LIPSCHITZ CLASSES AND QUASICONFORMAL MAPPINGS

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1. Introduction

Suppose that $D$ is a domain in the euclidean plane $\mathbb{R}^2$, that $f$ is analytic in $D$ and that $0<\alpha \leq 1$. If there exists a constant $m_1$ such that

$$|f(x_1)-f(x_2)| \leq m_1|x_1-x_2|^\alpha$$

in $D$, then it is not difficult to show that

$$|f'(x)| \leq m_2 d(x, \partial D)^{\alpha-1}$$

in $D$, where $d(x, \partial D)$ denotes the distance from $x$ to $\partial D$ and $m_2=m_1$. Conversely if $D$ is a disk, then by a well known theorem of Hardy and Littlewood [HL], (1.2) implies (1.1) with $m_1=(a/\alpha)m_2$, where $a$ is an absolute constant.

In a recent paper [GM] we observed that the Hardy—Littlewood theorem can be extended to a very large class of domains $D$, namely those which are uniform. (See Section 2 for the definition.) This fact can be viewed as the result of two implications. First, if (1.2) holds, then by the Hardy—Littlewood theorem,

$$|f(x_1)-f(x_2)| \leq m_3|x_1-x_2|^\alpha$$

in each disk $U \subset D$ where $m_3=(a/\alpha)m_2$. Second, if $D$ is uniform and if (1.3) holds in each disk $U \subset D$, then (1.1) holds in $D$ with $m_1=b m_3$ where $b$ depends only on $D$. The first step shows that (1.2) implies $f$ satisfies a uniform local Lipschitz condition in $D$ while the second step derives a global Lipschitz condition from the local condition whenever $D$ is uniform.

In the present paper, we consider both of these implications for domains $D$ in euclidean $n$-space $\mathbb{R}^n$ and functions $f: D \rightarrow \mathbb{R}^n$. In Section 2 we characterize the domains $D$ with the property that functions which satisfy a local Lipschitz condition in $D$ for some $\alpha$ always satisfy the corresponding global condition there; this class includes the uniform domains mentioned above. In Section 3 we study conditions

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which guarantee that a quasiconformal mapping \( f: D \to D' \) satisfies a local Lipschitz condition in \( D \). In particular, we obtain a geometric condition on \( D' \) which is necessary and sufficient for this to be the case whenever \( D \) satisfies the same condition; bounded uniform domains have this property.

2. \( \text{Lip}_\alpha \)-functions and \( \text{Lip}_\alpha \)-extension domains

Given a set \( A \) in \( \mathbb{R}^n \) we let \( \text{Lip}_\alpha(A), \ 0 < \alpha \leq 1 \), denote the Lipschitz class of mappings \( f: A \to \mathbb{R}^p \) satisfying for some \( m < \infty \)

\[
|f(x_1) - f(x_2)| \leq m|x_1 - x_2|^{\alpha}
\]

(2.1)

in \( A \). If \( D \) is a domain in \( \mathbb{R}^n \), then \( f: D \to \mathbb{R}^p \) belongs to the local Lipschitz class \( \text{loc Lip}_\alpha(D) \) if there exists a constant \( m < \infty \) such that (2.1) holds whenever \( x_1, x_2 \) lie in any open ball which is contained in \( D \).

In \( \text{Lip}_\alpha(D) \) and in \( \text{loc Lip}_\alpha(D) \) we shall use seminorms \( \| f \|_\alpha \) and \( \| f \|_{\alpha}^{\text{loc}} \), respectively, which mean the smallest \( m \) for which (2.1) holds in the corresponding set.

The class \( \text{Lip}_\alpha(A) \) does not depend on the set \( A \) since there is a bounded extension operator \( \text{Lip}_\alpha(A) \to \text{Lip}_\alpha(\mathbb{R}^n) \). To prove this observe that \( d(x, y) = |x - y|^{\alpha} \) defines a metric in \( \mathbb{R}^n \) and there is an extension operator \( \text{Lip}_1,\alpha(A) \to \text{Lip}_1,\alpha(\mathbb{R}^n) \) defined by

\[
f^*(y) = \inf \{ f(x) + md(x, y) : x \in A \}
\]

for real valued functions in the 1-Lipschitz class \( \text{Lip}_1,\alpha(A) \) with respect to the metric \( d \), cf. [McS]. Moreover, \( f^* \) has the same Lipschitz constant \( m \) as \( f \). For vector valued functions \( f \in \text{Lip}_\alpha(A) \), \( A \subset \mathbb{R}^n \), the Lipschitz constant may increase by a factor \( \alpha \geq 1 \) depending only on \( p \). For \( \alpha = 1 \), Kirszbraun's theorem yields \( \alpha = 1 \).

On the other hand, the metric structure of the domain \( D \) has an important influence on the class \( \text{loc Lip}_\alpha(D) \). A domain \( D \subset \mathbb{R}^n \) is called a \( \text{Lip}_\alpha \)-extension domain if there is a bounded extension operator \( A: \text{loc Lip}_\alpha(D) \to \text{Lip}_\alpha(D) \), i.e., there exists a constant \( a \) depending only on \( D \), \( \alpha \) and \( p \) such that \( \| f \|_{\alpha} \equiv a \| f \|_{\alpha}^{\text{loc}} \) for each \( f \in \text{loc Lip}_\alpha(D) \). The next theorem characterizes \( \text{Lip}_\alpha \)-extension domains.

2.2. Theorem. A domain \( D \) in \( \mathbb{R}^n \) is a \( \text{Lip}_\alpha \)-extension domain if and only if there is a constant \( M < \infty \) such that for all \( x_1, x_2 \in D \) there exists a rectifiable curve \( \gamma \) joining \( x_1 \) to \( x_2 \) in \( D \) with

\[
\int_{\gamma} d(x, \partial D)^{\alpha - 1} ds \leq M|x_1 - x_2|^{\alpha}.
\]

(2.3)

Proof. First we show that a \( \text{Lip}_\alpha \)-extension domain \( D \) satisfies (2.3). Fix \( x_0 \in D \) and let

\[
u(x) = \inf_{\mathcal{P}} \int_{\mathcal{P}} d(x, \partial D)^{\alpha - 1} ds
\]
where the infimum is taken over all curves joining $x_0$ to $x$ in $D$. Now let $x_1, x_2 \in D$ and let $\gamma$ be any curve joining $x_1$ to $x_2$. Fix a curve $\beta$ joining $x_0$ to $x_1$ in $D$. Then

$$u(x_2) \leq \int_{\gamma \cup \beta} d(x, \partial D)^{s-1} \, ds;$$

hence

$$u(x_2) \leq \inf_{\beta} \int_{\beta} d(x, \partial D)^{s-1} \, ds + \int_{\gamma} d(x, \partial D)^{s-1} \, ds$$

and thus

$$u(x_2) - u(x_1) \leq \int_{\gamma} d(x, \partial D)^{s-1} \, ds.$$

Reversing the roles of $x_1$ and $x_2$ yields

$$|u(x_1) - u(x_2)| \leq \int_{\gamma} d(x, \partial D)^{s-1} \, ds. \tag{2.4}$$

Next assume that $x_1, x_2$ belong to an open ball $U$ which is contained in $D$. Let $\gamma_1$ be the circular arc in $\overline{U}$ perpendicular to $\partial U$ and passing through $x_1, x_2$. Let $y_1$ and $y_2$ be the endpoints of $\gamma_1$ on $\partial U$ and let the points lie in the order $y_1, x_1, x_2, y_2$ on $\gamma_1$. We parametrize $\gamma_1$ by arc length $t$ measured from $y_1$. Let $\gamma$ be a subarc of $\gamma_1$ joining $x_1$ to $x_2$ and parametrized by arc length $s$ measured from $x_1$. If $l$ is the length of $\gamma$, then clearly

$$l \equiv \frac{\pi}{2} |x_1 - x_2|$$

and by plane geometry

$$\min (t, l_1 - t) \leq \frac{\pi}{2} d(\gamma_1(t), \partial U)$$

for all $t \in [0, l_1]$ where $l_1$ is the length of $\gamma_1$. Now fix $s \in [0, l]$ and then $t \in [0, l_1]$ such that $\gamma_1(t) = \gamma(s)$. Since $s \leq t$ and $l - s \leq l_1 - t$, we obtain

$$\min (s, l - s) \leq \min (t, l_1 - t) \leq \frac{\pi}{2} d(\gamma_1(t), \partial U) = \frac{\pi}{2} d(\gamma(s), \partial U).$$

Thus the above inequalities yield

$$\int_{\gamma} d(x, \partial D)^{s-1} \, ds \leq \int_{\gamma} d(x, \partial U)^{s-1} \, ds \leq \left(\frac{\pi}{2}\right)^{1-\alpha} \int_0^l \min (s, l-s)^{s-1} \, ds$$

$$= 2 \left(\frac{\pi}{2}\right)^{1-\alpha} \int_0^{l/2} s^{s-1} \, ds = \frac{\pi^{1-\alpha}}{\alpha} l^\alpha \leq \frac{\pi}{\alpha 2^\alpha} |x_1 - x_2|^\alpha.$$

Together with (2.4) this gives

$$|u(x_1) - u(x_2)| \leq m|x_1 - x_2|^\alpha,$$

$m = \pi \alpha^{-1} 2^{-\alpha}$. Hence $u$ belongs to $\text{loc Lip}_\alpha(D)$. By the assumption, $u \in \text{Lip}_\alpha(D)$ and $\|u\|_{\alpha}$ has an upper bound $M$ which is independent of $x_0$. The definition of $u$ now yields (2.3).
Next suppose that $D$ satisfies the condition (2.3), choose a positive constant $c \leq 1/2$ and suppose that
\begin{equation}
|f(x_1) - f(x_2)| \leq m|x_1 - x_2|^\gamma
\end{equation}
whenever $x_1, x_2 \in D$ with $|x_1 - x_2| \leq c \cdot d(x_1, \partial D)$. Fix $x_1, x_2 \in D$ and let $\gamma$ be a curve as in (2.3) and parametrized by arc length $s$ measured from $x_1$. Write $\gamma' = c/2$ and choose balls $B^n(y_1, r_1)$ as follows. Let $y_1 = x_1$, $r_1 = c' \cdot d(y_1, \partial D)$. Set $l_1 = \max \{ s \in [0, l] : \gamma(s) \in \overline{B^n(y_1, r_1)} \}$ where $l$ is the length of $\gamma$. If $y_i, r_i$ and $l_i$ have been chosen, $i = 1, 2, \ldots, k$, and $l_k < l$, set $y_{k+1} = \gamma(l_k)$, $r_{k+1} = c' \cdot d(y_{k+1}, \partial D)$ and $l_{k+1} = \max \{ s \in [0, l] : \gamma(s) \in \overline{B^n(y_{k+1}, r_{k+1})} \}$. After a finite number of steps, say $k$, $l_k = l$ and the process stops. Write $y_{k+1} = x_2$.

By (2.5)
\begin{equation}
|f(x_1) - f(x_2)| \leq \sum_{i=1}^k |f(y_i) - f(y_{i+1})| \leq m \sum_{i=1}^k |y_i - y_{i+1}|^\gamma.
\end{equation}
Let $l_0 = 0$ and for each $i = 1, 2, \ldots, k - 1$ let
\[ A_i = \{ s \in [l_{i-1}, l_i] : \gamma(s) \in \overline{B^n(y_i, r_i)} \}. \]
Then $A_i \subseteq [l_{i-1}, l_i]$ is a closed set and
\begin{equation}
\mathcal{M}_1(A_i) = r_i = |y_i - y_{i+1}|.
\end{equation}
Moreover, for $s \in A_i$
\[ d(\gamma(s), \partial D) \leq |\gamma(s) - y_i| - d(y_i, \partial D) \leq r_i + r_i / c' = r_i (1 + 1/c'), \]
and hence
\[ d(\gamma(s), \partial D)^{\gamma - 1} \leq (1 + 1/c')^{\gamma - 1} r_i^{\gamma - 1}. \]
Together with (2.7) this yields
\begin{equation}
\int \gamma d(x, \partial D)^{\gamma - 1} ds \leq \sum_{i=1}^{k-1} \int_{A_i} d(\gamma(s), \partial D)^{\gamma - 1} ds
\leq (1 + 1/c')^{\gamma - 1} \sum_{i=1}^{k-1} r_i^{\gamma - 1} \mathcal{M}_1(A_i)
\leq (1 + 1/c')^{\gamma - 1} \sum_{i=1}^{k-1} |y_i - y_{i+1}|^\gamma.
\end{equation}
To complete the proof suppose first that
\[ |x_1 - x_2| < |y_k - y_{k+1}| = |y_k - x_2|. \]
Then
\[ |x_1 - x_2| < |y_k - x_2| \equiv c' \cdot d(y_k, \partial D), \]
and hence
\[ d(x_1, \partial D) \leq d(y_k, \partial D) - |x_1 - x_2| - |x_2 - y_k| \equiv \frac{1}{2} d(y_k, \partial D) \]
since $c \leq 1/2$. This combined with the previous inequalities gives
\[ |x_1 - x_2| \leq 2c' d(x_1, \partial D) = cd(x_1, \partial D) \]
and we obtain
\begin{equation}
|f(x_1)-f(x_2)| \leq m|x_1-x_2|^s
\end{equation}
by (2.5). Suppose next that
\begin{equation}
|x_1-x_2| \geq |y_k-y_{k+1}|.
\end{equation}
Then (2.6), (2.8) and (2.10) with the assumption (2.3) yield
\begin{equation}
|f(x_1)-f(x_2)| \leq m(1+1/c)^1-s \int_{\gamma} d(x, \partial D)^{s^{-1}} ds + m|x_1-x_2|^s
\leq am|x_1-x_2|^s,
\end{equation}
where \(a=(1+2/c)^{1-s}M+1\).

Finally if \(f \in \text{loc Lip}_s(D)\), then \(f\) satisfies the condition (2.5) with \(c=1/2\) and \(m=\|f\|^{\text{loc}}\). Thus by (2.11)
\[\|f\|_x \leq a\|f\|^{\text{loc}}\]
where \(a=5^{1-s}M+1\), and this completes the proof.

In view of the opening remark Theorem 2.2 yields
\begin{equation}
2.12. \text{Corollary. There is a bounded extension operator from } \text{loc Lip}_s(D) \text{ into } \text{Lip}_s(R^n) \text{ if and only if } D \text{ satisfies the condition } (2.3).
\end{equation}

The proof for Theorem 2.2 also yields the following alternative characterization for the class \(\text{loc Lip}_s(D)\).

\begin{equation}
2.13. \text{Theorem. Suppose that } D \text{ is a domain in } R^n. \text{ Then } f: D \rightarrow R^p \text{ belongs to } \text{loc Lip}_s(D) \text{ if and only if there are constants } m<\infty \text{ and } 0<c<1 \text{ such that }
\end{equation}
\begin{equation}
|f(x_1)-f(x_2)| \leq m|x_1-x_2|^s
\end{equation}
whenever \(|x_1-x_2| \leq c d(x_1, \partial D)|.\)

\textit{Proof.} The necessity is immediate. For the sufficiency let \(U\) be an open ball in \(D\) and fix \(x_1, x_2 \in U\). As in the first part of the proof for Theorem 2.2 let \(\gamma\) be a circular arc in \(U\) joining \(x_1\) to \(x_2\) and we obtain
\[
\int_{\gamma} d(x, \partial U)^{s^{-1}} ds \leq \frac{\pi}{\alpha 2^s} |x_1-x_2|^s = M|x_1-x_2|^s.
\]
Next by replacing \(c\) by \(\min(c, 1/2)\) we see that we may assume without loss of generality \(c \leq 1/2\). Then (2.14) implies that
\[
|f(x_1)-f(x_2)| \leq m|x_1-x_2|^s
\]\nwhenever \(x_1, x_2 \in U\) with \(|x_1-x_2| \leq c d(x_1, \partial U)|, and we conclude from the second part of the proof of Theorem 2.2 that \(f \in \text{Lip}_s(U)\) with
\[\|f\|_x \leq ((1+2/c)^{1-s}M+1)m = m_1.\]
Thus \(f \in \text{loc Lip}_s(D)\) with \(\|f\|^{\text{loc}} \leq m_1.\).
The next theorem rules out the existence of inward directed cusps for \( n=2 \) and inward directed ridges for \( n \geq 3 \) in \( \text{Lip}_a \)-extension domains.

2.15. Theorem. Suppose that a domain \( D \) satisfies (2.3). Then there is a constant \( c<\infty \) depending only on \( \alpha \) and \( M \) such that for each \( x_0 \in \mathbb{R}^n \) and \( r>0 \) points in \( D \cap \overline{B}^n(x_0,r) \) can be joined in \( D \cap \overline{B}^n(x_0,cr) \).

Proof. Set \( c=2(M+1)^{1/\alpha}-1 \). Let \( x_0 \in \mathbb{R}^n \) and \( r>0 \). Choose points \( x_1, x_2 \) in \( D \cap \overline{B}^n(x_0, r) \) and let \( \gamma \) be a curve as in (2.3). We may assume that there is a point \( y \in \partial D \cap \overline{B}^n(x_0, r) \). Then

\[
\int_{\gamma} d(x, \partial D)^{\alpha-1} ds \geq \int_{\gamma} |x-y|^{\alpha-1} ds \geq \int_{\gamma} (|x-x_0|+r)^{\alpha-1} ds
\]

\[
\geq 2 \int_r^{|x-x_0|} (t+r)^{\alpha-1} dt = \frac{2r^\alpha}{\alpha} ((c+1)^\alpha-2^\alpha),
\]

where \( t=|x-x_0| \). On the other hand,

\[
\int_{\gamma} d(x, \partial D)^{\alpha-1} ds \leq M|x_1-x_2|^\alpha \leq 2^\alpha Mr^\alpha,
\]

which together with the previous inequality yields

\[
2^\alpha M \geq 2\alpha^{-1}((c+1)^\alpha-2^\alpha).
\]

This is impossible for the given \( c \). The proof is complete.

Next we point out a relatively large class of domains in \( \mathbb{R}^n \) which satisfy the condition (2.3) for all \( 0<\alpha \leq 1 \). A domain \( G \subset \mathbb{R}^n \) is said to be a John domain [MS] if there exist constants \( a, b, a>b>0 \), and a point \( x_0 \in G \), called a John center, such that each \( x \in G \) can be joined to \( x_0 \) by a rectifiable curve \( \gamma \) in \( G \) with

\[
l(\gamma) \equiv a,
\]

(2.16)

\[
b \frac{s}{l(\gamma)} \leq d(\gamma(s), \partial G)
\]

for \( 0 \leq s \leq l(\gamma) \). Here \( l(\gamma) \) denotes the arc length of \( \gamma \) and \( \gamma(s) \) its arc length representation with \( \gamma(0)=x \). A domain \( D \subset \mathbb{R}^n \) is called uniform if for some constants \( a, b \) each \( x_1, x_2 \in D \), \( x_1 \neq x_2 \), lie in a John domain \( G \subset D \) with constants \( a|x_1-x_2|, b |x_1-x_2| \).

If \( D \) is uniform, then (2.16) implies directly that each \( x_1, x_2 \in D \) can be joined by a rectifiable curve \( \gamma \) in \( D \) such that

\[
l(\gamma) \equiv a'|x_1-x_2|,
\]

(2.17)

\[
\min (s, l(\gamma)-s) \leq b'd(\gamma(s), \partial D)
\]

where \( a'=2a \) and \( b'=a/b \). Conversely, if each \( x_1, x_2 \in D \) can be joined by a curve \( \gamma \) satisfying (2.17), then \( D \) is uniform with constants \( a, b \) which depend only on \( a' \) and \( b' \). For other characterizations of uniform domains see [GO] and [M].
A domain $D \subset \mathbb{R}^n$ is said to be \textit{c-locally connected} if for each $x_0 \in \mathbb{R}^n$ and $r > 0$,
(i) points in $D \cap B^r(x_0, r)$ can be joined in $D \cap B^r(x_0, cr)$,
(ii) points in $D \setminus B^r(x_0, r)$ can be joined in $D \setminus B^r(x_0, r/c)$.
We say that $D$ is \textit{linearly locally connected} if it is \textit{c-locally connected} for some $c$. If $D$ is an $(a, b)$-uniform domain, then an elementary argument based on (2.17) shows that $D$ is \textit{c-locally connected} with
\[ c = 2 \max \left( 2a, \frac{a}{b} \right) + 1. \]
The next lemma will be needed in Section 3.

2.18. Lemma. \textit{Suppose that} $D$ \textit{is a bounded uniform domain. Then} $D$ \textit{is a John domain with constants}
\[ a_1 = 2a \text{ dia} (D), \quad b_1 = \frac{b^2 \text{ dia} (D)}{2a} \]
\textit{where} $a, b$ \textit{are the constants for} $D$.
\textit{Proof.} Since $D$ is bounded, there is $x_0 \in D$ with
\[ d(x_0, \partial D) = \max \{ d(x, \partial D) : x \in D \}. \]
Set $r = d(x_0, \partial D)$. Now
\begin{equation}
(2.19) \quad r \geq b \text{ dia} (D).
\end{equation}
To prove this let $\varepsilon > 0$ and choose $x_1, x_2 \in D$ such that
\[ |x_1 - x_2| > \text{ dia} (D) - \varepsilon. \]
Since $D$ is uniform, there is a John domain $G \subset D$ with constants $a|x_1 - x_2|, b|x_1 - x_2|$ containing $x_1$ and $x_2$ and a John center $y_0$ of $G$ which by (2.16) satisfies
\[ d(y_0, \partial D) \geq d(y_0, \partial G) \geq b|x_1 - x_2| > b(\text{ dia} (D) - \varepsilon). \]
Thus
\[ d(x_0, \partial D) \geq d(y_0, \partial D) > b(\text{ dia} (D) - \varepsilon) \]
and letting $\varepsilon \to 0$ we obtain (2.19).

Fix $x \in D$. Since $D$ is uniform, there is a rectifiable curve $\gamma$ joining $x$ to $x_0$ as in (2.17) and parametrized by arc length measured from $x$. Now
\begin{equation}
(2.20) \quad l(\gamma) \equiv 2a |x - x_0| \equiv 2a \text{ dia} (D).
\end{equation}
Let $0 \equiv s \leq l(\gamma)$. If $l(\gamma) - s < r/2$, then by (2.19)
\begin{equation}
(2.21) \quad d(\gamma(s), \partial D) \equiv r/2 \equiv \frac{b}{2} \text{ dia} (D) \equiv \frac{b \text{ dia} (D) s}{l(\gamma)} \equiv \frac{b^2 \text{ dia} (D)}{2a} \frac{s}{l(\gamma)}. \end{equation}
Next suppose \( l(\gamma) - s \geq r/2 \). Then by (2.17)

\[
(2.22) \quad d(\gamma(s), \partial D) \geq \frac{b}{a} \min(s, l(\gamma) - s) = \frac{b}{a} \min(s, r/2)
\]

and hence for \( s < r/2 \)

\[
(2.23) \quad d(\gamma(s), \partial D) \geq \frac{b}{a} s = \frac{b}{a} \frac{s}{l(\gamma)} = \frac{b}{a} \frac{s}{l(\gamma)} = \frac{b^2 \text{dia}(D)}{2a} \frac{s}{l(\gamma)}.
\]

On the other hand, if \( s \geq r/2 \), then (2.22) yields

\[
 d(\gamma(s), \partial D) \geq \frac{br}{a} \geq \frac{br}{2} \geq \frac{b(r/2 + s)}{a} = \frac{b}{a} \frac{s}{l(\gamma)} = \frac{b^2 \text{dia}(D)}{2a} \frac{s}{l(\gamma)}.
\]

This together with the inequalities (2.20), (2.21) and (2.23) yields the desired result.

2.24. Theorem. Uniform domains are Lip_\( \alpha \)-extension domains for all \( 0 < \alpha \leq 1 \).

Proof. By Theorem 2.2 it suffices to show that a uniform domain \( D \) satisfies the condition (2.3). Let \( x_1, x_2 \in D \) and let \( \gamma \) be a curve in \( D \) joining \( x_1 \) to \( x_2 \) and satisfying (2.17). Then

\[
\int_{\gamma} d(x, \partial D)^{\alpha-1} ds \equiv \left( \frac{a}{b} \right)^{1-\alpha} \int_{\gamma} \left( \min(s, l(\gamma) - s) \right)^{\alpha-1} ds
\]

\[
= 2 \left( \frac{a}{b} \right)^{1-\alpha} \int_0^{l(\gamma)/2} s^{\alpha-1} ds = \frac{(2a/b)^{1-\alpha}}{\alpha} l(\gamma)^{\alpha}
\]

and thus \( D \) satisfies the condition (2.3) with \( M = 2ax^{\alpha-1}b^{\alpha-1} \). The proof is complete.

We can generate a large class of Lip_\( \alpha \)-extension domains by combining Theorem 2.24 with the following observation.

2.25. Theorem. Suppose that \( M, m \) and \( k \) are fixed constants and \( D \) is the union of a family \( \mathcal{D} \) of Lip_\( \alpha \)-extension domains \( G \) which satisfy (2.3) with the same constant \( M \). Suppose also that for all \( x_1, x_2 \in D \) there exist domains \( G_1, \ldots, G_j \in \mathcal{D} \) with \( j \leq k \) and points \( y_1, \ldots, y_{j+1} \) such that \( x_1 = y_1, \ x_2 = y_{j+1} \) and

\[
y_i, y_{i+1} \in G_i, \quad |y_i - y_{i+1}| \leq m|x_1 - x_2|
\]

for \( i = 1, \ldots, j \). Then \( D \) is a Lip_\( \alpha \)-extension domain.

Proof. Given \( x_1, x_2 \in D \) and \( G_i, y_i \) as above, we can choose a curve \( \gamma_i \) joining \( y_i \) to \( y_{i+1} \) in \( G_i \subset D \) such that

\[
\int_{\gamma_i} d(x, \partial G_i)^{\alpha-1} ds \equiv \int_{\gamma_i} d(x, \partial D)^{\alpha-1} ds \equiv M |y_i - y_{i+1}|^{\alpha}.
\]
Then $\gamma = \gamma_1 + \ldots + \gamma_j$ is a curve joining $x_1$ to $x_2$ in $D$,

$$\int_{\gamma} d(x, \partial D)^{s-1} ds \leq \sum_{i=1}^{j} M |y_i - y_{i+1}|^s \leq M m^s k |x_1 - x_2|^s$$

and the desired conclusion follows from Theorem 2.2.

2.26. Examples. (a) Let $D$ be the domain $\{x \in \mathbb{R}^n: 0 < x_i < 1\}$ between two planes or the tube domain $\{x \in \mathbb{R}^n: |x_i| < 1, \ i = 1, \ldots, n-1\}$. Then it is easy to see that $D$ satisfies the condition (2.3) only for $\alpha = 1$ and hence $D$ is a Lip$_{\alpha}$-extension domain only for $\alpha = 1$.

(b) A domain in $\mathbb{R}^n$ is said to be a $K$-quasiball if it is the image of a ball under a $K$-quasiconformal mapping $f: \mathbb{R}^n \to \mathbb{R}^n$. It follows from [MS, Theorem 2.15] that every quasiball in $\mathbb{R}^n$ is a uniform domain. (See also [GO, Corollary 4].) Hence a quasiball is also a Lip$_{\alpha}$-extension domain for $0 < \alpha \leq 1$.

(c) There are Lip$_{\alpha}$-extension domains which are not uniform. To construct such a Jordan domain in $\mathbb{R}^3$ let $G_0$ denote the open triangle bounded by the lines

$$x = 0, \ y = 0, \ x - y = 1$$

and for $j = 1, 2, \ldots$ let $G_j$ be the open triangle bounded by

$$x = 2^{-2j}, \ y = 2^{-2j} - 2^{-4j}, \ x + y = 2^{-2j} - 2^{-4j}.$$

By Theorem 2.24, each triangle $G_j$ satisfies (2.3) with a constant $M$ independent of $j$ and it is not difficult to show that $D = \bigcup_{j=0}^{\infty} G_j$ is a Jordan domain which satisfies the hypotheses of Theorem 2.25 with $m < 16$ and $k = 3$. Thus $D$ is a Lip$_{\alpha}$-extension domain for $0 < \alpha \leq 1$. However, $D$ is not a quasidisk and hence not uniform. A similar construction in $\mathbb{R}^n, n \geq 3$, yields a Jordan domain $D$ which is a Lip$_{\alpha}$-extension domain for $0 < \alpha \leq 1$ but is not a quasiball. The next theorem clarifies the plane case.

2.27. Theorem. Suppose that $D \subset \mathbb{R}^2$ is a simply connected Lip$_{\alpha}$-extension domain with $\infty \in \partial D$. If $D' = \mathbb{R}^2 \setminus D$ is a Lip$_{\beta}$-extension domain, then $\partial D$ is a quasicircle in $\mathbb{R}^2$.

Proof. By Theorem 2.2 and Theorem 2.15 there is a constant $c$ such that the conclusion of Theorem 2.15 holds for $D$ and for $D'$. To prove the theorem it suffices to show, see [G, Lemma 4], that $D$ is $c'$-locally connected for some $c' > c$. Theorem 2.15 takes care of the first condition for the $c'$-local connectivity and we can now argue exactly as in the proof of Theorem 4.2 in [GM] to obtain the second.

2.28. Remark. If $J$ is a quasicircle in $\mathbb{R}^2$ which contains $\infty$, then its residual domains $D$ and $D'$ are quasidisks and hence uniform domains, cf. Example 2.26 (b). By Theorem 2.24, $D$ and $D'$ are Lip$_{\alpha}$-extension domains for all $0 < \alpha \leq 1$. This together with Theorem 2.27 gives a characterization of quasicircles: A Jordan curve $J$ in $\mathbb{R}^2$ which contains $\infty$ is a quasicircle if and only if its residual domains are Lip$_{\alpha}$-extension domains for some $\alpha$, $0 < \alpha \leq 1$. 

Lipschitz classes and quasiconformal mappings
3. Quasiconformal mappings and $\text{Lip}_a$-classes

It is well-known that a $K$-quasiconformal mapping $f: D \to \mathbb{R}^n$ is $K^{1/(1-n)}$-Lipschitzian on compact subsets of $D$. However, this does not imply that $f \in \text{loc Lip}_a(D)$ for some $0 < a \leq K^{1/(1-n)}$. On the other hand, in $\text{Lip}_a$-extension domains $D$, $f \in \text{loc Lip}_a(D)$ implies $f \in \text{Lip}_a(D)$. In this section we shall give two necessary and sufficient conditions for a quasiconformal mapping $f$ of $D$ to be in $\text{loc Lip}_a(D)$. The above result can then be used to conclude $f \in \text{Lip}_a(D)$ and, in particular, we employ this method to study quasiconformal mappings on uniform and John domains.

As we noted in Section 1, for $f$ analytic in a plane domain $D$ the condition

$$|f'(z)| \leq m d(z, \partial D)^{-1}$$

(3.1)

can be used on some domains $D$, e.g. on uniform domains, to conclude $f \in \text{Lip}_a(D)$; see [GM]. In this section we have replaced analytic functions and the property (3.1) by quasiconformal mappings and by $f \in \text{loc Lip}_a(D)$, respectively. Hence this section is, in a sense, a quasiconformal counterpart of [GM].

3.2. Quasiconformal maps and $\text{loc Lip}_a$. If $D \subset \mathbb{R}^n$ is a domain and $f: D \to \mathbb{R}^n$, then the boundary cluster set for $f$ is given by

$$\mathcal{C}(f, \partial D) = \bigcap \overline{\{f(U \cap D)\}}$$

where the intersection is taken over all neighborhoods $U$ of $\partial D$.

3.3. Lemma. Suppose that $f: D \to \mathbb{R}^n$ is in $\text{loc Lip}_a(D)$. Then

$$d(f(x), \mathcal{C}(f, \partial D)) \leq m d(x, \partial D)^a$$

for $x \in D$ where $m = \|f\|^\text{loc}_a$.

Proof. Fix $x_1 \in D$, choose $x_2 \in \partial D$ such that

$$|x_1 - x_2| = d(x_1, \partial D) = d$$

and let $U = B^n(x_1, d) \subset D$. Now

$$|f(y_1) - f(y_2)| \leq m |y_1 - y_2|^a$$

for $y_1, y_2 \in U$ where $m = \|f\|^\text{loc}_a$. Thus $f$ has a continuous extension to $U$, $f(x_2) \in \mathcal{C}(f, \partial D)$ and

$$d(f(x_1), \mathcal{C}(f, \partial D)) \leq |f(x_1) - f(x_2)| \leq m |x_1 - x_2|^a$$

$$= m d(x_1, \partial D)^a.$$
3.4. Theorem. Suppose that $f$ is a $K$-quasiconformal mapping of $D$ onto $D'$ and that $0 < \alpha \leq K^{1/(1-n)}$. Then $f \in \text{loc Lip}_\alpha(D)$ if and only if there exists a constant $M < \infty$ with
\begin{equation}
(3.5) \quad d(f(x), \partial D') \leq Md(x, \partial D)\alpha
\end{equation}
for all $x \in D$.

Proof. First suppose that $f$ belongs to $\text{loc Lip}_\alpha(D)$. Since $f$ is a homeomorphism
\[ C(f, \partial D) \subset f(\overline{D}) \setminus f(D) = \overline{D} \setminus D' = \partial D' \]
and hence by Lemma 3.3
\[ d(f(x), \partial D') \leq d(f(x), C(f, \partial D)) \leq Md(x, \partial D)\alpha \]
where $M = \|f\|_\alpha^\alpha$.

Next suppose that (3.5) holds. Since $0 < \alpha \leq K^{1/(1-n)}$, [GO, Lemma 2] implies that
\begin{equation}
\frac{|f(x_1) - f(x_2)|}{d(f(x_1), \partial D')} \leq 2\lambda_n^\alpha \left( \frac{|x_1 - x_2|}{d(x_1, \partial D)} \right)\alpha
\end{equation}
whenever $x_1 \in D$ and $|x_1 - x_2| \leq d(x_1, \partial D)$. Here $c = (2\lambda_n^\alpha)^{-1/\alpha}$ and $\lambda_n$ depends only on $n$. Thus (3.5) yields
\begin{align}
|f(x_1) - f(x_2)| &\leq c^{-\alpha} d(f(x_1), \partial D')d(x_1, \partial D)^{-\alpha} |x_1 - x_2|^\alpha \\
&\leq c^{-\alpha} M |x_1 - x_2|^\alpha
\end{align}
for $x_1 \in D$ and $|x_1 - x_2| \leq d(x_1, \partial D)$. By Theorem 2.13, $f \in \text{loc Lip}_\alpha(D)$ as desired.

3.6. Quasihyperbolic boundary condition. A domain $D$ in $\mathbb{R}^n$ satisfies a quasihyperbolic boundary condition if for some $x_0 \in D$ there exist constants $a$ and $b_0$ such that
\begin{equation}
k_D(x, x_0) \leq a \log \frac{1}{d(x, \partial D)} + b_0
\end{equation}
for all $x \in D$. (Cf. [BP, Theorem 1].) Here $k_D$ denotes the quasihyperbolic metric in $D$, i.e.
\begin{equation}
k_D(x_1, x_2) = \inf_\gamma \int_\gamma d(x, \partial D)^{-1} ds,
\end{equation}
where the infimum is taken over all rectifiable curves joining $x_1$ and $x_2$ in $D$. For the basic properties of this metric see [GP] and [GO]. Note that if $D$ satisfies (3.7) for some point $x_0 \in D$, then for any point $x_1 \in D$
\[ k_D(x_1, x_0) \leq k_D(x, x_0) + k_D(x_0, x_1) \leq a \log \frac{1}{d(x, \partial D)} + b_1 \]
for all $x \in D$ where $b_1 = b_0 + k_D(x_0, x_1)$. Thus whether or not $D$ satisfies (3.7) is independent of the choice of the point $x_0$. 
We first study some properties of domains satisfying a quasihyperbolic boundary condition and relate these domains to the domains mentioned in Section 2.

3.9. Lemma. If $D$ satisfies (3.7), then $D$ is bounded with

$$\text{dia}(D) \leq 2ae^{b_0/a}.$$ 

Proof. Let $x_1 \in D$ and let $\gamma$ be a quasihyperbolic geodesic joining $x_0$ and $x_1$ in $D$, cf. [GO, Lemma 1]. Suppose that $\gamma$ is parametrized by arc length measured from $x_0$. For each $s$ write $f(s) = k_D(\gamma(s), x_0)$. Then

$$f(s) = \int_0^s d(\gamma(t), \partial D)^{-1} \, dt$$

and thus $f'(s) = d(\gamma(s), \partial D)^{-1}$ for each $s$. Now by (3.7)

$$f(s) \equiv a \log d(\gamma(s), \partial D) + b_0 = a \log f'(s) + b_0$$

and hence

$$\log f'(s) \equiv \frac{1}{a} (f(s) - b_0).$$

This yields a differential inequality

$$(3.10) \quad f'(s) e^{\frac{1}{a}(b_0 - f(s))} \equiv 1$$

for $0 < s < l$ where $l$ is the length of $\gamma$. The inequality (3.10) can be written in the form

$$1 \equiv -a \frac{d}{ds} \left(e^{\frac{1}{a}(b_0 - f(s))}\right)$$

and by integration we obtain

$$l = \int_0^l ds \equiv -a \int_0^l \frac{d}{ds} \left(e^{\frac{1}{a}(b_0 - f(s))}\right) ds$$

$$= a \left(e^{\frac{1}{a}(b_0 - f(0))} - e^{\frac{1}{a}(b_0 - f(l))}\right) \equiv ae^{b_0/a}$$

since $f(0) = 0$. This holds for each $x_1 \in D$; hence

$$\text{dia}(D) \leq 2ae^{b_0/a}$$

as desired.

3.11. Lemma. If $D$ is a John domain, then $D$ satisfies a quasihyperbolic boundary condition.

Proof. Let $D$ be a John domain with constants $a, b$ and let $x_0$ be a John center for $D$. Next given $x_1 \in D$ let $\gamma$ denote a rectifiable curve joining $x_1$ to $x_0$ in $D$ which satisfies (2.16). We consider two cases.
Suppose first that
\begin{equation}
(3.12) \quad d(x_1, \partial D) \equiv \frac{a+b}{a} l(\gamma).
\end{equation}
Then by (3.12)
\[ d(x, \partial D) \equiv d(x_1, \partial D) - |x_1 - x| \equiv \frac{b}{a} l(\gamma) \]
for all \( x \in \gamma \) and hence
\[ k_D(x_1, x_0) \equiv \int_{\gamma} \frac{ds}{d(x, \partial D)} \equiv \frac{a}{b}. \]
On the other hand, \( d(x, \partial D) \leq a \) for all \( x \in D \) by (2.16) and thus
\[ k_D(x_1, x_0) \equiv \frac{a}{b} + \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} \equiv \frac{a}{b} \log \frac{1}{d(x_1, \partial D)} + \frac{a}{b} (\log a + 1). \]
Suppose next that (3.12) does not hold. Then we can choose a proper subarc \( \gamma_1 \) of \( \gamma \) with endpoints \( x_1 \) and \( x_2 \) so that
\[ d(x_1, \partial D) = \frac{a+b}{a} l(\gamma_1). \]
Then
\[ k_D(x_1, x_2) \equiv \frac{a}{b} \]
by what was proved above. If \( x \in \gamma \setminus \gamma_1 = \gamma_2 \), then by (2.16)
\[ d(x, \partial D) \equiv b \frac{\gamma(s)}{l(\gamma)} \equiv \frac{b}{a} s \]
where \( x = \gamma(s) \). Thus
\[ k_D(x_2, x_0) \equiv \int_{\gamma_2} \frac{ds}{d(x, \partial D)} \equiv \frac{a}{b} \int_{\gamma_1} \frac{ds}{s} \]
\[ = \frac{a}{b} \log \left( \frac{a}{d(x_1, \partial D)} \right) \equiv \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} + 1 \]
and we obtain
\[ k_D(x_1, x_0) \equiv k_D(x_1, x_2) + k_D(x_2, x_0) \]
\[ \equiv \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} + 1 + \frac{a}{b}. \]
Combining these two cases yields
\[ k_D(x_1, x_0) \equiv a' \log \frac{1}{d(x_1, \partial D)} + b'_0 \]
with
\[ a' = \frac{a}{b}, \quad b'_0 = \frac{a}{b} (\log a + 1) + 1. \]
3.13. Remark. The converse of Lemma 3.11 is false, i.e. there are domains which satisfy a quasihyperbolic boundary condition but which are not John domains. In fact, the plane domain \( D \) constructed in Example 2.26 (c) has this property. Since \( D \) is not a John domain, it remains to show that \( D \) satisfies a quasihyperbolic boundary condition. To see this let \( z_j \) denote the center of the maximum disk inscribed in \( G_j \) for \( j = 0, 1, \ldots \) and let \( w_j \) denote the midpoint of the line segment \( \partial G_0 \cap G_j \) for \( j = 1, 2, \ldots \). Fix \( z \in D \cap G_j \), let \( \gamma \) be the segment joining \( z_j \) to \( z \) and let \( s \) denote arc length measured along \( \gamma \) from \( z \). Then by elementary geometry

\[
d(\gamma(s), \partial D) \geq s/3
\]

for \( 0 \leq s \leq |z - z_j| \) and

\[
d(\gamma(s), \partial D) \geq d(z, \partial D)/2
\]

for \( 0 \leq s \leq d(z, \partial D)/2 \); these estimates imply that

\[
k_D(z, z_j) \leq 3 \log \frac{1}{d(z, \partial D)} + 3.
\]

Thus (3.7) holds for points \( z \in \overline{G}_0 \). If \( z \in G_j \cap \overline{G}_0 \), then (3.14) and the triangle inequality yield

\[
k_D(z, z_0) \leq k_D(z, z_j) + k_D(z_j, w_j) + k_D(w_j, z_0)
\]

\[
\leq 3 \log \frac{1}{d(z, \partial D)} + 6 \log \frac{1}{d(w_j, \partial D)} + 9 \leq 15 \log \frac{1}{d(z, \partial D)} + 9
\]

since

\[
d(z, \partial D) \leq \frac{1}{2} 2^{-2j} \left( \frac{1}{2} 2^{-4j} \right)^{1/2} = d(w_j, \partial D)^{1/2}.
\]

Hence \( D \) satisfies a quasihyperbolic boundary condition.

3.15. Remark. By Lemma 2.18 a bounded uniform domain is a John domain. Hence Lemma 3.11 holds for bounded uniform domains as well.

3.16. Quasihyperbolic boundary condition and quasiconformal mappings. For a quasiconformal mapping \( f: D \to D' \) we shall relate the condition \( f \in \text{loc Lip}_4(D) \) with quasihyperbolic boundary conditions on \( D \) and \( D' \).

3.17. Theorem. Suppose that \( f \) is a \( K \)-quasiconformal mapping of \( D \) onto \( D' \). If \( D' \) satisfies a quasihyperbolic boundary condition, then \( f \in \text{loc Lip}_4(D) \) for some \( 0 < \alpha \leq K^{1/(1-n)} \).

Proof. By Theorem 3.4 it suffices to show that there exist constants \( \alpha \) and \( M \) such that \( 0 < \alpha \leq K^{1/(1-n)} \) and

\[
d(f(x), \partial D') \leq M d(x, \partial D)^{\alpha}
\]
for all \( x \in D \). Fix \( x_0 \in D \) such that \( D' \) satisfies the quasihyperbolic boundary condition (3.7) at \( f(x_0) \). Now [GP, Lemma 2.1] and [GO, Theorem 3] yield

\[
\log \frac{d(x_0, \partial D)}{d(x, \partial D)} \leq k_D(x, x_0) \leq c(k_D(f(x), f(x_0)) + 1)
\]

where \( c = c(n, K) \). Thus

\[
\frac{d(x_0, \partial D)}{d(x, \partial D)} \leq \left( \frac{1}{d(f(x), \partial D')} \right)^{ac}
\]

and hence

\[
d(f(x), \partial D') \leq N d(x, \partial D)^\beta,
\]

where

\[
N = e^{(b_0 + 1)/a} d(x_0, \partial D)^{-\beta}, \quad \beta = 1/(ac).
\]

Set \( \alpha = \min(\beta, K^{1/(1-n)}) \) and \( M = \max(N, \text{dia}(D')) \). Observe that \( D' \) is bounded by Lemma 3.9. If now \( d(x, \partial D) \leq 1 \), then by the previous inequality

\[
d(f(x), \partial D') \leq N d(x, \partial D)^\beta \leq M d(x, \partial D)^\alpha
\]

while if \( d(x, \partial D) \geq 1 \), then

\[
d(f(x), \partial D') \leq \text{dia}(D') \leq M d(x, \partial D)^\alpha.
\]

Thus (3.18) holds as desired.

Lemma 3.11 gives the following corollary of Theorem 3.17.

3.19. Corollary. Suppose that \( f \) is a \( K \)-quasiconformal mapping of \( D \) onto a John domain \( D' \). Then \( f \in \text{loc Lip}_\alpha(D) \) for some \( 0 < \alpha \leq K^{1/(1-n)} \).

Next we shall study the converse of Theorem 3.17.

3.20. Lemma. Suppose that \( f \) is a \( K \)-quasiconformal mapping of \( D \) onto \( D' \) and that \( D \) satisfies a quasihyperbolic boundary condition. If \( f \in \text{loc Lip}_\alpha(D) \) for some \( 0 < \alpha \leq K^{1/(1-n)} \), then \( D' \) satisfies a quasihyperbolic boundary condition.

Proof. By Theorem 3.4 there is \( M \) such that

\[
d(f(x), \partial D') \leq M d(x, \partial D)^\alpha
\]

for all \( x \in D \). Fix \( x_0 \in D \) so that \( D \) satisfies (3.7) at \( x_0 \). By [GO, Theorem 3]

\[
k_{D'}(f(x), f(x_0)) \leq c(k_D(x, x_0) + 1)
\]

Therefore, \( D' \) satisfies the quasihyperbolic boundary condition.
where $c = c(n, K)$ and the inequality \((3.21)\) yields
\[
(3.23) \quad \frac{1}{d(x, \partial D)} \leq \left( \frac{M}{d(f(x), \partial D')} \right)^{1/\alpha};
\]
hence by \((3.22), (3.7)\) and \((3.23)\)
\[
k_{D'}(f(x), f(x_0)) \leq c \left( a \log \frac{1}{d(x, \partial D)} + b_0 + 1 \right)
\leq a' \log \frac{1}{d(f(x), \partial D')} + b'_0
\]
where $a' = a/\alpha$ and $b'_0 = a' \log M + c(b_0 + 1)$. Thus $D'$ satisfies a quasihyperbolic boundary condition at $f(x_0)$.

Theorem 3.17 and Lemma 3.20 give a characterization for $f \in \text{loc Lip}_\alpha(D)$ in terms of $D$ and $D'$.

3.24. Theorem. Suppose that $f$ is a $K$-quasiconformal mapping of $D$ onto $D'$ and that $D$ satisfies a quasihyperbolic boundary condition. Then $f \in \text{loc Lip}_\alpha(D)$ for some $0 < \alpha \leq K^{1/(1-n)}$ if and only if $D'$ satisfies a quasihyperbolic boundary condition.

3.25. Quasiconformal mappings and Lip\(_\alpha\). For a quasiconformal mapping $f: D \to D'$ we combine the results above with Section 2 to conclude that $f \in \text{Lip}_\alpha(D)$. The first theorem follows directly from Theorem 3.17.

3.26. Theorem. Suppose that $f$ is a $K$-quasiconformal mapping of $D$ onto $D'$ and that $D'$ satisfies a quasihyperbolic boundary condition. Then there is an $\alpha, 0 < \alpha \leq K^{1/(1-n)}$, such that $f \in \text{Lip}_\alpha(D)$ whenever $D$ is a Lip\(_\alpha\)-extension domain.

3.27. Corollary. Suppose that $f$ is a $K$-quasiconformal mapping of $D$ onto $D'$, that $D$ satisfies \((2.3)\) for all $0 < \alpha \leq K^{1/(1-n)}$ and that $D'$ satisfies a quasihyperbolic boundary condition. Then $f \in \text{Lip}_\alpha(D)$ for some $0 < \alpha \leq K^{1/(1-n)}$.

Proof. The corollary follows from Theorem 3.26 and Theorem 2.2.

Theorem 2.24, Lemma 3.11 and Corollary 3.27 yield

3.28. Theorem. Suppose that $f$ is a $K$-quasiconformal mapping of a uniform domain $D$ onto a John domain $D'$. Then $f \in \text{Lip}_\alpha(D)$ for some $0 < \alpha \leq K^{1/(1-n)}$.

3.29. Remark. An inspection of the proofs leading to Theorem 3.28 shows that $\alpha$ depends only on $n, K$ and the constants for $D'$ and that \(\|f\|_\alpha\) depends only on $n, K$, the constants for $D$ and $D'$ and $d(f^{-1}(y_0), \partial D)$ where $y_0$ is a John center of $D'$.

Finally we obtain the following far reaching extension of a well known theorem due to A. Mori [LV, Theorem II.3.2] from Theorem 3.28 and Lemma 2.18.
3.30. Corollary. Suppose that \( D \) and \( D' \) are bounded uniform domains in \( \mathbb{R}^n \) and that \( f \) is a \( K \)-quasiconformal mapping of \( D \) onto \( D' \). Then \( f \in \text{Lip}_\alpha(D) \) and \( f^{-1} \in \text{Lip}_\alpha(D') \) for some \( 0 < \alpha \leq K^{1/(1-n)} \), where \( \alpha \) depends only on \( n, K, D \) and \( D' \).

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