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CONCIRCULAR TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS

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Introduction. In this paper we deal with connected Riemannian manifolds (M, g), $(\overline{M}, \bar{g})$ of dimension n and of class C^3 at least. We shall denote their sectional curvature tensors by R, $\overline{R}$, their Ricci tensors by r, $\bar{r}$, and their scalar curvatures by S, $\bar{S}$. The space of C^2-vector fields on M will be denoted by $\Gamma(M)$.

With every strictly positive scalar function u of class C^2 on M, we associate the conformal deformation $g \mapsto u^{-2}g$ of (M, g). The curvature tensors of $(M, \bar{g}=u^{-2}g)$ are then given, in local coordinates, by:

$$\bar{r}_{ij}-r_{ij} = \frac{n-2}{u}u_{i,j}+\left(\frac{\Delta u}{u}-(n-1)\frac{\operatorname{grad}^2 u}{u^2}\right)g_{ij}, \tag{0.1}$$

$$u^2\overline{R}_{ijkl}-R_{ijkl} = -\frac{\operatorname{grad}^2 u}{u^2}(g_{ik}g_{jl}-g_{il}g_{jk}) \tag{0.2}$$

$$+\frac{1}{u}(g_{ik}u_{j,l}+g_{jl}u_{i,k}-g_{il}u_{j,k}-g_{jk}u_{i,l}),$$

$$u^{-2}\bar{S}-S = 2(n-1)\left(\frac{\Delta u}{u}-\frac{n}{2}\frac{\operatorname{grad}^2 u}{u^2}\right), \tag{0.3}$$

where the $u_{i,j}$ are the covariant derivatives of second order of u, and $\Delta u=g^{ij}u_{i,j}$. (These formulae follow from the usual ones by setting $u=e^{-\sigma}$.)

Definition. *The scalar function* u *and the associated conformal deformation* $g\mapsto u^{-2}g$ *are said to be concircular if there exists a scalar function* ϱ *such that*

$$u_{i,j} = \varrho g_{ij} \quad (i, j = 1, 2 \dots n). \tag{0.4}$$

Then, by setting $\tau=\varrho/u-(1/(2u^2))\operatorname{grad}^2 u$, the formulae (0.1), (0.2), (0.3) reduce to:

$$\bar{r}_{ij}-r_{ij} = 2(n-1)\tau g_{ij}, \tag{0.5}$$

$$u^2\overline{R}_{ijkl}-R_{ijkl} = 2\tau(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{0.6}$$

$$u^{-2}\bar{S}-S = 2n(n-1)\tau. \tag{0.7}$$

More generally, if (M, g) and $(\overline{M}, \bar{g})$ are two Riemannian manifolds of the same dimension n, a morphism $f\colon M\to\overline{M}$ is said to be *concircular* if it is conformal and if there exists a scalar function ϱ on M such that $u=|f'|^{-1}$ satisfy (0.4). In other

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words $f: (M, g) \to (\overline{M}, \bar{g})$ is concircular if, and only if, $g \mapsto f^*\bar{g}$ is a concircular deformation of (M, g).

A conformal morphism f such that $|f'|$ is constant will be called a *similarity;* a conformal deformation $g \mapsto u^{-2}g$ such that u is constant will be said to be a *homothety.*

Concircular transformations have been introduced by K. Yano [13] as conformal transformations preserving geodesic circles (curves whose normal parametrizations satisfy $d^3x/ds^3 = k(dx/ds)$ with $k = \text{const.}$). Later, W. O. Vogel [12] proved that every morphism preserving geodesic circles is necessarily conformal; and Y. Tashiro [9] gave a classification of complete Riemannian manifolds admitting a concircular field u (i.e. satisfying (0.4) but not necessarily >0).

Let us notice that the characterization of the sphere given by M. Obata [8] sets upon this classification. For other papers relative to concircular transformations, see [5, 8, 10]. An equation close to (0.4) has been studied by J. Lafontaine [7].

In this paper, we first review the main results concerning concircular deformations (Section 1); then we shall set some apparently new results, and examine the special case $\tau = 0$ (Sections 2, 3, 4). Most part of these results are extensible to pseudo-Riemannian manifolds.

1. Preliminaries. At first let us notice that, if $n \geqq 3$, either relation (0.5), (0.6) involves (0.4) with $\varrho = \tau u + (1/(2u)) \operatorname{grad}^2 u$. It easily follows:

Property 1.1. *Any conformal mapping between Einstein manifolds of dimension* $n \geqq 3$ *is concircular.*

Conversely if $f: (M, g) \to (\overline{M}, \bar{g})$ *is concircular, and if* (M, g) *is an Einstein space [resp. a space with constant sectional curvature], then so is* $(\overline{M}, \bar{g})$.

Property 1.2. *If* $u: M \to \mathbf{R}$ *is concircular (satisfying* (0.4)*) then:*

a) *For any constant* λ, $u + \lambda$ *is concircular; and* u^{-1} *is concircular on* $(M, u^{-2}g)$.

b) *There exists locally a function* G *such that*

$$\operatorname{grad}^2 u = G(u) \quad \text{and} \quad \varrho = \tfrac{1}{2} G'(u). \tag{1.1}$$

c) *For any vector fields* X, Y *on* M

$$R(X, Y). \operatorname{grad} u = (L_X \varrho) Y - (L_Y \varrho) X. \tag{1.2}$$

Proof. The first assertion a) is obvious; for $n \geqq 3$ the second one can be obtained by exchanging g and $\bar{g}$.

To prove b), we notice that, with intrinsic notations, (0.4) is equivalent with

$$\forall X \in \Gamma(M): \quad \nabla_X \operatorname{grad} u = \varrho X \tag{1.3}$$

which implies

$$\nabla_X \operatorname{grad}^2 u = 2\varrho X. \operatorname{grad} u = 2\varrho \nabla_X u$$

or, in other terms, $d(\operatorname{grad}^2 u) = 2\varrho\, du$.

Finally (1.2) follows from (1.3) by covariant differentiation.

Property 1.3. *In order that* $u: M\to \mathbf{R}$ *be concircular, it is necessary and sufficient that there exist local coordinates* (x_i), *with* $x_1=u$, *such that the metric of* M *be*

$$ds^2 = \frac{du^2}{G(u)} + G(u) \sum_{i,j\geqq 2} \gamma_{ij}(x_2, \ldots, x_n)\, dx_i\, dx_j \tag{1.4}$$

with $\varrho=(1/2)G'(u)$, *or equivalently:*

$$ds^2 = dv^2 + \varphi^2(v) \sum_{i,j\geqq 2} \gamma_{ij}(x_2, \ldots, x_n)\, dx_i\, dx_j \tag{1.5}$$

with $v=\int [G(u)]^{-1/2}\, du$ *and* $\varphi^2(v)=G(u)$.

Obviously, the function $G(u)=\mathrm{grad}^2\, u$ need not be defined at stationary points of u; but on a complete manifold, there are at most two such points (cf. [10]).

For the applications of concircular properties to Einstein or other special spaces, see [2], [5], [9], [10], [11], [12], [13].

2. Conformal properties. From the results reviewed in Section 1, we easily infer

Lemma 2.1. *If* $u: M\to \mathbf{R}$ *is concircular* (*i.e. satisfies* (0.4)), *then*

a) *the u-hypersurfaces (defined by* $u=$Const.*) are totally umbilical, of constant normal curvature* $\varrho/|\mathrm{grad}\, u|$.

b) *the integral curves of* grad u *are geodesics whose tangent at any point is an eigen direction of the Ricci tensor.*

The proof is classical; the last assertion follows from (1.2). In [4] we proved that the "conformal circles" of E. Cartan and K. Yano are the curves which, by a suitable conformal deformation of M can be changed into geodesics whose tangent is an eigen direction of the Ricci tensor. By looking for a converse of Lemma 2.1, we obtain:

Theorem 2.2. *Let* (M, g) *be a Riemannian manifold and* $u: M\to \mathbf{R}$ *a strictly positive scalar function having only isolated stationary points. In order that there exists a metric* $\bar{g}$, *conformal to* g, *such that* u *be concircular on* $(M, \bar{g})$, *it is necessary and sufficient that*

i) *the u-hypersurfaces be totally umbilical,*

ii) *their orthogonal trajectories be conformal circles.*

Proof. The necessity of these conditions follows from Lemma 2.1, since they are invariant under a conformal deformation.

Conversely we know (cf. [3]) that the condition i) implies the existence of local coordinates x_i, with $x_1=u$, such that the metric of M is

$$ds^2 = A^2\, du^2 + B^2 \sum_{i,j\geqq 2} \gamma_{ij}(x_2, \ldots, x_n)\, dx_i\, dx_j.$$

If the condition ii) is satisfied, the curves $x_i=$Const. $(i\geqq 2)$ are still conformal circles for the metric ds^2/A^2; and since they are geodesics for this new metric, $\partial/\partial u$ must be an eigen vector for the Ricci tensor. Now the $(1,j)$-components $(j\neq 1)$ of the Ricci tensor for ds^2/A^2 are given by

$$R_{1j} = -(n-2)\frac{\partial^2}{\partial u\,\partial x_j}\operatorname{Log}\left(\frac{B}{A}\right)$$

and therefore the condition ii) is equivalent with the existence of two positive functions α, β such that

$$\frac{B}{A} = \alpha(u)\beta(x_2, \ldots, x_n).$$

By setting $\bar{\gamma}_{ij}=\beta\gamma_{ij}$, we have

$$\frac{ds^2}{A^2} = du^2+\alpha^2(u)\sum_{i,j\geqq 2}\bar{\gamma}_{ij}(x_2, \ldots, x_n)\,dx_i\,dx_j$$

and the Property (1.3) shows that u is concircular for the metric $ds^2/(\alpha A^2)$.

Let us notice also that $\int \alpha\ du$ is concircular for ds^2/A^2. On the other hand we have:

Lemma 2.3. *If u is concircular on (M, g), and if θ is a differentiable function on $u(M)$, the only metrics conformally equivalent with g which admit $\theta(u)$ as a concircular field are given by $\bar{g}=C|\theta'|g$, with* $C=$Const.

Application: ***Conformally flat manifolds admitting a concircular deformation.*** Starting with $\boldsymbol{R}^n$ we have to look for functions $u>0$ such that the u-hypersurfaces be totally umbilical and their orthogonal trajectories be conformal circles: in other words, u-hypersurfaces are hyperspheres (or hyperplanes) whose orthogonal trajectories are circles (or straight lines); and they must belong to a bundle of spheres or hyperplanes. By using a Möbius transformation, we are brought back to the three typical bundles, respectively defined by

$$\text{i) } \Sigma x_i^2 = \text{Const.,}\quad \text{ii) } x_1 = \text{Const.,}\quad \text{iii) } \frac{x_2}{x_1} = \text{Const.}$$

Therefore, if u is concircular on a conformally flat manifold (M, g) there exist local coordinates x_i and a function θ of one variable such that

$$u = \theta(\Sigma x_i^2),\quad u = \theta(x_1)\quad \text{or}\quad u = \theta(x_2/x_1)$$

and, for a suitable choice of the function $\sigma(x_1, \ldots, x_n)$:

$$ds^2 = e^{2\sigma}\Sigma\, dx_i^2.$$

Now it can be directly checked that $\Sigma\, dx_i^2$ admits x_1 and $\Sigma\, x_i^2$ for concircular fields, while $(x_1^2+x_2^2)^{-1}\Sigma\, dx_i^2$ admits $\operatorname{arc\,tg}(x_2/x_1)$ as a concircular field. With help of Lemma 2.3, we can state:

Theorem 2.4. *The conformally flat metrics admitting a concircular field are locally given, by suitable choice of coordinates, by*

$$ds^2 = \varphi(\Sigma x_i^2)\Sigma\, dx_i^2, \quad ds^2 = \varphi(x_1)\Sigma\, dx_i^2$$

or

$$ds^2 = (x_1^2+x_2^2)^{-1}\varphi(\operatorname{arc\,tg} x_2/x_1)\Sigma\, dx_i^2$$

where φ *is an arbitrary function of one variable; and the associated concircular fields are the primitives of* φ.

This result completes Theorem 2 of [6]. Let us notice that, by setting $u=\operatorname{arc\,tg}(x_2/x_1)$ and $v=(1/2)\operatorname{Log}(x_1^2+x_2^2)$, the metrics of the third type can also be written

$$ds^2 = \varphi(u)(du^2+dv^2+e^{-2v}\textstyle\sum_{i\geqq 3} dx_i^2).$$

The global existence of concircular fields depends on the topology of the manifold.

3. A special case: quasi-similarities. Definition 3.1. *A conformal deformation [resp. a conformal morphism f] will be called quasi-homothetic [resp. a quasi-similarity] if there exists a scalar function* ϱ *such that the associated function u [resp. the function* $u=|f'|^{-1}$*] satisfies*

$$u_{i,j} = \varrho g_{ij} \quad \text{and} \quad \operatorname{grad}^2 u = 2\varrho u. \tag{3.1}$$

(In other terms: u is concircular and the associated function $\tau=\varrho/u-(1/(2u^2))\operatorname{grad}^2 u$ is null.)

If $n\geqq 3$, the conditions 3.1 express that the Ricci and sectional curvatures of M are transformed in the same way as under an homothety [resp. a similarity] of ratio u^{-1} (see formulae (0.5) and (0.6)). In particular, any conformal morphism of Ricci-flat manifolds (i.e. with Ricci curvature zero) of dimension $n\geqq 3$ is automatically a quasi-similarity. More precisely, we have:

Lemma 3.1. *Let* (M, g) *be a flat [resp. Ricci-flat] manifold of dimension* $n\geqq 3$, *and* $f\colon (M, g)\to(\overline{M}, \bar g)$ *a conformal morphism. In order that* $(f(M), \bar g)$ *be flat [resp. Ricci-flat] it is necessary and sufficient that* f *be a quasi-similarity.*

Quasi-similarities have been studied by us in [5]. Let us recall some results.

Lemma 3.2. *If* $u=M\to\mathbf{R}$ *satisfies* (3.1), *then* ϱ *is a constant; and for any vector fields* X, Y *on* M, *we have*

$$R(X, Y)\operatorname{grad} u = 0, \tag{3.2}$$

Lemma 3.3. *In order that $u: M \to \mathbf{R}$ be quasi-homothetic, it is necessary and sufficient that there exists a constant ϱ and local coordinates (x_i), with $x_1 = u$, such that:*

$$ds^2 = \frac{du^2}{2\varrho u} + u \sum_{i,j \geqq 2} \gamma_{ij}(x_2, \ldots, x_n)\, dx_i\, dx_j. \tag{3.3}$$

(This is a special case of Property 1.3.)

For brevity, we shall say that a quasi-homothety [resp. a quasi-similarity] is proper if the associated function u is not constant.

Theorem 3.4. *A complete Riemannian manifold does not admit any proper quasi-homothety or quasi-similarity.*

Proof. On any geodesic γ satisfying $dx/ds = \text{grad } u / |\text{grad } u|$, we have $du/ds = (2\varrho u)^{1/2}$ and $d^2u/ds^2 = \varrho$; hence, if $\varrho \neq 0$: $2u = \varrho(s - s_0)^2$; and if s could run from $-\infty$ to $+\infty$, u would take the value zero.

The case of $\mathbf{R}^n$ and of conformally flat manifolds. First of all, on $\mathbf{R}^n$, the only nonconstant solutions of (3.1) are the functions

$$u = \tfrac{1}{2}\varrho \Sigma (x_i - a_i)^2 \quad (\varrho, a_i = \text{Const.}) \tag{3.4}$$

This easy remark provides a very short proof of the theorem of Liouville in class C^3: namely, if U is an open set of $\mathbf{R}^n$ $(n \geqq 3)$ and if $f: U \to \mathbf{R}^n$ is a conformal (not necessarily injective) morphism, then, from Lemma 3.1, f is a quasi-similarity and $u = |f'|^{-1}$ satisfies (3.4) for some values of $\varrho, a_1, \ldots, a_n$. If $\varrho = 0$, $u = \text{Const.}$ and f is a similarity. If $\varrho \neq 0$, let be j the inversion $x \mapsto a + (2/\varrho)|x - a|^{-2}(x - a)$. Then we have $|f'(x)| = |j'(x)| = |j'(j(x))|^{-1}$, hence $|(f \circ j)'(x)| = 1$ and $f \circ j$ is an isometry. In both cases f is a Möbius transformation.

On another side, we may complete Lemma 2.3 and Theorem 2.4 by stating

Lemma 3.5. *If u is quasi-homothetic for (M, g), the only metrics, conformally equivalent with g, which admit a nonconstant function $\theta(u)$ as quasi-homothetic, are of the type $\bar{g} = Cu^{\lambda - 1}g$ with $C, \lambda = \text{Const.}$, $\lambda \neq 0$, the function θ then being $\theta(u) = ku^\lambda$ $(k = \text{Const.})$.*

Theorem 3.6. *The conformally flat metrics admitting a quasi-homothetic deformation are locally given, with suitable coordinates, by:*

$$ds^2 = [\Sigma(x_i^2)]^{\lambda - 1} \Sigma\, dx_i^2, \qquad ds^2 = e^{\lambda x_1} \Sigma\, dx_i^2 \quad \textit{or}$$

$$ds^2 = e^{\lambda u}(du^2 + dv^2 + e^{-2v} \textstyle\sum_{i \geqq 3} dx_i^2)$$

where $\lambda \neq 0$ is a constant.

Among these metrics there is no one of constant curvature $\neq 0$, as we could infer from (3.2).

4. Concircular deformations of submanifolds. The restriction to a submanifold of a concircular transformation is still conformal, but not necessarily concircular. In fact we have:

Theorem 4.1. *Let* $u: M \to \mathbf{R}$ *concircular. In order that the restriction of* u *to a submanifold* V *be concircular it is necessary and sufficient that at any point of* V

either grad u *be tangent to* V

or the normal component N *of* grad u *be an umbilical direction for* V.

Proof. We know that $v=u_{|V}$ is concircular if, and only if, there exists a function σ on V such that

$$\forall X \in \Gamma(V): \quad \overline{\nabla}_X (\operatorname{grad} v) = \sigma X, \tag{4.1}$$

where $\overline{\nabla}_X$ is the induced connection on V. Now $\operatorname{grad} v = T = \operatorname{grad} u - N$ is the tangential component of grad u, and $\overline{\nabla}_X T$ is the tangential component of $\nabla_X T$. By using (1.3) we see that the condition (4.1) is realized if, and only if, $\nabla_X N - (\varrho - \sigma) X$ is normal to V; and, if $N \neq 0$, this means that N is an umbilical direction for V.

This last condition is realized, in particular, if grad u is normal to V at any point; in that case v is constant and $\sigma = 0$.

If u is quasi-homothetic (which implies $\operatorname{grad}^2 u = 2\varrho u$ and $\varrho = \text{Const.}$) and if we want $v = u_{|V}$ to be quasi-homothetic, we have to set the additional condition $\operatorname{grad}^2 v = 2\sigma v$ with $\sigma = \text{Const.}$ This is realized if and only if $|T|^2/|N|^2 = \sigma/(e-\sigma) = \text{Const.}$, i.e. if the angle of grad u with V is constant. We can state:

Theorem 4.2. *Let* $f: M \to \overline{M}$ *be a quasi-similarity and* V *a submanifold of* M. *In order that the restriction of* f *to* V *be a quasi-similarity, it is necessary and sufficient that the field* $\operatorname{grad}(|f'|^{-1})$ *be tangent to* V, *or that it make a constant angle with* V *and its normal component be an umbilical direction for* V.

A special case: ***Hypersurfaces of an Einstein manifold.*** If V is an hypersurface satisfying the second condition stated in Theorem 4.1, then V is totally umbilical with scalar normal curvature $\lambda = (\sigma - \varrho)/|N|$. Now it is elementary to check that, in an Einstein space, the scalar normal curvature of a totally umbilical hypersurface is constant. In that case, the angular condition of Theorem 4.2 implies $|T| = \text{Const.}$ (since $(\sigma - \varrho)/|N| = \text{Const.}$), hence $\operatorname{grad}^2 v = 2\sigma v = \text{Const.}$, and $v = \text{Const.}$, which is realized if, and only if, $N = 0$. We can state:

Theorem 4.3. *Let* M *be an Einstein space, and* $u: M \to \mathbf{R}$ *a quasi-homothetic deformation. In order that the restriction of* u *to an hypersurface* V *be quasi-homothetic, it is necessary and sufficient that* V *be tangent or normal to the vector field* grad u.

By using a previous remark, we have:

Theorem 4.4. *Let* (M, g), $(\overline{M}, \bar{g})$, *be two Ricci-flat Riemannian manifolds of dimension* $n \geqq 4$, $f\colon M \to \overline{M}$ *a conformal morphism, and* V *a flat [resp. Ricci-flat] hypersurface of* M. *In order that* $f(V)$ *be flat [resp. Ricci-flat] it is necessary and sufficient that the field* grad $(|f'|^{-1})$ *be tangent or normal to* V.

In particular, the only flat hypersurfaces of $\boldsymbol{R}^n$ whose image, under an inversion with pole 0, is a flat hypersurface, are parts of cones with vertex 0 (this result being also true for $n=3$).

However these results cannot be extended to submanifolds of co-dimension $\geqq 2$, as is proved by the following counter-example which infirms an assertion of [1].

Example. Let V be the submanifold of $\boldsymbol{R}^{2p}$, image of the domain $(t_i > 0)$ of $\boldsymbol{R}^p$ under the imbedding f of components

$$f_k(t_1, \dots, t_p) = \frac{1}{\sqrt{2}} t_k \cos(\operatorname{Log} t_k) \quad f_{k+p} = \frac{1}{\sqrt{2}} t_k \sin(\operatorname{Log} t_k) \quad (1 \leqq k \leqq p).$$

We have $\sum_{i=1}^{2p} df_i^2 = \sum_{k=1}^{p} dt_k^2$, which proves that V is flat.

The image of V under the inversion $j\colon x \mapsto |x|^{-2}x$ admits the parametrization $g = j \circ f$, which satisfies

$$\sum_{i=1}^{2p} dg_i^2 = \sum_{i=1}^{2p} \frac{df_i^2}{|f|^4} = \frac{4}{|t|^4} \sum_{k=1}^{p} dt_k^2.$$

So $j(V)$ is flat, although V is neither tangent nor orthogonal to the field grad $|j'|^{-1}$. In fact the tangential and normal components of grad $|j'(x)|^{-1} = 2x$ at the point $x = f(t)$ are given by:

$$N_k = f_k + f_{k+p}, \quad N_{k+p} = f_{k+p} - f_k \quad (1 \leqq k \leqq p)$$

$$T_k = f_k - f_{k+p}, \quad T_{k+p} = f_k + f_{k+p} \quad (1 \leqq k \leqq p).$$

We can check that N is an umbilical direction for V, and that grad $|j'|^{-1}$ makes an angle of $\pi/4$ with V. Thus conditions of Theorem 4.2 are exactly fulfilled.

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