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COMPLEMENTS TO HAVIN'S THEOREM ON L^2-APPROXIMATION BY ANALYTIC FUNCTIONS

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Introduction. For a Borel set $E \subset \boldsymbol{C}$ denote by $L_a^2(E)$ the linear subset of $L^2(E)$ (with respect to 2-dimensional Lebesgue measure λ) consisting of those $f \in L^2(E)$ which can be extended to an analytic function in some open set containing E. It was shown by Havin [10] that $f \in L^2(E)$ belongs to the L^2-closure $\operatorname{cl} L_a^2(E)$ of $L_a^2(E)$ if and only if

$$\int f \overline{\partial \varphi}\, d\lambda = 0$$

for every Beppo Levi function φ on $\boldsymbol{C}$, (made "precise" in the sense of Deny and Lions [4]) for which

$$\varphi = 0 \text{ quasi everywhere in } \boldsymbol{C} \setminus E,$$

that is, everywhere in $\boldsymbol{C} \setminus E$ except in some set which locally is of zero outer logarithmic capacity. (We write $\partial = \partial/\partial x - i\partial/\partial y$.)

As a further necessary and sufficient condition for $f \in L^2(E)$ to belong to $\operatorname{cl} L_a^2(E)$, we find in Section 2 that there shall exist a function u which is *finely harmonic*, in the sense of [6], quasi everywhere in the interior E' of E in the Cartan *fine topology*, and such that

$$f = \partial u \text{ almost everywhere in } E',$$

in the sense of differentiability in the fine topology. In the affirmative case, u may be taken to be the restriction to E' of a Beppo Levi function U (precise, as above) on $\boldsymbol{C}$, and such that $\partial U = F$ a.e. in $\boldsymbol{C}$, where F denotes any prescribed extension of f to a function in $L^2(\boldsymbol{C})$. At the same time we show that the property of belonging to $\operatorname{cl} L_a^2(E)$ has a local character in the fine topology.

The key to our proof of Havin's theorem and these complements to it is the analogous result on Beppo Levi functions obtained in [8] (valid also in higher dimensions).

In Section 3 it is shown that the property of unique continuation of analytic functions gets lost when we pass from $L_a^2(E)$ to its L^2-closure. In particular, the functions in $\operatorname{cl} L_a^2(E)$ (for suitable E) are not all finely holomorphic q.e. in E' in the sense of [7]. To prove this we employ an example due to Lyons [11], showing that finely harmonic functions do not in general have the property of unique continuation.

Finally, in Section 4, we extend the preceding results to the case of a Borel set E in $\boldsymbol{R}^k$, $k = 1, 2, \ldots$. The functions f, φ, and u above are now replaced by exterior

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differential forms (currents), the homogeneous parts of which are of even degree in the case of f, say, and of odd degree in the case of φ and u. The operators ∂ and $\bar{\partial}$ are replaced by the restrictions of $d+\delta$ (exterior differentiation plus co-differentiation) to the forms that are "odd" and "even", respectively, in the above sense. For $k=2$ we recover the original case of Havin's theorem.

1. Spaces of Beppo Levi functions of a complex variable

For any domain (=connected open set) $\Omega\subset \boldsymbol{R}^2=\boldsymbol{C}$ we denote by $BL(\Omega)$ the vector space of all complex distributions $f\in\mathscr{D}'(\Omega)$ whose gradient ∇f is in $L^2(\Omega)$ with respect to Lebesgue measure λ on $\boldsymbol{C}$, see Deny and Lions [4], and further Schulze and Wildenhain [14, Kapitel IX]. Thus $BL(\Omega)$ is a prehilbert space with the degenerate inner product

$$(u, v)_1 = \int_\Omega \nabla u \overline{\nabla v}\, d\lambda.$$

The quotient space of $BL(\Omega)$ modulo the constants is a separable Hilbert space, denoted by $BL^{\cdot}(\Omega)$.

The elements of $BL(\Omega)$ are (equivalence classes modulo λ of) locally λ-integrable functions, called *Beppo Levi functions*. Each such equivalence class contains, however, a unique, much smaller class — an equivalence class modulo the *polar*[1]) sets, and consisting of *quasi-continuous*[2]) functions on Ω (called "fonctions précisées" in [4]). Recall that a function is quasi-continuous if and only if it is *finely continuous* (i.e., continuous with respect to the *fine topology*[3]) on $\boldsymbol{C}$) off some polar (hence finely closed and finely discrete) set, cf. [4, p. 356].

We denote by $BLD(\Omega)$ the vector space of all (equivalence classes modulo polar sets of) Beppo Levi—Deny functions, that is, quasi-continuous Beppo Levi functions on Ω. Considered as prehilbert spaces, $BL(\Omega)$ and $BLD(\Omega)$ are the same.

The vector space $\mathscr{D}(\Omega)$ of all infinitely differentiable functions of compact support in Ω is a subspace of $BL(\Omega)$ and of $BLD(\Omega)$. The Dirichlet seminorm

$$\|u\|_1 = (u, u)_1^{1/2} = \|\nabla u\|_{L^2(\Omega)}$$

is a *norm* on $\mathscr{D}(\Omega)$. Following [4] we denote by $\hat{\mathscr{D}}^1(\Omega)$ the *completion* of $\mathscr{D}(\Omega)$ in this norm. The Hilbert space $\hat{\mathscr{D}}^1(\Omega)$ may thus be identified naturally with the closure

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[1]) A *polar* set (in $\boldsymbol{C}$) is a set which locally is of outer logarithmic capacity 0.

[2]) A function φ: $\Omega\to\boldsymbol{C}$ $(\Omega\subset\boldsymbol{C})$ is called *quasi continuous* if Ω is the union of bounded open sets ω for which there exists, for every $\varepsilon>0$, an open set $\omega_\varepsilon\subset\omega$ of logarithmic capacity $<\varepsilon$ such that the restriction of f to $\omega\setminus\omega_\varepsilon$ is continuous.

[3]) The *fine topology* on $\boldsymbol{C}$ is the weakest topology for which all subharmonic functions (on open subsets of $\boldsymbol{C}$) are continuous. The fine topology is strictly stronger than the usual Euclidean topology on $\boldsymbol{C}$.

$B_0^{\boldsymbol{\cdot}}(\Omega)$ of $\mathscr{D}^{\boldsymbol{\cdot}}(\Omega)$ in $BL^{\boldsymbol{\cdot}}(\Omega)$, whereby $\mathscr{D}^{\boldsymbol{\cdot}}(\Omega)$ denotes the image of $\mathscr{D}(\Omega)$ under the canonical mapping $BL(\Omega)\to BL^{\boldsymbol{\cdot}}(\Omega)$.

For the particular case $\Omega=\boldsymbol{C}$ we have the following well-known

Lemma 1.1. a) 0 *is the only element of* $L^2(\boldsymbol{C})$ *represented by a function harmonic in all of* $\boldsymbol{C}$.

b) *The constants are the only elements of* $BL(\boldsymbol{C})$ *represented by a function harmonic in all of* $\boldsymbol{C}$.

c) $\mathscr{D}^{\boldsymbol{\cdot}}(\boldsymbol{C})$ *is norm dense in the Hilbert space* $BL^{\boldsymbol{\cdot}}(\boldsymbol{C})$. *In other words,* $BL_0^{\boldsymbol{\cdot}}(\boldsymbol{C})=BL^{\boldsymbol{\cdot}}(\boldsymbol{C})$.

Proof. a) If f is harmonic on $\boldsymbol{C}$ then $|f|^2$ is subharmonic on $\boldsymbol{C}$, hence $\leqq 0$ because, for any $a\in\boldsymbol{C}$, $r>0$,

$$|f(a)|^2 \leqq \frac{1}{\pi r^2}\int_{|z-a|<r}|f|^2\,d\lambda \leqq \frac{1}{\pi r^2}\|f\|^2_{L^2(\boldsymbol{C})},$$

which tends to 0 as $r\to\infty$.

b) Reduces to a) applied to the partial derivatives.

c) An element $u^{\boldsymbol{\cdot}}\in BL^{\boldsymbol{\cdot}}(\boldsymbol{C})$ is orthogonal to $\mathscr{D}^{\boldsymbol{\cdot}}(\boldsymbol{C})$ if and only if $u^{\boldsymbol{\cdot}}$ is represented by a harmonic element of $BL(\boldsymbol{C})$, i.e., a constant, by b). (Cf. [4, p. 318].) □

A domain $\Omega\subset\boldsymbol{C}$ is a *Green domain* (i.e., it has a Green function) if and only if $\complement\Omega$ is non-polar (Myrberg's theorem). When Ω is a Green domain, the natural injection $\mathscr{D}(\Omega)\to\mathscr{D}'(\Omega)$ extends uniquely to a continuous, linear, and injective mapping $\hat{\mathscr{D}}^1(\Omega)\to\mathscr{D}'(\Omega)$, [4, p. 350]. In this case $\hat{\mathscr{D}}^1(\Omega)$ will be identified with its image, which clearly is a vector subspace of $BL(\Omega)$, or let us rather say of $BLD(\Omega)$, since we shall understand that the elements of $\hat{\mathscr{D}}^1(\Omega)$ are (equivalence classes modulo polar sets of) quasi-continuous functions.

Recall that a function $u\in BLD(\Omega)$ (Ω a Green domain in $\boldsymbol{C}$) belongs to $\hat{\mathscr{D}}^1(\Omega)$ if and only if

(1) $$\operatorname*{fine\,lim}_{x\to y} u(x) = 0 \quad \text{q.e.[4]} \quad \text{for} \quad y\in\partial_{\text{fine}}\Omega,$$

where $\partial_{\text{fine}}\Omega$ denotes the boundary of Ω in the fine topology on $\boldsymbol{C}$. See [4, p. 359].

If a function u on $\boldsymbol{C}$ satisfies $u=0$ q.e. in $\boldsymbol{C}\setminus\Omega$ (Ω a Green domain), then

(2) $$u\in BLD(\boldsymbol{C}) \Leftrightarrow u|_\Omega\in\hat{\mathscr{D}}^1(\Omega).$$

Here $\Rightarrow$ follows from the above because u is finely continuous q.e. in $\boldsymbol{C}$. For $\Leftarrow$ see [4, pp. 355—359], [14, p. 308].

Definition. *A set* $E\subset\boldsymbol{C}$ *is called* quasi-analytic [quasi-coanalytic] *if* E *differs only by a polar set from some analytic* [*coanalytic*] *set.*

[4]) *Quasi everywhere* (q.e.) means: everywhere off some polar set, while *almost everywhere* (a.e.) means: everywhere off some λ-nullset.

Every finely closed set E is quasi-analytic, and even quasi Borel. In fact, by Kellogg's theorem, E differs only by a polar set from its base $b(E)$ which is an ordinary G_δ-set contained in E.[5]) Every quasi-(co)analytic set is Lebesgue measurable.

The interior of a set $E\subset C$ for the fine topology is denoted by E'.

Lemma 1.2. *For any set* $E\subset C$ *and any* $u\in BLD(C)$ *we have*

$$u = 0 \quad q.e. \text{ in } \quad C\setminus E \Rightarrow \nabla u = 0 \quad a.e. \text{ in } \quad C\setminus E'.$$

Proof. Since u is finely continuous q.e., the hypothesis $u=0$ q.e. in $\complement E$ implies $u=0$ q.e. in $\complement E'$. The conclusion $\nabla u=0$ a.e. in $\complement E'$ is obvious if $\complement E'$ is polar (hence a Lebesgue null set). If $\complement E'$ is non-polar it contains a compact non-polar set K. (This is because $\complement E'$ is finely closed, hence quasi-analytic and therefore "capacitable", cf. e.g. [2, Corollary 5.3.3].) Any component Ω of $\complement K$ is a Green domain, and $\partial_{\text{fine}}\Omega\subset\partial\Omega\subset K$. It follows that $u_{|\Omega}\in\hat{\mathcal{D}}^1(\Omega)$ by (1), because $u=0$ q.e. in $K(\subset\complement E')$. According to [8, corollaire, p. 142] it follows that $\nabla u=0$ a.e. in $\Omega\setminus E'$, and hence a.e. in $\complement(K\cup E')$. We conclude that $\nabla u=0$ a.e. in $\complement E'$ because we may replace K by $K\cap D_n$, $n=1, 2, \ldots$, where $D_n=\{z\in C \,|\, |z-a|\leqq 1/n\}$, $a\in C$ being chosen so that each $K\cap D_n$ is non-polar. $\square$

For any Green domain $\Omega\subset C$ and any $u\in BLD(\Omega)$ we have the canonical orthogonal decomposition [4, p. 322]

$$u = v+h, \quad v\in\hat{\mathcal{D}}^1(\Omega), \quad h \text{ harmonic in } \Omega, \tag{3}$$

and $(v|h)_1=0$.

Lemma 1.3. *Every Beppo Levi function u on an open set* $\Omega\subset C$ *such that u is* finely harmonic *q.e. in Ω, can be corrected (uniquely) on a polar set so as to become harmonic (hence finely harmonic) in the whole of Ω.*

Proof. We may suppose that Ω is a Green domain (othewise cover C by a sequence of Green domains, e.g. discs). In the decomposition (3) the function $v=u-h\in\hat{\mathcal{D}}^1(\Omega)$ is finely harmonic q.e. in Ω, and hence v and therefore u are harmonic (after correction on a polar set) according to [8, proposition 6]. $\square$ — As to finely harmonic functions see [6].

2. Havin's theorem and complements to it

The following key result is the 2-dimensional case of that part of [8, théorème 11] which deals with *finely harmonic* functions:

Theorem A. *Let* $\Omega\subset C$ *be a Green domain and* $E\subset\Omega$ *a quasi-coanalytic (e.g. Borel) set. For* $u\in\hat{\mathcal{D}}^1(\Omega)$ *the following are equivalent:*[6])

[5]) The *base* $b(E)$ of a set E is the set of points of C at which E is not thin (in the sense of Brelot), in other words, the finely derived set.

[6]) For *any* set $E\subset\Omega$ we have 1)$\Rightarrow$2)$\Leftrightarrow$3), cf. [8, remarque 1, p. 143].

1) *There exists a sequence* $(u_n)\subset\hat{\mathcal{D}}^1(\Omega)$ *converging to* u *in* $\hat{\mathcal{D}}^1(\Omega)$ *such that each* u_n *is harmonic in some open set* $\omega_n\supset E$.

2) $(u, \varphi)_1=0$ *for every* $\varphi\in\hat{\mathcal{D}}^1(\Omega)$ *such that* $\varphi=0$ *q.e. in* $\Omega\setminus E$ *(hence* $\nabla\varphi=0$ *a.e. in* $\Omega\setminus E$*).*[7])

3) u *is finely harmonic in the fine interior* E' *of* E.

In order to pass from this result to an extended version of Havin's theorem we need some preparations. Writing as usual $z=x+iy$ and (with a normalization convenient for our purpose)

$$\partial = \partial/\partial x - i\partial/\partial y, \quad \bar{\partial} = \partial/\partial x + i\partial/\partial y,$$

the Laplacian $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ becomes

$$\Delta = \bar{\partial}\partial = \partial\bar{\partial}.$$

Hence

$$\partial\bar{z}^{-1} = \partial\bar{\partial}\log|z| = \Delta\log|z| = 2\pi\varepsilon \tag{4}$$

(ε=the Dirac measure at 0). For any $u, v\in\hat{\mathcal{D}}^1(\Omega)$ (Ω a Green domain in $\boldsymbol{C}$) we have

$$(u, v)_1 = \int_\Omega \partial u\overline{\partial v}\, d\lambda = \int_\Omega \bar{\partial}u\partial\bar{v}\, d\lambda. \tag{5}$$

For $v\in\mathcal{D}(\Omega)$ this follows easily from $\bar{\partial}\partial=\Delta$. It extends by continuity to the general case.

Lemma 2.1. *The operator* ∂ *maps* $BL^\cdot(\boldsymbol{C})$ *isometrically onto* $L^2(\boldsymbol{C})$.

Proof. ∂ is isometric on $\Omega^\cdot(\boldsymbol{C})$, cf. the proof of (5), hence on $BL^\cdot(\boldsymbol{C})$ in view of Lemma 1.1c). If $f\in L^2(\boldsymbol{C})$ is orthogonal to the range of ∂ then $\bar{\partial}f=0$ in the distribution sense, hence $\partial\bar{\partial}f=\Delta f=0$, and so $f=0$ by Lemma 1.1a). □

For a Beppo Levi function w in an open set $\Omega\subset\boldsymbol{C}$ it is well known that the partial derivatives $\partial w/\partial x$, $\partial w/\partial y$ entering in the definition of ∂w and $\bar{\partial}w$ are the same in the distribution sense as in the classical sense, in which they exist a.e. and are of class $L^2(\Omega)$, cf. [4, p. 315]. If $w\in BLD(\Omega)$ then w is representable locally (q.e.) as the Riesz potential $|z|^{-1}*g$ (of order 1) of a function $g\in L^2_c(\boldsymbol{C})$, and hence w is likewise *finely differentiable* at almost every point z_0 of Ω. Moreover, the usual differential at (a.e.) $z_0\in\Omega$:

$$dw = (x-x_0)\frac{\partial w}{\partial x}(z_0)+(y-y_0)\frac{\partial w}{\partial y}(z_0),$$

is also the fine differential at z_0 because

$$|w(z)-w(z_0)-dw|/|z-z_0| \to 0$$

[7]) Cf. [8, corollaire, p. 142].

as $z \to z_0$ in the fine topology, or equivalently as $z \to z_0$ in the usual topology, though with z restricted to a suitable (punctured) fine neighbourhood of z_0; cf. Mizuta [12].

Every finely harmonic function u (real or complex valued) in a finely open set $U \subset \mathbf{C}$ is likewise finely differentiable a.e. in U. This is because every point $z_0 \in U$ has a fine neighbourhood $V \subset U$ such that $u_{|V}$ extends to a BLD function on $\mathbf{C}$, cf. [8, théorème 2].[8]) See also Davie and Øksendal [3, Theorem 6].

We are now prepared to state and prove Havin's theorem in an extended form.

Theorem 2. *Let E be a quasi-coanalytic subset of $\mathbf{C}$. For any $f \in L^2(E)$ the following are equivalent:*

1) *There exists a sequence $(f_n) \subset L^2(E)$ converging to f in norm in $L^2(E)$ such that each f_n admits an extension to a function F_n analytic in an open set $\omega_n \supset E$.*

2) *$\int_E f \overline{\partial \varphi}\, d\lambda = 0$ for every $\varphi \in BLD(\mathbf{C})$ such that $\varphi = 0$ q.e. in $\mathbf{C} \setminus E$ (hence $\partial \varphi = 0$ a.e. in $\mathbf{C} \setminus E$).*

3) *There exists a complex valued function u defined and finely harmonic q.e. in E' such that $\partial u = f$ a.e. in E' in the sense of differentiation in the fine topology.*

Remark 2.1. These three properties are of a *local* character in the fine topology; that is, each property is equivalent to the finely localized version of it, obtained by replacing E by $E \cap V$ (and f by $f_{|E \cap V}$) for every V belonging to some family $\mathscr{V}$ of finely open sets covering E, or just E'. In particular, it suffices to verify 2) for all $\varphi \in BLD(\mathbf{C})$ such that φ equals 0 off some compact subset of E'. (Recall that every fine neighbourhood of a point $z \in \mathbf{C}$ contains a fine neighbourhood of z which is compact in the usual topology.)

Remark 2.2. Suppose that $f \in L^2(E)$ has the equivalent properties 1), 2), 3), and let f also denote any extension of f of class $L^2(\mathbf{C})$. We may then arrange, in 1), that the analytic functions F_n in ω_n are square integrable and that $\|F_n - f_{|\omega_n}\|_{L^2(\omega_n)} \to 0$ as $n \to \infty$. And in 3) the function $u \in BLD(\mathbf{C})$ — uniquely determined q.e. in $\mathbf{C}$ (up to an additive constant) by $\partial u = f$ a.e. in $\mathbf{C}$ (cf. Lemma 2.1) — is finely harmonic q.e. in E'.

Proof of Theorem 2 and Remarks 2.1, 2.2. We shall denote by 1'), 2'), 3') the localized versions of 1), 2), 3) described in Remark 2.1. It is trivial that 1)$\Rightarrow$1'), 2)$\Rightarrow$2'), 3$\Rightarrow$3'), and so it remains to prove that 1')$\Rightarrow$2')$\Rightarrow$3')$\Rightarrow$3)$\Rightarrow$2)$\Rightarrow$1).

Ad 1')$\Rightarrow$2'). For every V from some covering $\mathscr{V}$ of E' as in the formulation of 1') in Remark 2.1, and for any $z \in E' \cap V$, choose W finely open and of compact closure $\overline{W}$ so that $z \in W \subset \overline{W} \subset E' \cap V$. For any $\varphi \in BLD(\mathbf{C})$ such that $\varphi = 0$ q.e. in $\complement W$, the support of φ as a distribution is compact and contained in $\overline{W} \subset E \cap V$.

[8]) In the first instance we obtain a bounded fine neighbourhood $V \subset U$ of z_0 such that $u_{|V}$ extends to a function of class $\hat{\mathscr{D}}^1(\Omega)$, Ω being a disc containing V. The further extension of this function by 0 in $\complement\Omega$ is of class $BLD(\mathbf{C})$ according to (2), Section 1.

With F_n analytic in $\omega_n \supset E\cap V$, and $F_{n|E\cap V} \to f$ in $L^2(E\cap V)$, we therefore obtain, noting that $\bar\partial\varphi=0$ a.e. in $\complement W$ by Lemma 1.2,

$$\int_W F_n \overline{\partial\varphi}\, d\lambda = \int_{\omega_n} F_n \overline{\partial\varphi}\, d\lambda = -\int_{\Omega_n} \bar\partial F_n \varphi\, d\lambda = 0.$$

It follows that $\int_W f\overline{\partial\varphi}\, d\lambda=0$, and this establishes 2′) because the above finely open sets W have the union E'.

Ad 2′)⇒3′). According to Lemma 2.1 there exists $u\in BLD(C)$ with $\partial u=f$ after extension of f by 0, say, in $\complement E$. For every V from some covering $\mathscr{V}$ of E' as in the formulation of 2′) in Remark 2.1, and for any $z\in E'\cap V$, choose W as above, and Ω as a bounded domain in C containing $\overline{W}$ (e.g., a disc). Consider the canonical decomposition $u_{|\Omega}=v+h$, cf. (3). Thus $v\in\hat{\mathscr{D}}^1(\Omega)$, and $\bar\partial\partial h=\Delta h=0$ in Ω. Now let $\varphi\in BLD(C)$ satisfy $\varphi=0$ q.e. in $C\setminus W$, in particular q.e. in $\complement\Omega$. Then $\varphi_{|\Omega}\in\hat{\mathscr{D}}^1(\Omega)$ according to (2), Section 1. Moreover, $\int_\Omega \partial h\overline{\partial\varphi}\, d\lambda=\int_\Omega \bar\partial\partial h\cdot\bar\varphi\, d\lambda=0$. Using (5), and noting that $\varphi=0$ q.e. in $\complement(E\cap V)$ in particular, we therefore obtain from 2′)

$$(v, \varphi|_\Omega)_1 = \int_\Omega \partial v\overline{\partial\varphi}\, d\lambda = \int_\Omega \partial u\overline{\partial\varphi}\, d\lambda = \int_C f\overline{\partial\varphi}\, d\lambda = 0$$

because $\partial\varphi=0$ a.e. in $\complement W(\supset\complement\Omega)$, cf. Lemma 1.2. Applying Theorem A to v, we find that v is finely harmonic q.e. in W, and so is therefore $u=v+h$ because h is harmonic and hence finely harmonic in Ω, cf. [6, Theorem 8.7]. — The above finely open sets W have the union E', and 3′) ensues.

Ad 3′)⇒3), with 3) amplified as in the latter part of Remark 2.2. Thus let $f\in L^2(C)$ extend the given f, and let $u\in BLD(C)$ satisfy $\partial u=f$ a.e. Then u is finely continuous in C off some polar set e. By 3′), E' is the union of a family of finely open sets V for each of which there exists a finely harmonic function u_V in V off some polar set e_V, such that $\partial u_V=f$ a.e. in V. Each point of $V\setminus(e\cup e_V)$ has a fine neighbourhood W in $V\setminus(e\cup e_V)$ such that u_V equals in W some function of class $BLD(C)$, as observed just before Theorem 2. The same is therefore true of $u-u_V$. Because $\partial(u-u_V)=f-f=0$ a.e. in $V\setminus(e\cup e_V)$, we infer from [7, Section 3] that $\overline{u-u_V}$ is finely harmonic (even finely holomorphic) in $V\setminus(e\cup e_V)$, and hence u is itself finely harmonic in $V\setminus(e\cup e_V)$. Invoking Doob's quasi Lindelöf principle [5], cf. also [2, exercise 7.2.6] and the sheaf property of fine harmonicity [6, p. 70], we conclude that u is indeed finely harmonic q.e. in E'.

Ad 3)⇒2). Since even 3′) implies the amplified version of 3), we may suppose that $u\in BLD(C)$ is finely harmonic q.e. in E', and that $\partial u=f$ after extending f by 0 in $\complement E$. If $\complement E'$ is polar, it follows from 3) by Lemma 1.3 and Lemma 1.1b) that u is constant (q.e.), hence $f=\partial u=0$ (a.e.), which implies 2). — Suppose next that $\complement E'$ is non-polar, and let $\varphi\in BLD(C)$ satisfy $\varphi=0$ q.e. in $\complement E$, hence $\varphi=0$ q.e. in $\complement E'$, by fine continuity. Proceeding as in the proof of Lemma 1.2 we choose a compact, non-polar set $K\subset\complement E'$. Every component Ω of $\complement K$ is a Green domain, and $\varphi_{|\Omega}\in\hat{\mathscr{D}}^1(\Omega)$. In the decomposition (3) of $u_{|\Omega}$, the function $v=u_{|\Omega}-h\in\hat{\mathscr{D}}^1(\Omega)$ is finely harmonic q.e. in E' (like u), and hence it follows from Theorem A, in view of (5), that $\int_\Omega \partial v\overline{\partial\varphi}\,d\lambda=(v, \varphi_{|\Omega})_1=0$. Since $h\in BL(\Omega)$ is harmonic in Ω, we have $\int_\Omega \partial h\overline{\partial\varphi}\,d\lambda=0$, as shown by approximating $\varphi_{|\Omega}\in\hat{\mathscr{D}}^1(\Omega)$ by functions of class $\mathscr{D}(\Omega)$. Consequently, $\int_\Omega \partial u\overline{\partial\varphi}\,d\lambda=0$. Adding over all components Ω of $\complement K$, and noting that $\partial\varphi=0$ a.e. in $\complement E'(\supset K\cup\complement E)$, by Lemma 1.2, we conclude that

$$\int_E f\overline{\partial\varphi}\,d\lambda=\int_{E'}\partial u\overline{\partial\varphi}\,d\lambda=\int_{\complement K}\partial u\overline{\partial\varphi}\,d\lambda=0.$$

Ad 2)⇒1), with 1) amplified as in Remark 2.2. If $\complement E$ is polar, it follows from 2) that $\int_C f\overline{\partial\varphi}\,d\lambda=0$ for every $\varphi\in\mathscr{D}(C)$. Hence f is analytic in C, and therefore $f=0$ in C, by Lemma 1.1a), which establishes 1). — Suppose next that the quasi-coanalytic set $\complement E$ is non-polar. Choose a compact, non-polar set $K\subset\complement E$ (cf. the proof of Lemma 1.2). Then $E\subset\complement K$, and it clearly suffices to prove 1) (in its amplified form) with E replaced by $E\cap\Omega$ for each component Ω of $\complement K$. For any $\varphi\in\hat{\mathscr{D}}^1(\Omega)$ such that $\varphi=0$ q.e. in $\Omega\setminus E$, the extension Φ of φ to C by $\Phi=0$ in $\complement\Omega$ satisfies $\Phi\in BLD(C)$, by (2), Section 1; and $\Phi=0$ q.e. in $\complement E$, hence $\partial\Phi=0$ a.e. in $\complement(E\cap\Omega)$, by Lemma 1.2. It follows from (3) that $\int_E f\overline{\partial\Phi}\,d\lambda=0$, and hence $\int_{E\cap\Omega} f\overline{\partial\varphi}\,d\lambda=0$, that is, $\int_\Omega f\overline{\partial\varphi}\,d\lambda=0$. Choose $u\in BLD(C)$ so that $\partial u=f$, cf. Lemma 2.1. In the canonical decomposition $u_{|\Omega}=v+h$ with $v\in\hat{\mathscr{D}}^1(\Omega)$ we obtain from 2) in view of (5)

$$(v, \varphi)_1=\int_\Omega \partial v\overline{\partial\varphi}\,d\lambda=\int_\Omega f\overline{\partial\varphi}\,d\lambda-\int_\Omega \partial h\overline{\partial\varphi}\,d\lambda=0,$$

because $h\in BL(\Omega)$ is harmonic in Ω. Applying Theorem A to $E\cap\Omega$ in place of E, we obtain a sequence $(v_n)\subset\hat{\mathscr{D}}^1(\Omega)$ converging to v in $\hat{\mathscr{D}}^1(\Omega)$ such that each v_n is harmonic in some open set ω_n, $E\cap\Omega\subset\omega_n\subset\Omega$. It follows that

$$u_n:=v_n+h\to v+h=u|_\Omega \quad\text{in}\quad BL^{\cdot}(\Omega),$$

and here u_n is harmonic in ω_n. Consequently, $f_n:=\partial u_n\in L^2(\Omega)$ is analytic in ω_n and converges to $\partial u_{|\Omega}=f_{|\Omega}$ in $L^2(\Omega)$. □

Remark 2.3. In property 3) in the above theorem, consider any two functions u_1, u_2 which are finely harmonic q.e. in E' and satisfy (in the sense of fine partial derivatives) $\partial u_1=\partial u_2=f$ a.e. in E'. The difference $\overline{u_1-u_2}$ is then finely harmonic in $E'\setminus e$ for some polar set $e\subset E'$, and the conjugate $\overline{u_1-u_2}$ is even *finely holomorphic* in $E'\setminus e$, cf. 1) in [7, définition 3], because $\bar\partial(\overline{u_1-u_2})=0$ a.e. in $E'\setminus e$. Hence $\overline{u_1-u_2}$ is finely differentiable everywhere in $E'\setminus e$, even in the complex sense, according to [7, théorème 10]. This implies that u_1 and u_2 are finely differentiable (in the real sense) at precisely the same points of $E'\setminus e$.

In order to be able to compare cl $L_a^2(E)$ and $L_a^2(E)^\perp$ (cf. the introduction) for different sets E we introduce the following

Notation. For any quasi-coanalytic set $E\subset C$ we denote by $L_a^2(C, E)$ the linear subset of $L^2(C)$ consisting of those $f\in L^2(C)$ which are holomorphic in some open set containing E. Thus $L^2_a(E)$ contains all restrictions to E of functions of class $L^2_a(\mathbf{C}, E)$.

The *closure* cl $L^2_a(\mathbf{C}, E)$ of $L^2_a(\mathbf{C}, E)$ within $L^2(\mathbf{C})$ is, by Remark 2.2, the set of all $f\in L^2(\mathbf{C})$ such that the restriction $f_{|E}$ belongs to the closure cl $L^2_a(E)$ of $L^2_a(E)$ within $L^2(E)$, in other words such that $f_{|E}$ has the equivalent properties 1), 2), 3) of Theorem 2. Using 2), we thus obtain

$$\mathrm{cl}\, L^2_a(\mathbf{C}, E) = \{\partial\varphi \mid \varphi\in BLD(\mathbf{C}),\ \varphi = 0 \text{ q.e. in } \complement E\}^{\perp},$$

where $\perp$ indicates orthogonal complement within $L^2(\mathbf{C})$. (Here we also make use of the fact that $\partial\varphi=0$ a.e. in $\complement E$ for any φ as stated in this formula, cf. Lemma 1.2.) It follows that

$$L^2_a(\mathbf{C}, E)^{\perp} = \{\partial\varphi \mid \varphi\in BLD(\mathbf{C}),\ \varphi = 0 \text{ q.e. in } \complement E\} \tag{6}$$

because the right hand member is a closed subspace of $L^2(\mathbf{C})$ in view of Lemma 2.1 and the known fact that $\{\varphi\in BLD(\mathbf{C}) \mid \varphi=0 \text{ q.e. in } \complement E\}^{\boldsymbol{\cdot}}$ is closed in $BL^{\boldsymbol{\cdot}}(\mathbf{C})$, cf. e.g. [14, p. 308].

As noted in the proof of 3)$\Rightarrow$2) in Theorem 2,

$$\varphi = 0 \text{ q.e. in } \complement E \Leftrightarrow \varphi = 0 \text{ q.e. in } \complement E'$$

when $\varphi\in BLD(\mathbf{C})$ (for then φ is finely continuous q.e.). Thus $L^2_a(\mathbf{C}, E)^{\perp}$ and hence cl $L^2_a(\mathbf{C}, E)$ only depend on the fine interior E' of the quasi-coanalytic set $E\subset\mathbf{C}$. Furthermore, these two orthogonal subspaces of $L^2(\mathbf{C})$ obviously depend only on the equivalence class of E modulo polar sets, in view of (6). More precisely we have

Lemma 2.2. *Let E and F be quasi-coanalytic subsets of $\mathbf{C}$. Then* cl $L^2_a(\mathbf{C}, E)=$ cl $L^2_a(\mathbf{C}, F)$ *holds if and only if the fine interiors E' and F' differ only by a polar set; or equivalently: if and only if $\complement E$ and $\complement F$ are thin at the same points: $b(\complement E)=b(\complement F)$.*

Proof. The following stronger result will be obtained:

$$F'\setminus E' \text{ polar} \Leftrightarrow L^2_a(\mathbf{C}, F)^{\perp} \subset L^2_a(\mathbf{C}, E)^{\perp}.$$

The implication$\Rightarrow$follows from (6) above. In proving the converse implication it suffices to consider the case where E and F are *finely open*. Suppose that $F\setminus E$ is non-polar, and choose an open disc Ω in $\mathbf{C}$ so that $A:=(F\setminus E)\cap\Omega$ is likewise non-polar. Then there exist two disjoint finely open subsets V and W of F such that $A\cap V$ and $A\cap W$ are both non-polar. Choose a finite strict Green potential $p=G\mu$ on Ω so that $\int G\mu\, d\mu<+\infty$, and put

$$u = p-\hat{R}_p^{\Omega\setminus V}$$

(balayage relative to Ω), cf. [2, Section 7.2]. Then $u\in\hat{\mathcal{D}}^1(\Omega)$, and so the extension φ of u to $\mathbf{C}$ obtained by putting $\varphi=0$ in $\complement\Omega$ belongs to $BLD(\mathbf{C})$, by (2), Section 1. Clearly, $\varphi=0$ q.e. in $\complement V$, in particular q.e. in $\complement F$, that is, $\partial\varphi\in L^2_a(\mathbf{C}, F)^{\perp}$, by (6). On the other hand, $\partial\varphi$ is not in $L^2_a(\mathbf{C}, E)^{\perp}$, for that would imply that φ were constant q.e. in $\complement E(\supset A)$ according to (6) combined with Lemma 2.1; and this is not the case. Indeed, $u>0$ in the non-polar set $A\cap V$ because p is strict [2, Proposition 7.2.2], while $u=\varphi=0$ q.e. in the non-polar set $A\cap W$ $(\subset\Omega\cap W\subset\Omega\setminus V)$. $\square$

Corollary (essentially due to Havin [10]). *Let E be a quasi-coanalytic subset of $\mathbf{C}$, and let F be an open set such that $F\subset E$ (for example $F=\mathring{E}$, the usual interior of E). In order that all functions $f\in L^2(E)$ such that f is analytic in F belong to* $\mathrm{cl}\, L_a^2(E)$, *it is necessary and sufficient that $E'\setminus F$ be polar.*

In particular, $\mathrm{cl}\, L_a^2(E)=L^2(E)$ *if and only if $E'=\emptyset$ (or equivalently: E' should be a Lebesgue null set).*

Proof. Since $F\subset E$ is open, we have $F\subset\mathring{E}\subset E'$, and $L_a^2(\mathbf{C}, F)$ is the set of functions in $L^2(\mathbf{C})$ which are analytic in F. This set is closed in $\mathscr{D}'(\mathbf{C})$ and hence in $L^2(\mathbf{C})$. Since $\mathrm{cl}\, L_a^2(\mathbf{C}, E)=\mathrm{cl}\, L_a^2(\mathbf{C}, E')$, the corollary follows from Lemma 2.2 in view of (6).

3. Relation to finely holomorphic functions

For every finely open set $U\subset\mathbf{C}$ we denote by $\mathcal{O}(U)$ the algebra of *finely holomorphic* functions $U\to\mathbf{C}$, cf. [7]. Every $f\in\mathcal{O}(U)$ has fine derivatives in the complex sense of all orders, and if U is a fine domain (=finely connected finely open set) the f is uniquely determined within $\mathcal{O}(U)$ by the values of f and all its derivatives at any prescribed point [7, théorèmes 10 et 14]. In particular, if $f(z)=0$ for all z in some fine neighbourhood of a point of a fine domain U, then $f(z)=0$ for all $z\in U$. Let us further write

$$\mathcal{O}^2(\mathbf{C}, U) = \{f\in L^2(\mathbf{C})\,|\, f \text{ is finely holomorphic in } U\}.$$

Returning to the quasi-coanalytic set $E\subset\mathbf{C}$ we have (in addition to the role of fine holomorphy described in Remark 2.3) the following inclusion relations:

$$L_a^2(\mathbf{C}, E)\subset\mathcal{O}^2(\mathbf{C}, E')\subset \mathrm{cl}\, L_a^2(\mathbf{C}, E), \tag{7}$$

showing that $L_a^2(\mathbf{C}, E)$ and $\mathcal{O}^2(\mathbf{C}, E')$ have the same closure in $L^2(\mathbf{C})$. The former relation (7) follows from the definition of $L_a^2(\mathbf{C}, E)$ together with the fact that every analytic function in an open set $\omega\subset\mathbf{C}$ is finely holomorphic in ω, and hence also in every finely open subset of ω (such as E' when $\omega\supset E$), cf. [7, p. 63]. The latter inclusion (7) follows from [7, proposition 16] (applied to $U=E'$) in view of Theorem 2 and the fact that $\mathrm{cl}\, L_a^2(\mathbf{C}, E)=\mathrm{cl}\, L_a^2(\mathbf{C}, E')$, as noted in Section 2 (before Lemma 2.2).

Each of the inclusion relations in (7) is *proper* for a suitable set E, even for a suitable *finely open* set E. As to the former inclusion this follows from the example [7, p. 74], taking $E=U'$. The latter inclusion is proper, as observed in [7, remarque 3, p. 81], or alternatively as a consequence of the following

Proposition 3. *There exists a bounded fine domain* $E\subset \boldsymbol{C}$ *and a function* $f\in \operatorname{cl} L_a^2(\boldsymbol{C}, E)\setminus \mathcal{O}^2(\boldsymbol{C}, E)$. *We may further arrange that* f *does not have the unique continuation property. Explicitly, we shall achieve that* $f=0$ *a.e. in some fine neighbourhood of a point of* E, *and yet* f *is not identically* 0 *a.e. in* E.

Proof. Let D denote the open unit disc in $\boldsymbol{C}$, and $D(z, r)$ the open disc of radius r centered at $z\in\boldsymbol{C}$. Following Lyons [11], in which is incorporated an idea due to M. Sakai, we apply Vitali's covering theorem to cover D up to a Lebesgue nullset by disjoint discs $D_n=D(z_n, r_n)\subset D$, $n\in\boldsymbol{N}$, and consider the probability measure

$$\mu = \sum\nolimits_{n\in N} r_n^2 \varepsilon_{z_r},$$

where ε_z denotes the Dirac measure at z. The logarithmic potential of μ is defined in $\boldsymbol{C}$ as the negative of

$$u(z) = \sum\nolimits_{n\in N} r_n^2 \log|z-z_n| \quad (\geqq -\infty), \quad z\in\boldsymbol{C}.$$

For $|z|>1$, $\log|z-\zeta|$ is a harmonic function of $\zeta\in D$, and hence

$$\log|z-z_n| = \frac{1}{\pi r_n^2}\int_{D_n} \log|z-\zeta|\, d\lambda(\zeta), \quad |z|>1.$$

It follows that

$$u(z) = \frac{1}{\pi}\sum\nolimits_{n\in N}\int_{D_n} \log|z-\zeta|\, d\lambda(\zeta) \tag{8}$$

$$= \frac{1}{\pi}\int_{D} \log|z-\zeta|\, d\lambda(\zeta) = \log|z|, \quad |z|>1.$$

The functions u and $z\mapsto\log|z|$ are subharmonic on $\boldsymbol{C}$. The set

$$e = \{z\in\boldsymbol{C} \mid u(z) = -\infty\}\cup\{0\}$$

is therefore polar, and clearly $\{z_n\}_{n\in N}\subset e\subset \bar{D}$. It follows that $\boldsymbol{C}\setminus e$ is a fine domain [6, Theorem 12.2], and that u and $z\mapsto\log|z|$ are *finely harmonic* in $\boldsymbol{C}\setminus e$, see [6, Theorem 8.10]. The function v defined on $\boldsymbol{C}\setminus e$ by

$$v(z) = u(z) - \log|z|$$

is thus finely harmonic. We have $v\equiv 0$ in $\boldsymbol{C}\setminus\bar{D}$, by (8), but not in all of $\boldsymbol{C}\setminus e$, for then the equality $u(z)=\log|z|$ would extend by fine continuity from $\boldsymbol{C}\setminus e$ to $\boldsymbol{C}$, noting that e has no finely interior points; but actually $\log(z_n) > -\infty = u(z_n)$ for every n such that $z_n\neq 0$. Because $\boldsymbol{C}\setminus e$ is a fine domain, we conclude that the fine interior of $\{z\in\boldsymbol{C}\setminus e \mid v(z)=0\}$ has a fine boundary point z^* relative to $\boldsymbol{C}\setminus e$. Since v is finely harmonic in $\boldsymbol{C}\setminus e$, there exists a finely open set E with $z^*\in E\subset\boldsymbol{C}\setminus e$ such that v coincides on E with a function $w\in BLD(\boldsymbol{C})$, see above (just before Theorem 2). We choose E as a fine domain, thus invoking the local connectedness of the fine topology, cf. [6, p. 92]. The function

$$f = \partial w \tag{9}$$

belongs to $\operatorname{cl} L_a^2(\boldsymbol{C}, E)$ according to Theorem 2, 3)$\Rightarrow$1).

It remains to show that $f_{|E}=(\partial v)_{|E}$ (cf. Lemma 1.2) does not have the unique continuation property. By the choice of z^* we have $v\not\equiv 0$ in E, but $v=0$ everywhere in some non-empty finely open set $V\subset E$, hence $f=\partial v=0$ a.e. in V. If $f=0$ a.e. in E then $\bar{v}$ is finely holomorphic in E according to 1) in [7, définition 3], and we are led to a contradiction by the unique continuation property of finely holomorphic functions [7, théorème 14]. □ — It can be shown that, necessarily, $|z^*|=1$.

4. An extension to higher dimensions

We replace $\boldsymbol{C}$ by $\boldsymbol{R}^k (k\in\boldsymbol{N})$ as space for the independent variable, now denoted by $x=(x^1, \dots, x^k)$. For any vector space $\mathscr{F}$ of real distributions on $\boldsymbol{R}^k$, or of (equivalence classes of) real functions on a given subset of $\boldsymbol{R}^k$, we denote by $\mathscr{F}_{\text{ext}}$ the vector space of currents, or differential forms,

$$f=\sum_{p=1}^{k}\sum_{i_1<\dots<i_p} f_{i_1,\dots,i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p} \tag{10}$$

with coefficients from $\mathscr{F}$, cf. [13]. We have the direct sum decomposition

$$\mathscr{F}_{\text{ext}}=\mathscr{F}_{\text{even}}+\mathscr{F}_{\text{odd}},$$

where $\mathscr{F}_{\text{even}}$, respectively $\mathscr{F}_{\text{odd}}$, denotes the set of those forms $f\in\mathscr{F}_{\text{ext}}$ that are even, respectively odd, in the sense that p is even, respectively odd, for every non-zero coefficient $f_{i_1,\dots,i_p}$ in the reduced expression (10) of f. The number of coefficients in (10) that are not a priori equal to 0 is 2^{k-1} for an even form f and likewise for an odd form.

If $\mathscr{F}$ is a (real) Hilbert space with an inner product $(\cdot|\cdot)$ then so is $\mathscr{F}_{\text{ext}}$, its inner product — likewise denoted $(\cdot|\cdot)$ — being given by

$$(f|g)=\sum_{p=1}^{k}\sum_{i_1<\dots<i_p}(f_{i_1,\dots,i_p}|g_{i_1,\dots,i_p}). \tag{11}$$

$\mathscr{F}_{\text{even}}$ and $\mathscr{F}_{\text{odd}}$ are then orthogonal to each other.

If the elements of $\mathscr{F}$ are differentiable (in some sense) we consider the differential operator $d+\delta$ on $\mathscr{F}_{\text{ext}}$, where d denotes exterior differentiation and δ denotes co-differentiation.[9]) Then $d+\delta$ carries even forms into odd forms, and vice versa. We further write

$$\partial = d+\delta, \quad \text{acting on} \quad \mathscr{F}_{\text{odd}},$$

$$\bar{\partial} = d+\delta, \quad \text{acting on} \quad \mathscr{F}_{\text{even}}.$$

Note that (when defined)

$$\bar{\partial}\partial,\ \partial\bar{\partial}\subset(d+\delta)^2=d\delta+\delta d=\Delta, \tag{12}$$

the Laplacian (acting coefficientwise). This shows that $d+\delta$ is *elliptic.*

[9]) The operator $d+\delta$ has been considered in an entirely different context by Gilkey [9]. Our δ and Δ correspond to $-\delta$ and $-\Delta$ in [13].

We proceed to extend the main results of the preceding sections to the present situation, which is more general — being, for $k=2$, nothing but a reformulation of the complex situation considered up till now, cf. below. Note, however, that instead of analytic functions F in open sets $w \subset \boldsymbol{C}$ now just speak of even differential forms F satisfying the generalized Cauchy–Riemann equation $\bar{\partial}F(=(d+\delta)F)=0$ in open sets $\omega \subset \boldsymbol{R}^k$. By ellipticity, such forms (currents) have real analytic coefficients in ω.

Actually, the proofs become simpler in the case $k \geqq 3$ because $\boldsymbol{R}^k$ then has a Green kernel, namely the Newtonian kernel. From Lemma 1.1c) (valid in any dimension) we obtain the identification

$$BLD(\boldsymbol{R}^k) = \hat{\mathscr{D}}^1(\boldsymbol{R}^k), \quad k \geqq 3.$$

Lemma 2.1 extends, showing that ∂ maps $[\hat{\mathscr{D}}^1(\boldsymbol{R}^k)]_{\mathrm{odd}}$ isometrically onto $[L^2(\boldsymbol{R}^k)]_{\mathrm{even}}$ $(k \geqq 3)$. In particular,

$$(u, v)_1 = (\partial u | \partial v), \quad u, v \in [\hat{\mathscr{D}}^1(\boldsymbol{R}^k)]_{\mathrm{odd}},$$

where $(\cdot, \cdot)$ and $(\cdot|\cdot)$ denote the inner products on $[\hat{\mathscr{D}}^1(\boldsymbol{R}^k)]_{\mathrm{ext}}$ and $[L^2(\boldsymbol{R}^k)]_{\mathrm{ext}}$ derived, as in (11), from the corresponding inner products on $\hat{\mathscr{D}}^1(\boldsymbol{R}^k)$ and $L^2(\boldsymbol{R}^k)$, respectively.

If we replace $\hat{\mathscr{D}}^1(\Omega)$ in [8] (in the case $\Omega=\boldsymbol{R}^k$, $k \geqq 3$) by $[\hat{\mathscr{D}}^1(\boldsymbol{R}^k)]_{\mathrm{odd}}$, then the part of [8, théorème 11] which deals with (finely) *harmonic* functions carries over with unchanged proof. Using this result in place of Theorem A above, Theorem 2 carries over as follows:

Theorem 4. *Let E be a quasi-coanalytic subset of $\boldsymbol{R}^k$, $k \in \boldsymbol{N}$. For any differential form $f \in [L^2(E)]_{\mathrm{even}}$ the following are equivalent:*

1) *There exists a sequence $(f_n) \subset [L^2(E)]_{\mathrm{even}}$ converging to f in norm in $[L^2(E)]_{\mathrm{even}}$ such that each f_n admits an extension to an even, real analytic differential form F_n for which $\bar{\partial}F_n=0$ in some open set $\omega_n \supset E$.*

2) *$(f|\partial\varphi)=0$ for every form $\varphi \in [BLD(\boldsymbol{R}^k)]_{\mathrm{odd}}$ such that $\varphi=0$ q.e.*[10]) *in $\complement E$ (hence $\partial\varphi=0$ a.e. in $\complement E$).*

3) *There exists an odd form u, with coefficients defined and finely harmonic q.e.*[10]) *in E', such that $\partial u=f$ a.e. in E' (in the sense of fine differentiation).*

The various remarks to Theorem 2 (with the exception of Remark 2.3 if $k \geqq 3$) carry over mutatis mutandis, and so do Lemma 2.2 and its corollary. As to Section 3, the construction in the proof of Proposition 3 carries over to produce an example of a fine domain $E \subset \boldsymbol{R}^k$, $k \geqq 2$, and a form $f \in [L^2(E)]_{\mathrm{even}}$ having the equivalent properties 1), 2), 3) in Theorem 4, but failing to have the property of unique continuation.

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[10]) When $k \geqq 3$, quasi-everywhere (q.e.) means: everywhere except in some polar set (=a set of zero outer Newton capacity).

In all of this, one may interchange throughout the terms "even" and "odd". In odd dimension k the two versions thus obtained are, however, equivalent via the involution *, cf. [13, p. 121]. For even $k\geqq 4$ the two versions do not appear to be equivalent in any natural sense.

For $k=2$, both versions reduce to the complex situation from Sections 1—3 via the following identifications (writing (x, y) for the generic point of $\boldsymbol{R}^2$):

$$f\,dx+g\,dy, \quad f+g\,dx\wedge dy \mapsto f+ig,$$

considered as a function of $z=x+iy$. Then ∂ and $\bar\partial$, as defined in the present section, correspond to ∂ and $\bar\partial$ from Section 2.

For $k=3$ we make the following identifications (writing (x, y, z) for the coordinates of the generic point of $\boldsymbol{R}^3$):

$$f_0\,dx\wedge dy\wedge dz+f_1\,dx+f_2\,dy+f_3\,dz \mapsto (f_0, f_1, f_2, f_3),$$
$$f_0+f_1\,dy\wedge dz+f_2\,dz\wedge dx+f_3\,dx\wedge dy \mapsto (f_0, f_1, f_2, f_3).$$

Then ∂ and $\bar\partial$ both correspond to the operator

$$D = \begin{pmatrix} 0 & \partial_x & \partial_y & \partial_z \\ \partial_x & 0 & -\partial_z & \partial_y \\ \partial_y & \partial_z & 0 & -\partial_x \\ \partial_z & -\partial_y & \partial_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & \mathrm{rot} \end{pmatrix},$$

where $\partial_x=\partial/\partial x$, etc. The operator D is studied, e.g., in Aržanyh [1]. — If we further identify $\boldsymbol{R}^4$ (as range space) with the quaternion field $\boldsymbol{H}$ as follows:

$$(a_0, a_1, a_2, a_3) \mapsto a_0+a_1 i+a_2 j+a_3 k,$$

then the operator D transforms into

$$Df = (\partial_x f)i+(\partial_y f)j+(\partial_z f)k$$

for functions f from $\boldsymbol{R}^3$ to $\boldsymbol{H}$.

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