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NONVANISHING UNIVALENT FUNCTIONS III

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In two previous papers [3, 4] we studied the class S_0 of functions f analytic, univalent, and nonvanishing in the unit disk $\boldsymbol{D}$, with $f(0)=1$. Among other things we investigated coefficient problems and the qualitative properties of support points. Here we continue this work with several new results. First we make the surprising observation that the asymptotic form of Littlewood's coefficient conjecture actually implies Littlewood's conjecture. Turning next to support points, we show that the arc Γ omitted by a support point of S_0 is always asymptotic to a line at infinity. Essentially the same argument gives the corresponding result for the Montel class M_r of functions f analytic and univalent in $\boldsymbol{D}$ with $f(0)=0$ and $f(r)=1$, where $0<r<1$. Finally, we show for the Montel class that if Γ has a maximal hyperbolic angle of $\pi/4$ at its tip, then under suitable restrictions it must be a half-line.

1. The Littlewood conjecture

Littlewood's conjecture, apparently weaker than the Bieberbach conjecture, was originally formulated for the class S of functions f analytic and univalent in $\boldsymbol{D}$ with $f(0)=0$ and $f'(0)=1$. It asserts that if a function $f(z)=z+c_2z^2+\dots$ in S omits a value ω, then $|c_n|\leqq 4|\omega|n$. As observed in [3], an equivalent formulation is that $|a_n|\leqq 4n$ for functions $f(z)=1+a_1z+a_2z^2+\dots$ in S_0. The conjectured extremal functions are rotations of the Koebe function

$$k_0(z)=\left(\frac{1+z}{1-z}\right)^2=1+4\sum_{n=1}^{\infty} nz^n.$$

The *asymptotic Littlewood conjecture* is that

$$\limsup_{n\to\infty}\frac{1}{n}A_n=4, \quad \text{where} \quad A_n=\sup_{f\in S_0}|a_n|.$$

Although this second form of the conjecture is apparently weaker, it turns out that the two are equivalent.

Theorem 1. *The asymptotic Littlewood conjecture implies the Littlewood conjecture.*

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The proof depends upon a slight variant of a lemma which Nehari [7] used to show that the asymptotic Bieberbach conjecture implies Littlewood's conjecture. (See also [1], p. 67.)

Lemma. *If* $f\in S_0$, *then the two functions*

$$F_\pm(z) = 2f(z^2)-1\pm 2\{f(z^2)[f(z^2)-1]\}^{1/2}$$

also belong to S_0.

Proof of Theorem. In view of the identity

$$f(z^2) = \tfrac{1}{2}+\tfrac{1}{4}[F_+(z)+F_-(z)],$$

the lemma tells us that

$$|a_n| \leqq \tfrac{1}{4}[A_{2n}+A_{2n}] = \tfrac{1}{2}A_{2n},$$

so that $A_n \leqq A_{2n}/2$. Iterating this inequality, we obtain

$$A_n \leqq 2^{-k}A_{2^k n}, \quad k = 1, 2, \ldots.$$

Thus if the asymptotic Littlewood conjecture is true, we can deduce that

$$\frac{1}{n}A_n \leqq \limsup_{k\to\infty}\frac{1}{2^k n}A_{2^k n} = 4,$$

or $A_n \leqq 4n$ for $n=1, 2, \ldots$, which is the Littlewood conjecture.

Because Hamilton [4] has shown that Littlewood's conjecture implies the asymptotic Bieberbach conjecture, it is a corollary that the asymptotic Littlewood conjecture is equivalent to the asymptotic Bieberbach conjecture.

2. Asymptotic half-lines

A *support point* of S_0 is a function which maximizes Re $\{L\}$ for some continuous linear functional L not constant on S_0. We showed in [3] that each support point f maps $\boldsymbol{D}$ onto the complement of a single analytic arc Γ which extends from 0 to ∞ monotonically with respect to the family of ellipses with foci at 0 and 1, and which satisfies the differential equation

$$\Phi(w)\frac{dw^2}{w(w-1)} > 0, \quad \text{where} \quad \Phi(w) = L\left(\frac{f(f-1)}{f-w}\right). \tag{1}$$

We found also that Γ makes an angle of less than $\pi/4$ with each of the confocal hyperbolas it meets. This angle is called the *hyperbolic angle* of Γ. It is well defined at each nonzero point on Γ, because there is a unique hyperbola with foci at 0 and 1 which meets Γ at this point.

We left open the question ([3], p. 205) whether $L(f(f-1))\neq 0$, or equivalently whether the quadratic differential has a simple pole at infinity. We now give an affirmative answer and deduce that Γ has an asymptotic half-line at infinity.

Theorem 2. *Let $F \in S_0$ be a support point which maximizes* Re $\{L\}$ *for some continuous linear functional L. Then $L(f(f-1)) \neq 0$. Furthermore, the arc Γ omitted by f is asymptotic to the half-line*

$$w = \frac{1}{3}\left\{1+\frac{L(f^2(f-1))}{L(f(f-1))}\right\} - L(f(f-1))t, \quad t \geqq 0, \tag{2}$$

at infinity.

Proof. To show that $L(f(f-1)) \neq 0$ we use the variation at infinity introduced by Schiffer [8] for the corresponding problem in S. We shall refer to the description of the method as it appears in [1], pp. 307—310. The quadratic differential (1) which determines Γ has a simple pole, a regular point, or a zero of finite order at infinity. It follows that Γ has an asymptotic direction at infinity: sgn $w \to e^{i\sigma}$ as $w \to \infty$ along Γ. Now truncate Γ to a subarc Γ_ϱ by removal of a section near the tip. Exactly as in [1], form the function

$$v(\omega;\ \varrho, \alpha) = \omega + \sum_{n=2}^{\infty} c_n(\varrho, \alpha)\omega^n,$$

which is analytic and univalent on the complement of a radial magnification of Γ_ϱ. As in [1], the asymptotic property of Γ allows an application of the Carathéodory convergence theorem to show that

$$v(\omega;\ \varrho, \alpha) \to k_\alpha(k_\beta^{-1}(\omega)) = \omega + \sum_{n=2}^{\infty} c_n(\alpha)\omega^n,$$

uniformly on compact subsets of $|\omega| < 1/4$, as $\varrho \to \infty$. Here k_α is the rotated Koebe function $k_\alpha(z) = z(1-e^{i\alpha}z)^{-2}$, and $\beta = \pi - \sigma$. Thus $c_n(\varrho, \alpha) \to c_n(\alpha)$, $n = 2, 3, \ldots$. The function

$$V(w) = V(w;\ \varrho, \alpha) = \varrho v(w/\varrho;\ \varrho, \alpha) = w + \sum_{n=2}^{\infty} b_n w^n, \quad b_n = c_n(\varrho, \alpha)\varrho^{1-n},$$

is analytic and univalent on $C - \Gamma_\varrho$ and in particular on the range of f. Also $V(0) = 0$, and so $V(1) \neq 0$. Thus the function $U(w) = V(w)/V(1)$ is analytic and univalent on the range of f, with $U(0) = 0$ and $U(1) = 1$. It follows that $f^* = U \circ f \in S_0$, so Re $\{L(f^*)\} \leqq$ Re $\{L(f)\}$.

The function Φ in (1) has the expansion

$$\Phi(w) = \sum_{n=1}^{\infty} L(f^n(1-f))w^{-n}$$

near infinity. Because $\Phi(w) \not\equiv 0$, not all of the coefficients can vanish. Suppose $L(f^n(1-f)) = 0$ for $n = 1, 2, \ldots, m-2$ but $L(f^{m-1}(1-f)) \neq 0$. In other words,

$$L(f) = L(f^2) = \ldots = L(f^{m-1}) \neq L(f^m), \quad m \geqq 3. \tag{3}$$

On the other hand

$$L(f^*) = \{1 + \sum_{n=2}^{\infty} b_n\}^{-1}\{L(f) + \sum_{n=2}^{\infty} b_n L(f^n)\},$$

and so

$$L(f^*) - L(f) = \{1 + \sum_{n=2}^{\infty} b_n\}^{-1} \sum_{n=m}^{\infty} b_n [L(f^n) - L(f)]$$

under the assumption (3). Now multiply by ϱ^{m-1} and let $\varrho\to\infty$ to conclude from the extremal property of f that

$$\operatorname{Re}\{c_m(\alpha)[L(f^m)-L(f)]\}\leqq 0. \tag{4}$$

The rest of the proof proceeds exactly as in [1]. With suitable choices of α we can conclude from (4) that $L(f^m)-L(f)=0$, contradicting our assumption. (It is here that the condition $m\geqq 3$ is used.) Thus $L(f(f-1))\neq 0$.

We can now show that Γ is asymptotic to the half-line (2) at infinity. It is convenient to write

$$\Phi(w)=\sum_{n=1}^{\infty}\alpha_n w^{-n},\quad \alpha_n=L(f^n(1-f)).$$

We have shown that $\alpha_1\neq 0$. Let Γ be given a parametrization $w=w(t)$, $0<t<T$, such that $w(t)\to\infty$ as $t\to 0$ and the differential equation (1) has the form

$$\frac{1}{w(w-1)}\Phi(w)\left(\frac{dw}{dt}\right)^2=1.$$

The transformation $w=v^{-2}$ then carries Γ to an analytic curve

$$v=\beta_1 t+\beta_3 t^3+\dots,\quad -T<t<T,$$

through the origin which satisfies

$$\frac{4}{v^2(1-v^2)}\Phi(1/v^2)\left(\frac{dv}{dt}\right)^2=1,$$

or

$$(\alpha_1+(\alpha_1+\alpha_2)\beta_1^2 t^2+\dots)(\beta_1^2+6\beta_1\beta_3 t^2+\dots)=\frac{1}{4}.$$

Equating coefficients, we find

$$\alpha_1\beta_1^2=\frac{1}{4},\quad (\alpha_1+\alpha_2)\beta_1^4+6\alpha_1\beta_1\beta_3=0. \tag{5}$$

Observe now that

$$w=v^{-2}=\lambda t^{-2}+\mu+O(t^2),\quad t\to 0,$$

where

$$\lambda=\beta_1^{-2},\quad \mu=-2\beta_1^{-3}\beta_3.$$

This means that Γ is asymptotic to the half-line

$$w=\mu+\lambda s,\quad s\geqq 0,$$

at infinity. The equations (5) give

$$\lambda=4\alpha_1,\quad \mu=\frac{1}{3}\left(1+\frac{\alpha_2}{\alpha_1}\right).$$

Thus the asymptotic half-line has the form (2), and the proof is complete.

3. The Montel class

Very little is changed if instead of S_0 we consider the Montel class M_r of functions f analytic and univalent in $\boldsymbol{D}$ with $f(0)=0$ and $f(r)=1$. Both classes are preserved under composition with univalent functions which fix both 0 and 1. The variation developed in [3] for S_0 therefore applies also to M_r. The same considerations show that each support point f of M_r, maximizing Re $\{L\}$ for some continuous linear functional L, must map $\boldsymbol{D}$ onto the complement of an analytic arc Γ which again satisfies the differential equation (1). Furthermore, Γ is monotonic with respect to ellipses with foci at 0 and 1, the only difference being that Γ now extends from a point $w_0 \neq 0$ to infinity. An omitted-value transformation allows us to deduce, exactly as in the case of S_0, that Re $\{\Phi(w)\}>0$ for all points $w \neq w_0$ on Γ, so that Γ has hyperbolic angle less than $\pi/4$ except perhaps at its tip.

Furthermore, the variation at infinity used in the proof of Theorem 2 applies equally to M_r, because again it consists of composition with univalent functions fixing 0 and 1. We conclude that $L(f(f-1)) \neq 0$ for support points $f \in M_r$, and that the omitted arc Γ is asymptotic to the half-line (2) at infinity.

4. Maximum hyperbolic angle

For the arcs Γ omitted by the support points of the Montel class, it is not known whether the bound $\pi/4$ on the hyperbolic angle is best possible, or whether a hyperbolic angle of $\pi/4$ can be realized at the terminal point w_0. In analogy with the investigation [2] of the corresponding problem for the class S, we shall now show under mild additional assumptions that this cannot happen unless Γ is a half-line. We begin with the following theorem.

Theorem 3. *Let f be a support point of M_r maximizing* Re $\{L\}$, *and suppose that its omitted arc Γ has a hyperbolic angle of $\pi/4$ at its tip w_0. Then in addition to* (1) *the arc Γ satisfies the differential equation*

$$w_0(w_0-1)\frac{\Phi(w)-\Phi(w_0)}{(w-w_0)^2}\frac{dw^2}{w(w-1)} > 0. \tag{6}$$

Proof. In view of the differential equation (1), the hypothesis of maximal hyperbolic angle is equivalent to supposing that Re $\{\Phi(w_0)\}=0$. (See [3], p. 204.) By the linearity of L, it then follows that Re $\{L(f)\}=$Re $\{L(g)\}$, where

$$g = \frac{(1-w_0)\psi}{f-w_0} \in M_r.$$

Thus g is also a support point so its omitted arc $\tilde{\Gamma}$ satisfies

$$\Psi(\omega)\frac{d\omega^2}{\omega(\omega-1)} > 0, \quad \Psi(\omega) = L\left(\frac{g(g-1)}{g-\omega}\right). \tag{7}$$

We shall see that this equation for $\tilde{\Gamma}$ can be transformed to the second differential equation (6) for Γ. By the definition of g, the mapping

$$\omega = \frac{(1-w_0)w}{w-w_0} \tag{8}$$

sends Γ to $\tilde{\Gamma}$. The identity

$$\frac{(1-\omega)g}{g-\omega} = \frac{(1-w)f}{f-w}$$

is easily verified. Note also that

$$f - \frac{(1-w)f}{f-w} = \frac{f(f-1)}{f-w}$$

and

$$g - \frac{(1-\omega)g}{g-\omega} = \frac{g(g-1)}{g-\omega}.$$

These relations give

$$\Phi(w_0) = L\left(\frac{f(f-1)}{f-w_0}\right) = L(f-g) = \Phi(w) - \Psi(\omega),$$

so that $\Psi(\omega) = \Phi(w) - \Phi(w_0)$. Also, a calculation based on (8) gives

$$\frac{d\omega^2}{\omega(\omega-1)} = \frac{w_0(w_0-1)}{(w-w_0)^2}\frac{dw^2}{w(w-1)}.$$

Substituting these expressions into (7), we obtain (6), and the proof is complete.

If $\Phi(w_0)=0$, we can divide (1) by (6) to obtain

$$\frac{(w-w_0)^2}{w_0(w_0-1)} > 0, \quad w \in \Gamma, \quad w \neq w_0.$$

This implies that Γ is a half-line.

For the case of a point-evaluation functional

$$L(f) = e^{-i\sigma} f(\zeta), \quad \zeta \in \mathbf{D}, \quad \zeta \neq 0,\ r,$$

we find in a similar manner that $\operatorname{Re}\{\Phi(w_0)\}=0$ implies

$$\frac{(B-w_0)}{w_0(w_0-1)}(w-w_0) > 0, \quad w \in \Gamma, \quad w \neq w_0,$$

where $B=f(\zeta)$. Thus Γ is a half-line. It must therefore coincide (near infinity) with its asymptotic half-line (2), which takes the form

$$w = \frac{1}{3}(B+1) - e^{-i\sigma}B(B-1)t, \quad t \geqq 0.$$

The general case cannot be handled so easily. We shall assume that L is a functional of *rational type,* meaning that $L(f(f-1)/(f-w))$ is a rational function of w for each $f\in M_r$. Our result is again analogous to that obtained in [2] for the class S.

Theorem 4. *Let L be a continuous linear functional of rational type, not constant on the Montel space M_r. Let f maximize* $\operatorname{Re}\{L\}$ *over M_r, and suppose that the arc Γ omitted by f has a hyperbolic angle of $\pi/4$ at its tip. Suppose further that $\Phi(w)=L(f(f-1)/(f-w))$ has no simple zero at 0 or 1 and no double zero elsewhere. Then Γ is a half-line.*

Proof. We shall use the method introduced in [2], with some of the details omitted because the reasoning is quite similar. We may assume that $\Phi(w_0)\neq 0$, because (as noted above) the theorem is true if $\Phi(w_0)=0$. Let z_0 and z_1 be the points on the unit circle where $f(z_0)=w_0$ and $f(z_1)=\infty$. By Theorem 3, the arc Γ satisfies both (1) and (6). A standard argument (cf. [2]) allows us to conclude that $w=f(z)$ satisfies the two differential equations

$$\Phi(w)\frac{(w')^2}{w(w-1)}=\frac{1}{z^2}Q(z) \tag{9}$$

and

$$w_0(w_0-1)\frac{\Phi(w)-\Phi(w_0)}{(w-w_0)^2}\frac{(w')^2}{w(w-1)}=\frac{1}{z^2}R(z), \tag{10}$$

where Q and R are rational functions. Apart from double zeros of Q at z_0 and of R at z_1, both functions are real, negative, and finite on the unit circle. (By Theorem 2, Φ has a simple zero at infinity.) Equation (10) shows that $\Phi'(w_0)\neq 0$, since $R(z_0)\neq 0$.

Dividing (9) by (10), and recalling that Φ is rational, we see that f is an algebraic function satisfying

$$\frac{(w-w_0)^2\Phi(w)}{\Phi(w)-\Phi(w_0)}=w_0(w_0-1)\frac{Q(z)}{R(z)},\quad w=f(z). \tag{11}$$

But $\Phi(w_0)\neq 0$ and $\Phi'(w_0)\neq 0$, so (11) has the form

$$\frac{\Phi(w_0)}{\Phi'(w_0)}(w-w_0)+O((w-w_0)^2)=O((z-z_0)^2)$$

near the point $(z_0, w_0)\in \boldsymbol{C}^2$. Thus the equation (11) has a *unique* local solution $w=F(z)$ such that $F(z_0)=w_0$. In particular, F is an analytic continuation of f to a full neighborhood of z_0.

The differential equations (9) and (10) give a simultaneous global analytic continuation of f to a possibly multiple-valued algebraic function F which satisfies (11). In order to prove that F is rational, we have only to show that it is single-valued. For this it suffices to show that $F(z_0)=w_0$ in every sheet of the continuation.

By what we observed above, this will show that in every analytic continuation along a closed path beginning at z_0 and ending at the same point, F will return to its initial function element.

If some continuation produces a value $F(z_0)\neq w_0$, then the algebraic function F has one of the two local structures

$$w = F(z) = w_1+c(z-z_0)^\alpha+\ldots,\quad c\neq 0,\quad \alpha>0,\quad w_1\neq w_0; \tag{12}$$

$$w = F(z) = c(z-z_0)^{-\alpha}+\ldots,\quad c\neq 0,\quad \alpha>0. \tag{13}$$

In each case F must be a local solution to both differential equations (9) and (10).

Suppose first that F has the local form (12) with $w_1=0$. If Φ has a zero of order $n\geqq 1$ at the origin, then (9) gives $\alpha=4/(n+1)$, while (10) implies $\alpha=2$. Thus $n=1$ and Φ has a simple zero at the origin, contrary to hypothesis. If Φ is analytic at the origin and $\Phi(0)\neq 0$, then (9) gives $\alpha=4$ while (10) gives $\alpha\leqq 2$. If Φ has a pole of order $m\geqq 1$ at 0, then (9) gives $(1-m)\alpha=4$, which is impossible.

Next suppose that F has the local form (12) with $w_1=1$. If Φ has a zero of order $n\geqq 1$ at 1, then (9) gives $\alpha=4/(n+1)$ while (10) gives $\alpha=2$. Thus $n=1$ and Φ has a simple zero at 1, contrary to hypothesis. If Φ is analytic at 1 and $\Phi(1)\neq 0$, then (9) gives $\alpha=4$ while (10) gives $\alpha\leqq 2$. If Φ has a pole of order $m\geqq 1$ at 1, then (9) gives $(1-m)\alpha=4$, which is impossible.

Suppose now that F has the local form (12) with $w_1\neq 0, 1, w_0$. If Φ has a zero of order $n\geqq 1$ at w_1, then (9) gives $\alpha=4/(n+2)$ while (10) gives $\alpha=1$. Thus $n=2$, contradicting our hypothesis that Φ has no double zero except perhaps at 0 or 1. If Φ is analytic at w_1 and $\Phi(w_1)\neq 0$, then (9) gives $\alpha=2$ while (10) gives $\alpha\leqq 1$. If Φ has a pole of order $m\geqq 1$ at w_1, then (9) gives $(2-m)\alpha=4$ while (10) gives $(2-m)\alpha=2$.

Finally, suppose that F has the local form (13). Then since Φ has a simple zero at infinity, (9) gives $\alpha=4$ while (10) gives $\alpha=1$. (Theorem 2 is used here.)

Having eliminated all cases to the contrary, we have now shown that $F(z_0)=w_0$ in every continuation. Thus F is single-valued and is therefore a rational function. In other words, the support point f is a rational function which maps $\boldsymbol{D}$ univalently onto the complement of an analytic arc Γ extending to infinity. But as Srebro [9] has shown, this implies that Γ is a half-line. This completes the proof.

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