COMPLEMENTS TO HAVIN'S THEOREM
ON $L^2$-APPROXIMATION BY ANALYTIC FUNCTIONS

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Introduction. For a Borel set $E \subset \mathbb{C}$ denote by $L^2_0(E)$ the linear subset of $L^2(E)$ (with respect to 2-dimensional Lebesgue measure $\lambda$) consisting of those $f \in L^2(E)$ which can be extended to an analytic function in some open set containing $E$. It was shown by Havin [10] that $f \in L^2(E)$ belongs to the $L^2$-closure $\text{cl} L^2_0(E)$ of $L^2_0(E)$ if and only if

$$\int f \overline{\partial} \varphi \, d\lambda = 0$$

for every Beppo Levi function $\varphi$ on $\mathbb{C}$, (made "precise" in the sense of Deny and Lions [4]) for which

$$\varphi = 0 \text{ quasi everywhere in } \mathbb{C} \setminus E,$$

that is, everywhere in $\mathbb{C} \setminus E$ except in some set which locally is of zero outer logarithmic capacity. (We write $\partial = \partial/\partial x - i \partial/\partial y$.)

As a further necessary and sufficient condition for $f \in L^2(E)$ to belong to $\text{cl} L^2_0(E)$, we find in Section 2 that there shall exist a function $u$ which is finely harmonic, in the sense of [6], quasi everywhere in the interior $E'$ of $E$ in the Cartan fine topology, and such that

$$f = \partial u \text{ almost everywhere in } E',$$

in the sense of differentiability in the fine topology. In the affirmative case, $u$ may be taken to be the restriction to $E'$ of a Beppo Levi function $U$ (precise, as above) on $\mathbb{C}$, and such that $\partial U = F$ a.e. in $\mathbb{C}$, where $F$ denotes any prescribed extension of $f$ to a function in $L^2(\mathbb{C})$. At the same time we show that the property of belonging to $\text{cl} L^2_0(E)$ has a local character in the fine topology.

The key to our proof of Havin's theorem and these complements to it is the analogous result on Beppo Levi functions obtained in [8] (valid also in higher dimensions).

In Section 3 it is shown that the property of unique continuation of analytic functions gets lost when we pass from $L^2_0(E)$ to its $L^2$-closure. In particular, the functions in $\text{cl} L^2_0(E)$ (for suitable $E$) are not all finely holomorphic q.e. in $E'$ in the sense of [7]. To prove this we employ an example due to Lyons [11], showing that finely harmonic functions do not in general have the property of unique continuation.

Finally, in Section 4, we extend the preceding results to the case of a Borel set $E$ in $\mathbb{R}^k$, $k=1, 2, \ldots$. The functions $f, \varphi,$ and $u$ above are now replaced by exterior

differential forms (currents), the homogeneous parts of which are of even degree in the case of \( f \), say, and of odd degree in the case of \( \varphi \) and \( u \). The operators \( \partial \) and \( \bar{\partial} \) are replaced by the restrictions of \( d+\delta \) (exterior differentiation plus co-differentiation) to the forms that are "odd" and "even", respectively, in the above sense. For \( k=2 \) we recover the original case of Havin's theorem.

1. Spaces of Beppo Levi functions of a complex variable

For any domain (=connected open set) \( \Omega \subset \mathbb{R}^2 = \mathbb{C} \) we denote by \( BL(\Omega) \) the vector space of all complex distributions \( f \in \mathcal{D}'(\Omega) \) whose gradient \( \nabla f \) is in \( L^2(\Omega) \) with respect to Lebesgue measure \( \lambda \) on \( \mathbb{C} \), see Deny and Lions [4], and further Schulze and Wildenhain [14, Kapitel IX]. Thus \( BL(\Omega) \) is a prehilbert space with the degenerate inner product

\[
(u, v)_1 = \int_\Omega \nabla u \nabla v \, d\lambda.
\]

The quotient space of \( BL(\Omega) \) modulo the constants is a separable Hilbert space, denoted by \( BL^*(\Omega) \).

The elements of \( BL(\Omega) \) are (equivalence classes modulo \( \lambda \) of) locally \( \lambda \)-integrable functions, called Beppo Levi functions. Each such equivalence class contains, however, a unique, much smaller class — an equivalence class modulo the polar \(^1\) sets, and consisting of quasi-continuous \(^2\) functions on \( \Omega \) (called "fonctions précisées" in [4]). Recall that a function is quasi-continuous if and only if it is finely continuous (i.e., continuous with respect to the fine topology\(^3\)) on \( \mathbb{C} \) off some polar (hence finely closed and finely discrete) set, cf. [4, p. 356].

We denote by \( BLD(\Omega) \) the vector space of all (equivalence classes modulo polar sets of) Beppo Levi - Deny functions, that is, quasi-continuous Beppo Levi functions on \( \Omega \). Considered as prehilbert spaces, \( BL(\Omega) \) and \( BLD(\Omega) \) are the same.

The vector space \( \mathcal{D}(\Omega) \) of all infinitely differentiable functions of compact support in \( \Omega \) is a subspace of \( BL(\Omega) \) and of \( BLD(\Omega) \). The Dirichlet seminorm

\[
\|u\|_1 = (u, u)^{1/2} = \|\nabla u\|_{L^2(\Omega)}
\]

is a norm on \( \mathcal{D}(\Omega) \). Following [4] we denote by \( \mathcal{D}^1(\Omega) \) the completion of \( \mathcal{D}(\Omega) \) in this norm. The Hilbert space \( \mathcal{D}^1(\Omega) \) may thus be identified naturally with the closure

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\(^1\) A polar set (in \( \mathbb{C} \)) is a set which locally is of outer logarithmic capacity 0.

\(^2\) A function \( \varphi: \Omega \to \mathbb{C} (\Omega \subset \mathbb{C}) \) is called quasi continuous if \( \Omega \) is the union of bounded open sets \( \omega \) for which there exists, for every \( \varepsilon > 0 \), an open set \( \omega_\varepsilon \subset \omega \) of logarithmic capacity \( < \varepsilon \) such that the restriction of \( \varphi \) to \( \omega \setminus \omega_\varepsilon \) is continuous.

\(^3\) The fine topology on \( \mathbb{C} \) is the weakest topology for which all subharmonic functions (on open subsets of \( \mathbb{C} \)) are continuous. The fine topology is strictly stronger than the usual Euclidean topology on \( \mathbb{C} \).
\(L^3\)-approximation by analytic functions

For the particular case \(\Omega = C\) we have the following well-known

Lemma 1.1. a) 0 is the only element of \(L^2(C)\) represented by a function harmonic in all of \(C\).

b) The constants are the only elements of \(BL(C)\) represented by a function harmonic in all of \(C\).

c) \(D'(C)\) is norm dense in the Hilbert space \(BL^*(C)\). In other words, \(BL_0^*(C) = BL^*(C)\).

Proof. a) If \(f\) is harmonic on \(C\) then \(|f|^2\) is subharmonic on \(C\), hence \(\equiv 0\) because, for any \(a \in C\), \(r > 0\),

\[
|f(a)|^2 \leq \frac{1}{\pi r^2} \int_{|x-a|<r} |f|^2 \, d\lambda = \frac{1}{\pi r^2} \|f\|^2_{L^2(C)},
\]

which tends to 0 as \(r \to \infty\).

b) Reduces to a) applied to the partial derivatives.

c) An element \(u' \in BL'(C)\) is orthogonal to \(D'(C)\) if and only if \(u'\) is represented by a harmonic element of \(BL(C)\), i.e., a constant, by b). (Cf. [4, p. 318].) \(\square\)

A domain \(\Omega \subset C\) is a Green domain (i.e., it has a Green function) if and only if \(C\Omega\) is non-polar (Myrberg's theorem). When \(\Omega\) is a Green domain, the natural injection \(D(\Omega) \to D'(\Omega)\) extends uniquely to a continuous, linear, and injective mapping \(\hat{D}^1(\Omega) \to D'(\Omega)\), [4, p. 350]. In this case \(\hat{D}^1(\Omega)\) will be identified with its image, which clearly is a vector subspace of \(BL(\Omega)\), or let us rather say of \(BL(D(\Omega))\), since we shall understand that the elements of \(\hat{D}^1(\Omega)\) are (equivalence classes modulo polar sets of) quasi-continuous functions.

Recall that a function \(u \in BLD(\Omega)\) (\(\Omega\) a Green domain in \(C\)) belongs to \(D^1(\Omega)\) if and only if

\[
\lim_{x \to y} u(x) = 0 \text{ q.e.}^4 \quad \text{for} \quad y \notin \partial_{\text{fine}} \Omega,
\]

where \(\partial_{\text{fine}} \Omega\) denotes the boundary of \(\Omega\) in the fine topology on \(C\). See [4, p. 359].

If a function \(u\) on \(C\) satisfies \(u=0\) q.e. in \(C \setminus \Omega\) (\(\Omega\) a Green domain), then

\[
u \in BLD(C) \iff u|_{\Omega} \in \hat{D}^1(\Omega).
\]

Here \(\Rightarrow\) follows from the above because \(u\) is finely continuous q.e. in \(C\). For \(\Leftarrow\) see [4, pp. 355—359], [14, p. 308].

Definition. A set \(E \subset C\) is called quasi-analytic [quasi-coanalytic] if \(E\) differs only by a polar set from some analytic [coanalytic] set.

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\(^4\) Quasi everywhere (q.e.) means: everywhere off some polar set, while almost everywhere (a.e.) means: everywhere off some \(\lambda\)-nullset.
Every finely closed set $E$ is quasi-analytic, and even quasi Borel. In fact, by Kellogg’s theorem, $E$ differs only by a polar set from its base $b(E)$ which is an ordinary $G_δ$-set contained in $E$.\(^5\) Every quasi-(co)analytic set is Lebesgue measurable.

The interior of a set $E \subseteq C$ for the fine topology is denoted by $E'$.

**Lemma 1.2.** For any set $E \subseteq C$ and any $u \in BLD(C)$ we have

\[ u = 0 \text{ q.e. in } C \setminus E \Rightarrow \nabla u = 0 \text{ a.e. in } C \setminus E'. \]

**Proof.** Since $u$ is finely continuous q.e., the hypothesis $u=0$ q.e. in $CE$ implies $u=0$ q.e. in $CE'$. The conclusion $\nabla u=0$ a.e. in $CE'$ is obvious if $CE'$ is polar (hence a Lebesgue null set). If $CE'$ is non-polar it contains a compact non-polar set $K$. (This is because $CE'$ is finely closed, hence quasi-analytic and therefore “capacitable”, cf. e.g. [2, Corollary 5.3.3].) Any component $\Omega$ of $CK$ is a Green domain, and $\partial_{\text{fine}} \Omega \subseteq \partial \Omega \subseteq K$. It follows that $u|_{\Omega} \in \hat{\Omega}^1(\Omega)$ by (1), because $u=0$ q.e. in $K(\subset CE')$. According to [8, corollaire, p. 142] it follows that $\nabla u=0$ a.e. in $\Omega \setminus E'$, and hence a.e. in $C(K \cup E')$. We conclude that $\nabla u=0$ a.e. in $CE'$ because we may replace $K$ by $K \cap D_n$, $n=1, 2, \ldots$, where $D_n=\{z \in C \mid |z-a| \leq 1/n\}$, $a \in C$ being chosen so that each $K \cap D_n$ is non-polar. \(\Box\)

For any Green domain $\Omega \subseteq C$ and any $u \in BLD(\Omega)$ we have the canonical orthogonal decomposition [4, p. 322]

\[ u = v + h, \quad v \in \hat{\Omega}^1(\Omega), \quad h \text{ harmonic in } \Omega, \]

and $(v|h)_1=0$.

**Lemma 1.3.** Every Beppo Levi function $u$ on an open set $\Omega \subseteq C$ such that $u$ is finely harmonic q.e. in $\Omega$, can be corrected (uniquely) on a polar set so as to become harmonic (hence finely harmonic) in the whole of $\Omega$.

**Proof.** We may suppose that $\Omega$ is a Green domain (otherwise cover $C$ by a sequence of Green domains, e.g. discs). In the decomposition (3) the function $v=u-h \in \hat{\Omega}^1(\Omega)$ is finely harmonic q.e. in $\Omega$, and hence $v$ and therefore $u$ are harmonic (after correction on a polar set) according to [8, proposition 6]. \(\Box\) — As to finely harmonic functions see [6].

## 2. Havin’s theorem and complements to it

The following key result is the 2-dimensional case of that part of [8, théorème 11] which deals with finely harmonic functions:

**Theorem A.** Let $\Omega \subseteq C$ be a Green domain and $E \subseteq \Omega$ a quasi-coanalytic (e.g. Borel) set. For $u \in \hat{\Omega}^1(\Omega)$ the following are equivalent:\(^6\)

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\(^5\) The base $b(E)$ of a set $E$ is the set of points of $C$ at which $E$ is not thin (in the sense of Brelot), in other words, the finely derived set.

\(^6\) For any set $E \subseteq \Omega$ we have $1) \Rightarrow 2) \Leftrightarrow 3)$, cf. [8, remarque 1, p. 143].
1) There exists a sequence \((u_n) \subset \mathcal{D}^1(\Omega)\) converging to \(u\) in \(\mathcal{D}^1(\Omega)\) such that each \(u_n\) is harmonic in some open set \(\omega_n \supset E\).

2) \((u, \varphi)_1 = 0\) for every \(\varphi \in \mathcal{D}^1(\Omega)\) such that \(\varphi = 0\) q.e. in \(\Omega \setminus E\) (hence \(\nabla \varphi = 0\) a.e. in \(\Omega \setminus E\)).

3) \(u\) is finely harmonic in the fine interior \(E'\) of \(E\).

In order to pass from this result to an extended version of Havin's theorem we need some preparations. Writing as usual \(z = x + iy\) and (with a normalization convenient for our purpose)

\[ \partial = \partial/\partial x - i \partial/\partial y, \quad \bar{\partial} = \partial/\partial x + i \partial/\partial y, \]

the Laplacian \(\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2\) becomes

\[ \Delta = \bar{\partial} \partial - \partial \bar{\partial}. \]

Hence

\[ (\partial \bar{\partial})^{-1} = \partial \bar{\partial} \log |z| = \Delta \log |z| = 2\pi \delta \]

(\(\delta\) = the Dirac measure at 0). For any \(u, v \in \mathcal{D}^1(\Omega)\) (\(\Omega\) a Green domain in \(\mathbb{C}\)) we have

\[ (u, v)_1 = \int_{\Omega} \partial u \bar{\partial} v \, d\lambda = \int_{\Omega} \bar{\partial} u \partial v \, d\lambda. \]

For \(v \in \mathcal{D}(\Omega)\) this follows easily from \(\bar{\partial} \partial = \Delta\). It extends by continuity to the general case.

**Lemma 2.1.** The operator \(\partial\) maps \(BL^*(\mathbb{C})\) isometrically onto \(L^2(\mathbb{C})\).

**Proof.** \(\partial\) is isometric on \(\Omega^*(\mathbb{C})\), cf. the proof of (5), hence on \(BL^*(\mathbb{C})\) in view of Lemma 1.1c). If \(f \in L^2(\mathbb{C})\) is orthogonal to the range of \(\partial\) then \(\bar{\partial} f = 0\) in the distribution sense, hence \(\bar{\partial} \partial f = \Delta f = 0\), and so \(f = 0\) by Lemma 1.1a). \(\square\)

For a Beppo Levi function \(w\) in an open set \(\Omega \subset \mathbb{C}\) it is well known that the partial derivatives \(\partial w/\partial x, \partial w/\partial y\) entering in the definition of \(\partial w\) and \(\bar{\partial} w\) are the same in the distribution sense as in the classical sense, in which they exist a.e. and are of class \(L^1(\Omega)\), cf. [4, p. 315]. If \(w \in BLD(\Omega)\) then \(w\) is representable locally (q.e.) as the Riesz potential \(|z|^{-1} * g\) (of order 1) of a function \(g \in L^1_+(\mathbb{C})\), and hence \(w\) is likewise *finely differentiable* at almost every point \(z_0\) of \(\Omega\). Moreover, the usual differential at (a.e.) \(z_0 \in \Omega\):

\[ dw = (x - x_0) \frac{\partial w}{\partial x}(z_0) + (y - y_0) \frac{\partial w}{\partial y}(z_0), \]

is also the fine differential at \(z_0\) because

\[ |w(z) - w(z_0) - dw|/|z - z_0| \to 0 \]

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1) Cf. [8, corollaire, p. 142].
as \( z \to z_0 \) in the fine topology, or equivalently as \( z \to z_0 \) in the usual topology, though with \( z \) restricted to a suitable (punctured) fine neighbourhood of \( z_0 \); cf. Mizuta [12].

Every finely harmonic function \( u \) (real or complex valued) in a finely open set \( U \subset C \) is likewise finely differentiable a.e. in \( U \). This is because every point \( z_0 \in U \) has a fine neighbourhood \( V \subset U \) such that \( u|_V \) extends to a \( BLD \) function on \( C \), cf. [8, théorème 2].\(^8\) See also Davie and Øksendal [3, Theorem 6].

We are now prepared to state and prove Havin’s theorem in an extended form.

**Theorem 2.** Let \( E \) be a quasi-coanalytic subset of \( C \). For any \( f \in L^2(E) \) the following are equivalent:

1) There exists a sequence \( (f_n) \subset L^2(E) \) converging to \( f \) in norm in \( L^2(E) \) such that each \( f_n \) admits an extension to a function \( F_n \) analytic in an open set \( \omega_n \supset E \).

2) \( \int_E f \overline{\partial \varphi} \, d\lambda = 0 \) for every \( \varphi \in BLD(C) \) such that \( \varphi = 0 \) q.e. in \( C \setminus E \) (hence \( \partial \varphi = 0 \) a.e. in \( C \setminus E \)).

3) There exists a complex valued function \( u \) defined and finely harmonic q.e. in \( E' \) such that \( \partial u = f \) a.e. in \( E' \) in the sense of differentiation in the fine topology.

**Remark 2.1.** These three properties are of a local character in the fine topology; that is, each property is equivalent to the finely localized version of it, obtained by replacing \( E \) by \( E \cap V \) (and \( f \) by \( f|_{E \cap V} \)) for every \( V \) belonging to some family \( \mathcal{V} \) of finely open sets covering \( E \), or just \( E' \). In particular, it suffices to verify 2) for all \( \varphi \in BLD(C) \) such that \( \varphi = 0 \) off some compact subset of \( E' \). (Recall that every fine neighbourhood of a point \( z \in C \) contains a fine neighbourhood of \( z \) which is compact in the usual topology.)

**Remark 2.2.** Suppose that \( f \in L^2(E) \) has the equivalent properties 1), 2), 3), and let \( f \) also denote any extension of \( f \) of class \( L^2(C) \). We may then arrange, in 1), that the analytic functions \( F_n \) in \( \omega_n \) are square integrable and that \( \|F_n - f|_{\omega_n}\|_{L^2(\omega_n)} \to 0 \) as \( n \to \infty \). And in 3) the function \( u \in BLD(C) \) — uniquely determined q.e. in \( C \) (up to an additive constant) by \( \partial u = f \) a.e. in \( C \) (cf. Lemma 2.1) — is finely harmonic q.e. in \( E' \).

**Proof of Theorem 2 and Remarks 2.1, 2.2.** We shall denote by 1), 2), 3) the localized versions of 1), 2), 3) described in Remark 2.1. It is trivial that 1)\( \Rightarrow \) 1'), 2)\( \Rightarrow \) 2'), 3)\( \Rightarrow \) 3'), and so it remains to prove that 1')\( \Rightarrow \) 2')\( \Rightarrow \) 3')\( \Rightarrow \) 2)\( \Rightarrow \) 1).

Ad 1')\( \Rightarrow \) 2'). For every \( V \) from some covering \( \mathcal{V} \) of \( E' \) as in the formulation of 1') in Remark 2.1, and for any \( z \in E' \cap V \), choose \( W \) finely open and of compact closure \( \overline{W} \) so that \( z \in W \subset \overline{W} \subset E' \cap V \). For any \( \varphi \in BLD(C) \) such that \( \varphi = 0 \) q.e. in \( CW \), the support of \( \varphi \) as a distribution is compact and contained in \( \overline{W} \subset E' \cap V \).

\(^8\) In the first instance we obtain a bounded fine neighbourhood \( V \subset U \) of \( z_0 \) such that \( u|_V \) extends to a function of class \( \hat{O}^1(\Omega) \), \( \Omega \) being a disc containing \( V \). The further extension of this function by 0 in \( C\Omega \) is of class \( BLD(C) \) according to (2), Section 1.
With \( F_n \) analytic in \( \omega_n \supset E \cap V \), and \( F_n|_{E \cap V} \to f \) in \( L^2(E \cap V) \), we therefore obtain, noting that \( \partial \varphi = 0 \) a.e. in \( CW \) by Lemma 1.2,
\[
\int_W F_n \partial \varphi \, d\lambda = \int_{\omega_n} F_n \partial \varphi \, d\lambda = -\int_{\omega_n} \partial F_n \varphi \, d\lambda = 0.
\]
It follows that \( \int_W f \partial \varphi \, d\lambda = 0 \), and this establishes 2') because the above finely open sets \( W \) have the union \( E' \).

Ad 2')\( \Rightarrow \)3'). According to Lemma 2.1 there exists \( u \in BLD(C) \) with \( \partial u = f \) after extension of \( f \) by 0, say, in \( CE \). For every \( V \) from some covering \( \mathcal{V} \) of \( E' \) as in the formulation of 2') in Remark 2.1, and for any \( z \in E' \cap V \), choose \( W \) as above, and \( \Omega \) as a bounded domain in \( C \) containing \( \overline{W} \) (e.g., a disc). Consider the canonical decomposition \( u|_{\Omega} = v + h \), cf. (3). Thus \( v \in \hat{A}(\Omega) \), and \( \partial \bar{h} = \Delta h = 0 \) in \( \Omega \). Now let \( \varphi \in BLD(C) \) satisfy \( \varphi = 0 \) q.e. in \( C \setminus W \), in particular q.e. in \( \Omega \). Then \( \varphi|_{\Omega} \in \hat{A}(\Omega) \) according to (2), Section 1. Moreover, \( \int_{\Omega} \partial h \partial \varphi \, d\lambda \) = \( \int_{\Omega} \partial \bar{h} \cdot \bar{\varphi} \, d\lambda \) = 0. Using (5), and noting that \( \varphi = 0 \) q.e. in \( C \setminus V \) in particular, we therefore obtain from 2')
\[
(v, \varphi|_{\Omega}) = \int_{\Omega} \partial v \partial \varphi \, d\lambda = \int_{\Omega} \partial u \partial \varphi \, d\lambda = \int_C f \partial \varphi \, d\lambda = 0
\]
but \( \partial \varphi = 0 \) a.e. in \( CW(\supset C \Omega) \), cf. Lemma 1.2. Applying Theorem A to \( v \), we find that \( v \) is finely harmonic q.e. in \( W \), and so is therefore \( u = v + h \) because \( h \) is harmonic and hence finely harmonic in \( \Omega \), cf. [6, Theorem 8.7]. — The above finely open sets \( W \) have the union \( E' \), and 3') ensues.

Ad 3')\( \Rightarrow \)3), with 3) amplified as in the latter part of Remark 2.2. Thus let \( f \in L^2(C) \) extend the given \( f \), and let \( u \in BLD(C) \) satisfy \( \partial u = f \) a.e. Then \( u \) is finely continuous in \( C \) off some polar set \( e \). By 3'), \( E' \) is the union of a family of finely open sets \( V \) for each of which there exists a finely harmonic function \( u_V \) in \( V \) off some polar set \( e_V \), such that \( \partial u_V = f \) a.e. in \( V \). Each point of \( V \setminus (e \cup e_V) \) has a fine neighbourhood \( W \) in \( V \setminus (e \cup e_V) \) such that \( u_V \) equals in \( W \) some function of class \( BLD(C) \), as observed just before Theorem 2. The same is therefore true of \( u - u_V \). Because \( \partial(u - u_V) = f - f = 0 \) a.e. in \( V \setminus (e \cup e_V) \), we infer from [7, Section 3] that \( \overline{u - u_V} \) is finely harmonic (even finely holomorphic) in \( V \setminus (e \cup e_V) \), and hence \( u \) is itself finely harmonic in \( V \setminus (e \cup e_V) \). Invoking Doob's quasi Lindelöf principle [5], cf. also [2, exercise 7.2.6] and the sheaf property of fine harmonicity [6, p. 70], we conclude that \( u \) is indeed finely harmonic q.e. in \( E' \).

Ad 3')\( \Rightarrow \)2). Since even 3') implies the amplified version of 3), we may suppose that \( u \in BLD(C) \) is finely harmonic q.e. in \( E' \), and that \( \partial u = f \) after extending \( f \) by 0 in \( CE \). If \( CE' \) is polar, it follows from 3) by Lemma 1.3 and Lemma 1.1b) that \( u \) is constant (q.e.), hence \( f = \partial u = 0 \) (a.e.), which implies 2). — Suppose next that \( CE' \) is non-polar, and let \( \varphi \in BLD(C) \) satisfy \( \varphi = 0 \) q.e. in \( CE \), hence \( \varphi = 0 \) q.e. in \( CE' \), by fine continuity. Proceeding as in the proof of Lemma 1.2 we choose a compact, non-polar set \( K \subset CE' \). Every component \( \Omega \) of \( CK \) is a Green domain, and \( \varphi|_{\Omega} \in \hat{A}(\Omega) \). In the decomposition (3) of \( u|_{\Omega} \), the function \( v = u|_{\Omega} - h \in \hat{A}(\Omega) \) is
finely harmonic q.e. in \( E' \) (like \( u \)), and hence it follows from Theorem A, in view of (5), that \( \int_{\Omega} \partial u \overline{\partial \varphi} \, d\lambda = (v, \varphi)|_{\Omega} = 0 \). Since \( h \in BL(\Omega) \) is harmonic in \( \Omega \), we have \( \int_{\Omega} \partial h \overline{\partial \varphi} \, d\lambda = 0 \), as shown by approximating \( \varphi_{\Omega} \in \mathcal{D}^1(\Omega) \) by functions of class \( \mathcal{D}(\Omega) \). Consequently, \( \int_{\Omega} \partial u \overline{\partial \varphi} \, d\lambda = 0 \). Adding over all components \( \Omega \) of \( \mathbf{K} \), and noting that \( \partial \varphi = 0 \) a.e. in \( \mathbf{C}E' \subset \mathbf{C}K \), by Lemma 1.2, we conclude that

\[
\int_E f \overline{\partial \varphi} \, d\lambda = \int_{E'} \partial u \overline{\partial \varphi} \, d\lambda = \int_{\mathbf{C}K} \partial u \overline{\partial \varphi} \, d\lambda = 0.
\]

Ad 2) \( \Rightarrow \) 1), with 1) amplified as in Remark 2.2. If \( \mathbf{C}E \) is polar, it follows from 2) that \( \int_E f \overline{\partial \varphi} \, d\lambda = 0 \) for every \( \varphi \in \mathcal{D}(\mathbf{C}) \). Hence \( f \) is analytic in \( \mathbf{C} \), and therefore \( f = 0 \) in \( \mathbf{C} \), by Lemma 1.1a), which establishes 1). — Suppose next that the quasi-coanalytic set \( \mathbf{C}E \) is non-polar. Choose a compact, non-polar set \( \mathbf{K} \subset \mathbf{C}E \) (cf. the proof of Lemma 1.2). Then \( \mathbf{E} \subset \mathbf{C}K \), and it clearly suffices to prove 1) (in its amplified form) with \( E \) replaced by \( E \cap \Omega \) for each component \( \Omega \) of \( \mathbf{C}K \). For any \( \varphi \in \mathcal{D}^1(\Omega) \) such that \( \varphi = 0 \) q.e. in \( \Omega \setminus E \), the extension \( \Phi \) of \( \varphi \) to \( \mathbf{C} \) by \( \Phi = 0 \) in \( \mathbf{C} \Omega \) satisfies \( \Phi \in BLD(\mathbf{C}) \), by (2), Section 1; and \( \Phi = 0 \) q.e. in \( \mathbf{C}E \), hence \( \partial \Phi = 0 \) a.e. in \( \mathbf{C}(E \cap \Omega) \), by Lemma 1.2. It follows from (3) that \( \int_E f \overline{\partial \Phi} \, d\lambda = 0 \), and hence \( \int_{E \cap \Omega} f \overline{\partial \varphi} \, d\lambda = 0 \), that is, \( \int_{\Omega} f \overline{\partial \varphi} \, d\lambda = 0 \). Choose \( u \in BLD(\mathbf{C}) \) so that \( \partial u = f \), cf. Lemma 2.1. In the canonical decomposition \( u|_{\Omega} = v + h \) with \( v \in \mathcal{D}^1(\Omega) \) we obtain from 2) in view of (5)

\[
(v, \varphi)_1 = \int_{\Omega} \partial u \overline{\partial \varphi} \, d\lambda = \int_{\Omega} f \overline{\partial \varphi} \, d\lambda - \int_{\Omega} \partial h \overline{\partial \varphi} \, d\lambda = 0,
\]

because \( h \in BL(\Omega) \) is harmonic in \( \Omega \). Applying Theorem A to \( E \cap \Omega \) in place of \( E \), we obtain a sequence \((v_n) \subset \mathcal{D}^1(\Omega)\) converging to \( v \) in \( \mathcal{D}^1(\Omega) \) such that each \( v_n \) is harmonic in some open set \( \omega_n \), \( E \cap \Omega \subset \omega_n \subset \Omega \). It follows that

\[
u_n := v_n + h \to v + h = u|_{\Omega} \quad \text{in} \quad BL'(\Omega),
\]

and here \( u_n \) is harmonic in \( \omega_n \). Consequently, \( f_n := \partial u_n \in L^2(\Omega) \) is analytic in \( \omega_n \) and converges to \( \partial u|_{\Omega} = f|_{\Omega} \) in \( L^2(\Omega) \). \( \square \)

Remark 2.3. In property 3) in the above theorem, consider any two functions \( u_1, u_2 \) which are finely harmonic q.e. in \( E' \) and satisfy (in the sense of fine partial derivatives) \( \partial u_1 = \partial u_2 = f \) a.e. in \( E' \). The difference \( \overline{u_1 - u_2} \) is then finely harmonic in \( E' \setminus e \) for some polar set \( e \subset E' \), and the conjugate \( \overline{u_1 - u_2} \) is even finely holomorphic in \( E' \setminus e \), cf. 1) in [7, définition 3], because \( \overline{\partial (u_1 - u_2)} = 0 \) a.e. in \( E' \setminus e \). Hence \( \overline{u_1 - u_2} \) is finely differentiable everywhere in \( E' \setminus e \), even in the complex sense, according to [7, théorème 10]. This implies that \( u_1 \) and \( u_2 \) are finely differentiable (in the real sense) at precisely the same points of \( E' \setminus e \).

In order to be able to compare \( \text{cl} L^2_s(E) \) and \( L^2_s(E)^{\perp} \) (cf. the introduction) for different sets \( E \) we introduce the following

**Notation.** For any quasi-coanalytic set \( E \subset C \) we denote by \( L^2_s(C, E) \) the linear subset of \( L^2(C) \) consisting of those \( f \in L^2(C) \) which are holomorphic in some open
set containing $E$. Thus $L^2(E)$ contains all restrictions to $E$ of functions of class $L^2(C, E)$.

The closure $\text{cl} L^2(C, E)$ of $L^2(C, E)$ within $L^2(C)$ is, by Remark 2.2, the set of all $f \in L^2(C)$ such that the restriction $f|_E$ belongs to the closure $\text{cl} L^2(E)$ of $L^2(E)$ within $L^2(E)$, in other words such that $f|_E$ has the equivalent properties 1), 2), 3) of Theorem 2. Using 2), we thus obtain

$$\text{cl} L^2(C, E) = \{ \partial \varphi : \varphi \in \text{BLD}(C), \varphi = 0 \text{ q.e. in } CE \}^\perp,$$

where $\perp$ indicates orthogonal complement within $L^2(C)$. (Here we also make use of the fact that $\partial \varphi = 0$ a.e. in $CE$ for any $\varphi$ as stated in this formula, cf. Lemma 1.2.)

It follows that

$$(6) \quad L^2(C, E)^\perp = \{ \partial \varphi : \varphi \in \text{BLD}(C), \varphi = 0 \text{ q.e. in } CE \}$$

because the right hand member is a closed subspace of $L^2(C)$ in view of Lemma 2.1 and the known fact that $\{ \varphi \in \text{BLD}(C) | \varphi = 0 \text{ q.e. in } CE \}$ is closed in $BL'(C)$, cf. e.g. [14, p. 308].

As noted in the proof of 3)$\Rightarrow$2) in Theorem 2,

$$\varphi = 0 \text{ q.e. in } CE \Leftrightarrow \varphi = 0 \text{ q.e. in } CE'$$

when $\varphi \in \text{BLD}(C)$ (for then $\varphi$ is finely continuous q.e.). Thus $L^2(C, E)^\perp$ and hence $\text{cl} L^2(C, E)$ only depend on the fine interior $E'$ of the quasi-coanalytic set $E \subset C$. Furthermore, these two orthogonal subspaces of $L^2(C)$ obviously depend only on the equivalence class of $E$ modulo polar sets, in view of (6). More precisely we have

**Lemma 2.2.** Let $E$ and $F$ be quasi-coanalytic subsets of $C$. Then $\text{cl} L^2(C, E) = \text{cl} L^2(C, F)$ holds if and only if the fine interiors $E'$ and $F'$ differ only by a polar set; or equivalently: if and only if $CE$ and $CF$ are thin at the same points: $b(CE) = b(CF)$.

**Proof.** The following stronger result will be obtained:

$$F' \setminus E' \text{ polar } \Leftrightarrow L^2(C, F)^\perp \subset L^2(C, E)^\perp.$$

The implication $\Rightarrow$ follows from (6) above. In proving the converse implication it suffices to consider the case where $E$ and $F$ are finely open. Suppose that $F' \setminus E'$ is non-polar, and choose an open disc $\Omega$ in $C$ so that $A := (F' \setminus E) \cap \Omega$ is likewise non-polar. Then there exist two disjoint finely open subsets $V$ and $W$ of $F$ such that $A \cap V$ and $A \cap W$ are both non-polar. Choose a finite strict Green potential $p = G\mu$ on $\Omega$ so that $\int G\mu \, d\mu = +\infty$, and put

$$u = p - \hat{R}_p^{\alpha \setminus V}$$

(balayage relative to $\Omega$), cf. [2, Section 7.2]. Then $u \in \mathbb{D}^1(\Omega)$, and so the extension $\varphi$ of $u$ to $C$ obtained by putting $\varphi = 0$ in $C\Omega$ belongs to $\text{BLD}(C)$, by (2), Section 1. Clearly, $\varphi = 0$ q.e. in $C\Omega$, in particular q.e. in $CF$, that is, $\partial \varphi \in L^2(C, F)^\perp$, by (6). On the other hand, $\partial \varphi$ is not in $L^2(C, E)^\perp$, for that would imply that $\varphi$ were constant
q.e. in $\mathbb{C}E(\supset A)$ according to (6) combined with Lemma 2.1; and this is not the case. Indeed, $u=0$ in the non-polar set $A \cap V$ because $p$ is strict [2, Proposition 7.2.2], while $u=\varphi=0$ q.e. in the non-polar set $A \cap W (\subset \Omega \cap W \subset \Omega \setminus V)$. □

Corollary (essentially due to Havin [10]). Let $E$ be a quasi-coanalytic subset of $\mathbb{C}$, and let $F$ be an open set such that $F \subset E$ (for example $F=\mathring{E}$, the usual interior of $E$). In order that all functions $f \in L^2(E)$ such that $f$ is analytic in $F$ belong to $\text{cl } L^2_a(E)$, it is necessary and sufficient that $E' \setminus F$ be polar.

In particular, $\text{cl } L^2_a(E)=L^2(E)$ if and only if $E'=\emptyset$ (or equivalently: $E'$ should be a Lebesgue null set).

Proof. Since $F \subset E$ is open, we have $F \subset \mathring{E} \subset E'$, and $L^2_a(C,F)$ is the set of functions in $L^2(C)$ which are analytic in $F$. This set is closed in $\mathcal{D}'(C)$ and hence in $L^2(C)$. Since $\text{cl } L^2_a(C,E) = \text{cl } L^2_a(C,E')$, the corollary follows from Lemma 2.2 in view of (6).

3. Relation to finely holomorphic functions

For every finely open set $U \subset \mathbb{C}$ we denote by $\mathcal{O}(U)$ the algebra of finely holomorphic functions $U \to \mathbb{C}$, cf. [7]. Every $f \in \mathcal{O}(U)$ has fine derivatives in the complex sense of all orders, and if $U$ is a fine domain (= finely connected finely open set) the $f$ is uniquely determined within $\mathcal{O}(U)$ by the values of $f$ and all its derivatives at any prescribed point [7, théorèmes 10 et 14]. In particular, if $f(z)=0$ for all $z$ in some fine neighbourhood of a point of a fine domain $U$, then $f(z)=0$ for all $z \in U$. Let us further write

$$\mathcal{O}^a(C,U) = \{ f \in L^2(C) | f \text{ is finely holomorphic in } U \}.$$  

Returning to the quasi-coanalytic set $E \subset \mathbb{C}$ we have (in addition to the role of fine holomorphy described in Remark 2.3) the following inclusion relations:

(7) $$L^2_a(C,E) \subset \mathcal{O}^a(C,E') \subset \text{cl } L^2_a(C,E),$$

showing that $L^2_a(C,E)$ and $\mathcal{O}^a(C,E')$ have the same closure in $L^2(C)$. The former relation (7) follows from the definition of $L^2_a(C,E)$ together with the fact that every analytic function in an open set $\omega \subset C$ is finely holomorphic in $\omega$, and hence also in every finely open subset of $\omega$ (such as $E'$ when $\omega \supset E$), cf. [7, p. 63]. The latter inclusion (7) follows from [7, proposition 16] (applied to $U=E'$) in view of Theorem 2 and the fact that $\text{cl } L^2_a(C,E) = \text{cl } L^2_a(C,E')$, as noted in Section 2 (before Lemma 2.2).

Each of the inclusion relations in (7) is proper for a suitable set $E$, even for a suitable finely open set $E$. As to the former inclusion this follows from the example [7, p. 74], taking $E=U'$. The latter inclusion is proper, as observed in [7, remarque 3, p. 81], or alternatively as a consequence of the following
Proposition 3. There exists a bounded fine domain $E \subset \mathbb{C}$ and a function $f \in \text{cl} \mathbb{L}_{u}^{2}(C, E) \setminus \mathbb{C}^{2}(C, E)$. We may further arrange that $f$ does not have the unique continuation property. Explicitly, we shall achieve that $f = 0$ a.e. in some fine neighbourhood of a point of $E$, and yet $f$ is not identically $0$ a.e. in $E$.

Proof. Let $D$ denote the open unit disc in $C$, and $D(z, r)$ the open disc of radius $r$ centered at $z \in C$. Following Lyons [11], in which is incorporated an idea due to M. Sakai, we apply Vitali’s covering theorem to cover $D$ up to a Lebesgue nullset by disjoint discs $D_{n} = D(z_n, r_n) \subset D$, $n \in N$, and consider the probability measure

$$
\mu = \sum_{n \in N} r_{n}^{2} \delta_{z_{n}},
$$

where $\delta_{z}$ denotes the Dirac measure at $z$. The logarithmic potential of $\mu$ is defined in $C$ as the negative of

$$
u(z) = \sum_{n \in N} r_{n}^{2} \log |z - z_{n}| \quad (\equiv -\infty), \quad z \in C.
$$

For $|z| > 1$, $\log |z - \zeta|$ is a harmonic function of $\zeta \in D$, and hence

$$
\log |z - z_{n}| = \frac{1}{\pi r_{n}^{2}} \int_{D_{n}} \log |z - \zeta| \, d\lambda(\zeta), \quad |z| > 1.
$$

It follows that

$$
(8) \quad u(z) = \frac{1}{\pi} \sum_{n \in N} \int_{D_{n}} \log |z - \zeta| \, d\lambda(\zeta) = \frac{1}{\pi} \int_{D} \log |z - \zeta| \, d\lambda(\zeta) = \log |z|, \quad |z| > 1.
$$

The functions $u$ and $z \mapsto \log |z|$ are subharmonic on $C$. The set

$$
e = \{z \in C | u(z) = -\infty\} \cup \{0\}
$$

is therefore polar, and clearly $\{z_{n}\}_{n \in N} \subset e \subset \bar{D}$. It follows that $C \setminus e$ is a fine domain [6, Theorem 12.2], and that $u$ and $z \mapsto \log |z|$ are finely harmonic in $C \setminus e$, see [6, Theorem 8.10]. The function $v$ defined on $C \setminus e$ by

$$
v(z) = u(z) - \log |z|
$$

is thus finely harmonic. We have $v \equiv 0$ in $C \setminus \bar{D}$, by (8), but not in all of $C \setminus e$, for then the equality $u(z) = \log |z|$ would extend by fine continuity from $C \setminus e$ to $C$, noting that $e$ has no finely interior points; but actually $\log(z_n) > -\infty = u(z_n)$ for every $n$ such that $z_n \neq 0$. Because $C \setminus e$ is a fine domain, we conclude that the fine interior of $\{z \in C \setminus e | v(z) = 0\}$ has a fine boundary point $z^*$ relative to $C \setminus e$. Since $v$ is finely harmonic in $C \setminus e$, there exists a finely open set $E$ with $z^* \in E \subset C \setminus e$ such that $v$ coincides on $E$ with a function $w \in BLD(C)$, see above (just before Theorem 2). We choose $E$ as a fine domain, thus invoking the local connectedness of the fine topology, cf. [6, p. 92]. The function

$$
f = \partial w
$$

belongs to $\text{cl} \mathbb{L}_{u}^{2}(C, E)$ according to Theorem 2, 3) $\Rightarrow$ 1).
It remains to show that \( f_{1|E} = (\partial v)|_E \) (cf. Lemma 1.2) does not have the unique continuation property. By the choice of \( z^\ast \) we have \( v \neq 0 \) in \( E \), but \( v = 0 \) everywhere in some non-empty finely open set \( V \subset E \), hence \( f = \partial v = 0 \) a.e. in \( V \). If \( f = 0 \) a.e. in \( E \) then \( \bar{v} \) is finely holomorphic in \( E \) according to 1) in [7, définition 3], and we are led to a contradiction by the unique continuation property of finely holomorphic functions [7, théorème 14]. \( \Box \) — It can be shown that, necessarily, \( |z^\ast| = 1 \).

4. An extension to higher dimensions

We replace \( C \) by \( R^k (k \in \mathbb{N}) \) as space for the independent variable, now denoted by \( x = (x^1, \ldots, x^k) \). For any vector space \( \mathcal{F} \) of real distributions on \( R^k \), or of (equivalence classes of) real functions on a given subset of \( R^k \), we denote by \( \mathcal{F}_{\text{ext}} \) the vector space of currents, or differential forms,

\[
 f = \sum_{p=1}^{k} \sum_{i_1, \ldots, i_p} f_{i_1, \ldots, i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}
\]

with coefficients from \( \mathcal{F} \), cf. [13]. We have the direct sum decomposition

\[
 \mathcal{F}_{\text{ext}} = \mathcal{F}_{\text{even}} + \mathcal{F}_{\text{odd}},
\]

where \( \mathcal{F}_{\text{even}} \), respectively \( \mathcal{F}_{\text{odd}} \), denotes the set of those forms \( f \in \mathcal{F}_{\text{ext}} \) that are even, respectively odd, in the sense that \( p \) is even, respectively odd, for every non-zero coefficient \( f_{i_1, \ldots, i_p} \) in the reduced expression (10) of \( f \). The number of coefficients in (10) that are not a priori equal to 0 is \( 2^{k-1} \) for an even form \( f \) and likewise for an odd form.

If \( \mathcal{F} \) is a (real) Hilbert space with an inner product \( (\cdot | \cdot) \) then so is \( \mathcal{F}_{\text{ext}} \), its inner product — likewise denoted \( (\cdot | \cdot) \) — being given by

\[
 (f | g) = \sum_{p=1}^{k} \sum_{i_1, \ldots, i_p} (f_{i_1, \ldots, i_p} | g_{i_1, \ldots, i_p}).
\]

\( \mathcal{F}_{\text{even}} \) and \( \mathcal{F}_{\text{odd}} \) are then orthogonal to each other.

If the elements of \( \mathcal{F} \) are differentiable (in some sense) we consider the differential operator \( d + \delta \) on \( \mathcal{F}_{\text{ext}} \), where \( d \) denotes exterior differentiation and \( \delta \) denotes co-differentiation.\(^9\) Then \( d + \delta \) carries even forms into odd forms, and vice versa. We further write

\[
 \partial = d + \delta, \quad \bar{\partial} = d + \delta, \quad \text{acting on } \mathcal{F}_{\text{odd}},
\]

\[
 \bar{\partial} = d + \delta, \quad \text{acting on } \mathcal{F}_{\text{even}}.
\]

Note that (when defined)

\[
 \bar{\partial} \partial, \bar{\partial} \bar{\partial} \subset (d + \delta)^2 = d\delta + \delta d = \Delta,
\]

the Laplacian (acting coefficientwise). This shows that \( d + \delta \) is elliptic.

\(^9\) The operator \( d + \delta \) has been considered in an entirely different context by Gilkey [9]. Our \( \delta \) and \( \Delta \) correspond to \( -\delta \) and \( -\Delta \) in [13].
We proceed to extend the main results of the preceding sections to the present situation, which is more general—being, for \( k=2 \), nothing but a reformulation of the complex situation considered up till now, cf. below. Note, however, that instead of analytic functions \( F \) in open sets \( w \subset C \) now just speak of even differential forms \( F \) satisfying the generalized Cauchy–Riemann equation \( \overline{\partial} F = (d + \delta) F = 0 \) in open sets \( \omega \subset \mathbb{R}^k \). By ellipticity, such forms (currents) have real analytic coefficients in \( \omega \).

Actually, the proofs become simpler in the case \( k \geq 3 \) because \( \mathbb{R}^k \) then has a Green kernel, namely the Newtonian kernel. From Lemma 1.1c) (valid in any dimension) we obtain the identification

\[
BLD(\mathbb{R}^k) = \mathcal{L}^1(\mathbb{R}^k), \quad k \geq 3.
\]

Lemma 2.1 extends, showing that \( \partial \) maps \( [\mathcal{L}^1(\mathbb{R}^k)]_{\text{odd}} \) isometrically onto \( [L^2(\mathbb{R}^k)]_{\text{even}} \) \((k \geq 3)\). In particular,

\[
(u, v)_1 = (\partial u | \partial v), \quad u, v \in [\mathcal{L}^1(\mathbb{R}^k)]_{\text{odd}},
\]

where \((\cdot, \cdot)\) and \((\cdot | \cdot)\) denote the inner products on \( [\mathcal{L}^1(\mathbb{R}^k)]_{\text{ext}} \) and \( [L^2(\mathbb{R}^k)]_{\text{ext}} \) derived, as in (11), from the corresponding inner products on \( \mathcal{L}^1(\mathbb{R}^k) \) and \( L^2(\mathbb{R}^k) \), respectively.

If we replace \( \mathcal{L}^1(\Omega) \) in [8] (in the case \( \Omega = \mathbb{R}^k \), \( k \geq 3 \)) by \( [\mathcal{L}^1(\mathbb{R}^k)]_{\text{odd}} \), then the part of [8, théorème 11] which deals with (finely) harmonic functions carries over with unchanged proof. Using this result in place of Theorem A above, Theorem 2 carries over as follows:

**Theorem 4.** Let \( E \) be a quasi-coanalytic subset of \( \mathbb{R}^k, k \in \mathbb{N} \). For any differential form \( f \in [L^2(E)]_{\text{even}} \) the following are equivalent:

1) There exists a sequence \( (f_n) \subset [L^2(E)]_{\text{even}} \) converging to \( f \) in norm in \( [L^2(E)]_{\text{even}} \) such that each \( f_n \) admits an extension to an even, real analytic differential form \( F_n \) for which \( \overline{\partial} F_n = 0 \) in some open set \( \omega_n \supset E \).

2) \( (f | \partial f) = 0 \) for every form \( \varphi \in [BLD(\mathbb{R}^k)]_{\text{odd}} \) such that \( \varphi = 0 \) q.e.\(^{10} \) in \( CE \) (hence \( \partial \varphi = 0 \) a.e. in \( CE \)).

3) There exists an odd form \( u \), with coefficients defined and finely harmonic q.e.\(^{10} \) in \( E' \), such that \( \partial u = f \) a.e. in \( E' \) (in the sense of fine differentiation).

The various remarks to Theorem 2 (with the exception of Remark 2.3 if \( k \geq 3 \)) carry over mutatis mutandis, and so do Lemma 2.2 and its corollary. As to Section 3, the construction in the proof of Proposition 3 carries over to produce an example of a fine domain \( E \subset \mathbb{R}^k, k \geq 2 \), and a form \( f \in [L^2(E)]_{\text{even}} \) having the equivalent properties 1), 2), 3) in Theorem 4, but failing to have the property of unique continuation.

\(^{10} \) When \( k \geq 3 \), quasi-everywhere (q.e.) means: everywhere except in some polar set (= a set of zero outer Newton capacity).
In all of this, one may interchange throughout the terms "even" and "odd". In odd dimension \( k \) the two versions thus obtained are, however, equivalent via the involution \( * \), cf. [13, p. 121]. For even \( k \geq 4 \) the two versions do not appear to be equivalent in any natural sense.

For \( k=2 \), both versions reduce to the complex situation from Sections 1—3 via the following identifications (writing \((x, y)\) for the generic point of \( \mathbb{R}^2 \)):

\[
fdx + gdy, \quad f + gdx \wedge dy \mapsto f + ig,
\]

considered as a function of \( z=x+iy \). Then \( \partial \) and \( \overline{\partial} \), as defined in the present section, correspond to \( \partial \) and \( \overline{\partial} \) from Section 2.

For \( k=3 \) we make the following identifications (writing \((x, y, z)\) for the coordinates of the generic point of \( \mathbb{R}^3 \)):

\[
\begin{align*}
f_0dx \wedge dy \wedge dz + f_1dx + f_2dy + f_3dz & \mapsto (f_0, f_1, f_2, f_3), \\
f_0 + f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy & \mapsto (f_0, f_1, f_2, f_3).
\end{align*}
\]

Then \( \partial \) and \( \overline{\partial} \) both correspond to the operator

\[
D = \begin{pmatrix}
0 & \partial_x & \partial_y & \partial_z \\
\partial_x & 0 & -\partial_z & \partial_y \\
\partial_y & \partial_z & 0 & -\partial_x \\
-\partial_y & \partial_x & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & \text{div} \\
0 & \text{grad rot}
\end{pmatrix},
\]

where \( \partial_x = \partial/\partial x \), etc. The operator \( D \) is studied, e.g., in Aržanyh [1]. — If we further identify \( \mathbb{R}^4 \) (as range space) with the quaternion field \( \mathbb{H} \) as follows:

\[
(a_0, a_1, a_2, a_3) \mapsto a_0 + a_1 i + a_2 j + a_3 k,
\]

then the operator \( D \) transforms into

\[
Df = (\partial_x f)i + (\partial_y f)j + (\partial_z f)k
\]

for functions \( f \) from \( \mathbb{R}^3 \) to \( \mathbb{H} \).

References


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