2 \ dt \right\} \ d\mu(\bar{v})
\]

and by equation (1.10)

\[
(\Gamma_q \Gamma_p F)(\bar{y}) = \int_{L_2^w} \exp \left\{ i \sum_{j=1}^r \int_a^b v_j(t) \frac{dy_j(t)}{dt} \ dt \right\} \exp \left\{ \frac{p}{2i} \sum_{j=1}^r \int_a^b [v_j(t)]^2 \ dt \right\} \cdot \exp \left\{ \frac{q}{2i} \sum_{j=1}^r \int_a^b [v_j(t)]^2 \ dt \right\} \ d\mu(\bar{v}) = (\Gamma_{p+q} F)(\bar{y}).
\]

Equation (1.12) is given by the definition of the sequential Fourier—Feynman transform and equation (1.13) follows from equation (1.11) by setting \( q = -p \). Finally
we establish the isometric property of $\Gamma_p$. If $F \in \mathcal{S}$ and $G = \Gamma_p F$, then by equations (1.8), (1.7) and (1.1)

$$\|G\| = \|\Gamma_p F\| = \|\tau\| \leqslant \mu = \|F\|.$$  

Also by (1.13)

$$F = \Gamma_{-p} G$$

and

$$\|F\| \leqslant \|G\|,$$

and the theorem is proved.

In conclusion, for the case $v=1, \quad p = -1$, the definition of the sequential Fourier—Feynman transform given in this paper is similar in form to the definition of the analytic Fourier—Feynman transform $T$ given by Brue [3]. Brue defines the transform, $TF$, of a functional $F$, in terms of the analytic Feynman integral of $F$; namely he lets

$$(TF)(y) \equiv \int_C^{anf-1} F(x+y) \, dx$$

whenever the right hand side exists for all $y \in C$. He then proceeds to establish the existence of the transform $T$ and its inverse $T^*$ for several large classes of functionals. There is also a formal similarity to the definitions given in [2] and [4], but they involve more complicated forms of the analytic Feynman integral.

References


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