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LIPSCHITZ CLASSES AND QUASICONFORMAL MAPPINGS

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1. Introduction

Suppose that D is a domain in the euclidean plane R^2, that f is analytic in D and that $0<\alpha\leqq 1$. If there exists a constant m_1 such that

$$|f(x_1)-f(x_2)| \leqq m_1|x_1-x_2|^\alpha \tag{1.1}$$

in D, then it is not difficult to show that

$$|f'(x)| \leqq m_2 d(x, \partial D)^{\alpha-1} \tag{1.2}$$

in D, where $d(x, \partial D)$ denotes the distance from x to ∂D and $m_2=m_1$. Conversely if D is a disk, then by a well known theorem of Hardy and Littlewood [HL], (1.2) implies (1.1) with $m_1=(a/\alpha)m_2$ where a is an absolute constant.

In a recent paper [GM] we observed that the Hardy—Littlewood theorem can be extended to a very large class of domains D, namely those which are uniform. (See Section 2 for the definition.) This fact can be viewed as the result of two implications. First, if (1.2) holds, then by the Hardy—Littlewood theorem,

$$|f(x_1)-f(x_2)| \leqq m_3|x_1-x_2|^\alpha \tag{1.3}$$

in each disk $U\subset D$ where $m_3=(a/\alpha)m_2$. Second, if D is uniform and if (1.3) holds in each disk $U\subset D$, then (1.1) holds in D with $m_1=b\, m_3$ where b depends only on D. The first step shows that (1.2) implies f satisfies a uniform *local* Lipschitz condition in D while the second step derives a *global* Lipschitz condition from the local condition whenever D is uniform.

In the present paper, we consider both of these implications for domains D in euclidean n-space R^n and functions $f\colon D\to R^p$. In Section 2 we characterize the domains D with the property that functions which satisfy a local Lipschitz condition in D for some α always satisfy the corresponding global condition there; this class includes the uniform domains mentioned above. In Section 3 we study conditions

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which guarantee that a quasiconformal mapping $f\colon D\to D'$ satisfies a local Lipschitz condition in D. In particular, we obtain a geometric condition on D' which is necessary and sufficient for this to be the case whenever D satisfies the same condition; bounded uniform domains have this property.

2. Lip_α-functions and Lip_α-extension domains

Given a set A in R^n we let $\mathrm{Lip}_\alpha(A)$, $0<\alpha\leqq 1$, denote the Lipschitz class of mappings $f\colon A\to R^p$ satisfying for some $m<\infty$

$$|f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha \tag{2.1}$$

in A. If D is a domain in R^n, then $f\colon D\to R^p$ belongs to the local Lipschitz class $\mathrm{loc\,Lip}_\alpha(D)$ if there exists a constant $m<\infty$ such that (2.1) holds whenever x_1, x_2 lie in any open ball which is contained in D.

In $\mathrm{Lip}_\alpha(D)$ and in $\mathrm{loc\,Lip}_\alpha(D)$ we shall use seminorms $\|f\|_\alpha$ and $\|f\|_\alpha^{\mathrm{loc}}$, respectively, which mean the smallest m for which (2.1) holds in the corresponding set.

The class $\mathrm{Lip}_\alpha(A)$ does not depend on the set A since there is a bounded extension operator $\mathrm{Lip}_\alpha(A)\to\mathrm{Lip}_\alpha(R^n)$. To prove this observe that $d(x,y)=|x-y|^\alpha$ defines a metric in R^n and there is an extension operator $\mathrm{Lip}_{1,d}(A)\to\mathrm{Lip}_{1,d}(R^n)$ defined by

$$f^*(y)=\inf\{f(x)+md(x,y)\colon x\in A\}$$

for real valued functions in the 1-Lipschitz class $\mathrm{Lip}_{1,d}(A)$ with respect to the metric d, cf. [McS]. Moreover, f^* has the same Lipschitz constant m as f. For vector valued functions $f\in\mathrm{Lip}_\alpha(A)$, $A\subset R^n$, the Lipschitz constant may increase by a factor $\varkappa\geqq 1$ depending only on p. For $\alpha=1$, Kirszbraun's theorem yields $\varkappa=1$.

On the other hand, the metric structure of the domain D has an important influence on the class $\mathrm{loc\,Lip}_\alpha(D)$. A domain $D\subset R^n$ is called a *Lip_α-extension domain* if there is a bounded extension operator $\Lambda\colon \mathrm{loc\,Lip}_\alpha(D)\to\mathrm{Lip}_\alpha(D)$, i.e., there exists a constant a depending only on D, α and p such that

$$\|f\|_\alpha \leqq a\|f\|_\alpha^{\mathrm{loc}}$$

for each $f\in\mathrm{loc\,Lip}_\alpha(D)$. The next theorem characterizes Lip_α-extension domains.

2.2. Theorem. *A domain D in R^n is a Lip_α-extension domain if and only if there is a constant $M<\infty$ such that for all $x_1, x_2\in D$ there exists a rectifiable curve γ joining x_1 to x_2 in D with*

$$\int_\gamma d(x,\partial D)^{\alpha-1}\,ds \leqq M|x_1-x_2|^\alpha. \tag{2.3}$$

Proof. First we show that a Lip_α-extension domain D satisfies (2.3). Fix $x_0\in D$ and let

$$u(x)=\inf_\beta\int_\beta d(x,\partial D)^{\alpha-1}\,ds$$

where the infimum is taken over all curves joining x_0 to x in D. Now let $x_1, x_2 \in D$ and let γ be any curve joining x_1 to x_2. Fix a curve β joining x_0 to x_1 in D. Then

$$u(x_2) \leqq \int_{\beta+\gamma} d(x, \partial D)^{\alpha-1}\, ds;$$

hence

$$u(x_2) \leqq \inf_{\beta} \int_{\beta} d(x, \partial D)^{\alpha-1}\, ds + \int_{\gamma} d(x, \partial D)^{\alpha-1}\, ds$$

and thus

$$u(x_2) - u(x_1) \leqq \int_{\gamma} d(x, \partial D)^{\alpha-1}\, ds.$$

Reversing the roles of x_1 and x_2 yields

$$|u(x_1) - u(x_2)| \leqq \int_{\gamma} d(x, \partial D)^{\alpha-1}\, ds. \tag{2.4}$$

Next assume that x_1, x_2 belong to an open ball U which is contained in D. Let γ_1 be the circular arc in $\bar{U}$ perpendicular to ∂U and passing through x_1, x_2. Let y_1 and y_2 be the endpoints of γ_1 on ∂U and let the points lie in the order y_1, x_1, x_2, y_2 on γ_1. We parametrize γ_1 by arc length t measured from y_1. Let γ be a subarc of γ_1 joining x_1 to x_2 and parametrized by arc length s measured from x_1. If l is the length of γ, then clearly

$$l \leqq \frac{\pi}{2} |x_1 - x_2|$$

and by plane geometry

$$\min(t, l_1 - t) \leqq \frac{\pi}{2} d(\gamma_1(t), \partial U)$$

for all $t \in [0, l_1]$ where l_1 is the length of γ_1. Now fix $s \in [0, l]$ and then $t \in [0, l_1]$ such that $\gamma_1(t) = \gamma(s)$. Since $s \leqq t$ and $l - s \leqq l_1 - t$, we obtain

$$\min(s, l-s) \leqq \min(t, l_1 - t) \leqq \frac{\pi}{2} d(\gamma_1(t), \partial U) = \frac{\pi}{2} d(\gamma(s), \partial U).$$

Thus the above inequalities yield

$$\int_{\gamma} d(x, \partial D)^{\alpha-1}\, ds \leqq \int_{\gamma} d(x, \partial U)^{\alpha-1}\, ds \leqq \left(\frac{\pi}{2}\right)^{1-\alpha} \int_0^l \min(s, l-s)^{\alpha-1}\, ds$$

$$= 2\left(\frac{\pi}{2}\right)^{1-\alpha} \int_0^{l/2} s^{\alpha-1}\, ds = \frac{\pi^{1-\alpha}}{\alpha} l^{\alpha} \leqq \frac{\pi}{\alpha 2^{\alpha}} |x_1 - x_2|^{\alpha}.$$

Together with (2.4) this gives

$$|u(x_1) - u(x_2)| \leqq m |x_1 - x_2|^{\alpha},$$

$m = \pi\alpha^{-1}2^{-\alpha}$. Hence u belongs to $\operatorname{loc}\operatorname{Lip}_{\alpha}(D)$. By the assumption, $u \in \operatorname{Lip}_{\alpha}(D)$ and $\|u\|_{\alpha}$ has an upper bound M which is independent of x_0. The definition of u now yields (2.3).

Next suppose that D satisfies the condition (2.3), choose a positive constant $c \leqq 1/2$ and suppose that

$$|f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha \tag{2.5}$$

whenever $x_1, x_2 \in D$ with $|x_1-x_2| \leqq c\, d(x_1, \partial D)$. Fix $x_1, x_2 \in D$ and let γ be a curve as in (2.3) and parametrized by arc length s measured from x_1. Write $c'=c/2$ and choose balls $B^n(y_i, r_i)$ as follows. Let $y_1=x_1$, $r_1=c'd(y_1, \partial D)$. Set $l_1=\max\{s\in[0, l]: \gamma(s)\in \bar{B}^n(y_1, r_1)\}$ where l is the length of γ. If y_i, r_i and l_i have been chosen, $i=1, 2, \ldots, k$, and $l_k<l$, set $y_{k+1}=\gamma(l_k)$, $r_{k+1}=c'd(y_{k+1}, \partial D)$ and $l_{k+1}=\max\{s\in[0, l]: \gamma(s)\in \bar{B}^n(y_{k+1}, r_{k+1})\}$. After a finite number of steps, say k, $l_k=l$ and the process stops. Write $y_{k+1}=x_2$.

By (2.5)

$$|f(x_1)-f(x_2)| \leqq \sum_{i=1}^k |f(y_i)-f(y_{i+1})| \leqq m \sum_{i=1}^k |y_i-y_{i+1}|^\alpha. \tag{2.6}$$

Let $l_0=0$ and for each $i=1, 2, \ldots, k-1$ let

$$A_i = \{s\in[l_{i-1}, l_i]: \gamma(s)\in \bar{B}^n(y_i, r_i)\}.$$

Then $A_i \subset [l_{i-1}, l_i]$ is a closed set and

$$m_1(A_i) \geqq r_i = |y_i-y_{i+1}|. \tag{2.7}$$

Moreover, for $s\in A_i$

$$d(\gamma(s), \partial D) \leqq |\gamma(s)-y_i|+d(y_i, \partial D) \leqq r_i+r_i/c' = r_i(1+1/c'),$$

and hence

$$d(\gamma(s), \partial D)^{\alpha-1} \geqq (1+1/c')^{\alpha-1} r_i^{\alpha-1}.$$

Together with (2.7) this yields

$$\begin{aligned} \int_\gamma d(x, \partial D)^{\alpha-1}\, ds &\geqq \sum_{i=1}^{k-1} \int_{A_i} d(\gamma(s), \partial D)^{\alpha-1}\, ds \\ &\geqq (1+1/c')^{\alpha-1} \sum_{i=1}^{k-1} r_i^{\alpha-1} m_1(A_i) \\ &\geqq (1+1/c')^{\alpha-1} \sum_{i=1}^{k-1} |y_i-y_{i+1}|^\alpha. \end{aligned} \tag{2.8}$$

To complete the proof suppose first that

$$|x_1-x_2| < |y_k-y_{k+1}| = |y_k-x_2|.$$

Then

$$|x_1-x_2| < |y_k-x_2| \leqq c'd(y_k, \partial D),$$

and hence

$$d(x_1, \partial D) \geqq d(y_k, \partial D)-|x_1-x_2|-|x_2-y_k| \geqq \tfrac{1}{2} d(y_k, \partial D)$$

since $c \leqq 1/2$. This combined with the previous inequalities gives

$$|x_1-x_2| \leqq 2c'd(x_1, \partial D) = c\,d(x_1, \partial D)$$

and we obtain

$$|f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha \tag{2.9}$$

by (2.5). Suppose next that

$$|x_1-x_2| \geqq |y_k-y_{k+1}|. \tag{2.10}$$

Then (2.6), (2.8) and (2.10) with the assumption (2.3) yield

$$\begin{aligned} |f(x_1)-f(x_2)| &\leqq m(1+1/c')^{1-\alpha}\int_\gamma d(x, \partial D)^{\alpha-1}\,ds+m|x_1-x_2|^\alpha \\ &\leqq am|x_1-x_2|^\alpha, \end{aligned} \tag{2.11}$$

where $a=(1+2/c)^{1-\alpha}M+1$.

Finally if $f\in \operatorname{loc}\operatorname{Lip}_\alpha(D)$, then f satisfies the condition (2.5) with $c=1/2$ and $m=\|f\|_\alpha^{\mathrm{loc}}$. Thus by (2.11)

$$\|f\|_\alpha \leqq a\|f\|_\alpha^{\mathrm{loc}}$$

where $a=5^{1-\alpha}M+1$, and this completes the proof.

In view of the opening remark Theorem 2.2 yields

2.12. Corollary. *There is a bounded extension operator from* $\operatorname{loc}\operatorname{Lip}_\alpha(D)$ *into* $\operatorname{Lip}_\alpha(R^n)$ *if and only if* D *satisfies the condition* (2.3).

The proof for Theorem 2.2 also yields the following alternative characterization for the class $\operatorname{loc}\operatorname{Lip}_\alpha(D)$.

2.13. Theorem. *Suppose that* D *is a domain in* R^n. *Then* $f\colon D\to R^p$ *belongs to* $\operatorname{loc}\operatorname{Lip}_\alpha(D)$ *if and only if there are constants* $m<\infty$ *and* $0<c<1$ *such that*

$$|f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha \tag{2.14}$$

whenever $|x_1-x_2|\leqq c\, d(x_1, \partial D)$.

Proof. The necessity is immediate. For the sufficiency let U be an open ball in D and fix $x_1, x_2\in U$. As in the first part of the proof for Theorem 2.2 let γ be a circular arc in U joining x_1 to x_2 and we obtain

$$\int_\gamma d(x, \partial U)^{\alpha-1}\,ds \leqq \frac{\pi}{\alpha 2^\alpha}|x_1-x_2|^\alpha = M|x_1-x_2|^\alpha.$$

Next by replacing c by $\min(c, 1/2)$ we see that we may assume without loss of generality $c\leqq 1/2$. Then (2.14) implies that

$$|f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha$$

whenever $x_1, x_2\in U$ with $|x_1-x_2|\leqq cd(x_1, \partial U)$, and we conclude from the second part of the proof of Theorem 2.2 that $f\in\operatorname{Lip}_\alpha(U)$ with

$$\|f\|_\alpha \leqq ((1+2/c)^{1-\alpha}M+1)m = m_1.$$

Thus $f\in\operatorname{loc}\operatorname{Lip}_\alpha(D)$ with $\|f\|_\alpha^{\mathrm{loc}}\leqq m_1$.

The next theorem rules out the existence of inward directed cusps for $n=2$ and inward directed ridges for $n \geqq 3$ in Lip_α-extension domains.

2.15. Theorem. *Suppose that a domain D satisfies* (2.3). *Then there is a constant $c<\infty$ depending only on α and M such that for each $x_0 \in R^n$ and $r>0$ points in $D \cap \bar{B}^n(x_0, r)$ can be joined in $D \cap \bar{B}^n(x_0, cr)$.*

Proof. Set $c=2(M+1)^{1/\alpha}-1$. Let $x_0 \in R^n$ and $r>0$. Choose points x_1, x_2 in $D \cap \bar{B}^n(x_0, r)$ and let γ be a curve as in (2.3). We may assume that there is a point $y \in \partial D \cap \bar{B}^n(x_0, r)$. Suppose that γ is not contained in $D \cap \bar{B}^n(x_0, c\,r)$. Then

$$\int_\gamma d(x, \partial D)^{\alpha-1}\, ds \geqq \int_\gamma |x-y|^{\alpha-1}\, ds \geqq \int_\gamma (|x-x_0|+r)^{\alpha-1}\, ds$$

$$\geqq 2\int_r^{cr} (t+r)^{\alpha-1}\, dt = \frac{2r^\alpha}{\alpha}((c+1)^\alpha - 2^\alpha),$$

where $t=|x-x_0|$. On the other hand,

$$\int_\gamma d(x, \partial D)^{\alpha-1}\, ds \leqq M|x_1-x_2|^\alpha \leqq 2^\alpha M r^\alpha,$$

which together with the previous inequality yields

$$2^\alpha M \geqq 2\alpha^{-1}((c+1)^\alpha - 2^\alpha).$$

This is impossible for the given c. The proof is complete.

Next we point out a relatively large class of domains in R^n which satisfy the condition (2.3) for all $0<\alpha \leqq 1$. A domain $G \subset R^n$ is said to be a *John domain* [MS] if there exist constants a, b, $\infty > a \geqq b > 0$, and a point $x_0 \in G$, called a *John center*, such that each $x \in G$ can be joined to x_0 by a rectifiable curve γ in G with

$$l(\gamma) \leqq a, \tag{2.16}$$
$$b\frac{s}{l(\gamma)} \leqq d(\gamma(s), \partial G)$$

for $0 \leqq s \leqq l(\gamma)$. Here $l(\gamma)$ denotes the arc length of γ and $\gamma(s)$ its arc length representation with $\gamma(0)=x$. A domain $D \subset R^n$ is called *uniform* if for some constants a, b each $x_1, x_2 \in D$, $x_1 \neq x_2$, lie in a John domain $G \subset D$ with constants $a|x_1-x_2|$, $b\,|x_1-x_2|$.

If D is uniform, then (2.16) implies directly that each $x_1, x_2 \in D$ can be joined by a rectifiable curve γ in D such that

$$l(\gamma) \leqq a'|x_1-x_2|, \tag{2.17}$$
$$\min(s, l(\gamma)-s) \leqq b' d(\gamma(s), \partial D)$$

where $a'=2a$ and $b'=a/b$. Conversely, if each $x_1, x_2 \in D$ can be joined by a curve γ satisfying (2.17), then D is uniform with constants a, b which depend only on a' and b'. For other characterizations of uniform domains see [GO] and [M].

A domain $D\subset R^n$ is said to be *c-locally connected* if for each $x_0\in R^n$ and $r>0$,

(i) points in $D\cap \bar{B}^n(x_0, r)$ can be joined in $D\cap \bar{B}^n(x_0, cr)$,

(ii) points in $D\setminus B^n(x_0, r)$ can be joined in $D\setminus B^n(x_0, r/c)$.

We say that D is *linearly locally connected* if it is c-locally connected for some c. If D is an (a, b)-uniform domain, then an elementary argument based on (2.17) shows that D is c-locally connected with

$$c = 2\max\left(2a, \frac{a}{b}\right)+1.$$

The next lemma will be needed in Section 3.

2.18. Lemma. *Suppose that D is a bounded uniform domain. Then D is a John domain with constants*

$$a_1 = 2a \operatorname{dia}(D), \quad b_1 = \frac{b^2 \operatorname{dia}(D)}{2a}$$

where a, b are the constants for D.

Proof. Since D is bounded, there is $x_0\in D$ with

$$d(x_0, \partial D) = \max\{d(x, \partial D)\colon x\in D\}.$$

Set $r=d(x_0, \partial D)$. Now

$$r \geqq b \operatorname{dia}(D). \tag{2.19}$$

To prove this let $\varepsilon>0$ and choose $x_1, x_2\in D$ such that

$$|x_1-x_2| > \operatorname{dia}(D)-\varepsilon.$$

Since D is uniform, there is a John domain $G\subset D$ with constants $a|x_1-x_2|$, $b|x_1-x_2|$ containing x_1 and x_2 and a John center y_0 of G which by (2.16) satisfies

$$d(y_0, \partial D) \geqq d(y_0, \partial G) \geqq b|x_1-x_2| > b(\operatorname{dia}(D)-\varepsilon).$$

Thus

$$d(x_0, \partial D) \geqq d(y_0, \partial D) > b(\operatorname{dia}(D)-\varepsilon)$$

and letting $\varepsilon\to 0$ we obtain (2.19).

Fix $x\in D$. Since D is uniform, there is a rectifiable curve γ joining x to x_0 as in (2.17) and parametrized by arc length measured from x. Now

$$l(\gamma) \leqq 2a|x-x_0| \leqq 2a \operatorname{dia}(D). \tag{2.20}$$

Let $0\leqq s\leqq l(\gamma)$. If $l(\gamma)-s<r/2$, then by (2.19)

$$d(\gamma(s), \partial D) \geqq r/2 \geqq \frac{b}{2}\operatorname{dia}(D) \geqq \frac{b \operatorname{dia}(D)}{2}\frac{s}{l(\gamma)} \geqq \frac{b^2 \operatorname{dia}(D)}{2a}\frac{s}{l(\gamma)}. \tag{2.21}$$

Next suppose $l(\gamma)-s \geqq r/2$. Then by (2.17)

$$d(\gamma(s), \partial D) \geqq \frac{b}{a} \min(s, l(\gamma)-s) \geqq \frac{b}{a} \min(s, r/2) \tag{2.22}$$

and hence for $s<r/2$

$$d(\gamma(s), \partial D) \geqq \frac{b}{a} s = \frac{b l(\gamma)}{a} \frac{s}{l(\gamma)} \geqq \frac{b(r/2+s)}{a} \frac{s}{l(\gamma)} \geqq \frac{b^2 \operatorname{dia}(D)}{2a} \frac{s}{l(\gamma)}. \tag{2.23}$$

On the other hand, if $s \geqq r/2$, then (2.22) yields

$$d(\gamma(s), \partial D) \geqq \frac{b}{a} \frac{r}{2} \geqq \frac{b}{a} \frac{r}{2} \frac{s}{l(\gamma)} \geqq \frac{b^2 \operatorname{dia}(D)}{2a} \frac{s}{l(\gamma)}.$$

This together with the inequalities (2.20), (2.21) and (2.23) yields the desired result.

2.24. Theorem. *Uniform domains are* Lip_α*-extension domains for all* $0<\alpha \leqq 1$.

Proof. By Theorem 2.2 it suffices to show that a uniform domain D satisfies the condition (2.3). Let $x_1, x_2 \in D$ and let γ be a curve in D joining x_1 to x_2 and satisfying (2.17). Then

$$\begin{aligned}
\int_\gamma d(x, \partial D)^{\alpha-1}\, ds &\leqq \left(\frac{a}{b}\right)^{1-\alpha} \int_0^{l(\gamma)} (\min(s, l(\gamma)-s))^{\alpha-1}\, ds \\
&= 2\left(\frac{a}{b}\right)^{1-\alpha} \int_0^{l(\gamma)/2} s^{\alpha-1}\, ds = \frac{(2a/b)^{1-\alpha}}{\alpha} l(\gamma)^\alpha \\
&\leqq \frac{(2a/b)^{1-\alpha}}{\alpha} (2a)^\alpha |x_1-x_2|^\alpha = \frac{2ab^{\alpha-1}}{\alpha} |x_1-x_2|^\alpha
\end{aligned}$$

and thus D satisfies the condition (2.3) with $M=2a\alpha^{-1}b^{\alpha-1}$. The proof is complete.

We can generate a large class of Lip_α-extension domains by combining Theorem 2.24 with the following observation.

2.25. Theorem. *Suppose that* M, m *and* k *are fixed constants and* D *is the union of a family* $\mathcal{D}$ *of* Lip_α*-extension domains* G *which satisfy* (2.3) *with the same constant* M. *Suppose also that for all* $x_1, x_2 \in D$ *there exist domains* $G_1, \ldots, G_j \in \mathcal{D}$ *with* $j \leqq k$ *and points* $y_1, \ldots, y_{j+1}$ *such that* $x_1=y_1$, $x_2=y_{j+1}$ *and*

$$y_i, y_{i+1} \in G_i, \quad |y_i-y_{i+1}| \leqq m|x_1-x_2|$$

for $i=1, \ldots, j$. *Then* D *is a* Lip_α*-extension domain.*

Proof. Given $x_1, x_2 \in D$ and G_i, y_i as above, we can choose a curve γ_i joining y_i to y_{i+1} in $G_i \subset D$ such that

$$\int_{\gamma_i} d(x, \partial D)^{\alpha-1}\, ds \leqq \int_{\gamma_i} d(x, \partial G_i)^{\alpha-1}\, ds \leqq M|y_i-y_{i+1}|^\alpha.$$

Then $\gamma=\gamma_1+\ldots+\gamma_j$ is a curve joining x_1 to x_2 in D,

$$\int_\gamma d(x, \partial D)^{\alpha-1}\, ds \leqq \sum_{i=1}^{j} M|y_i-y_{i+1}|^\alpha \leqq Mm^\alpha k|x_1-x_2|^\alpha$$

and the desired conclusion follows from Theorem 2.2.

2.26. Examples. (a) Let D be the domain $\{x\in R^n: 0<x_n<1\}$ between two planes or the tube domain $\{x\in R^n: |x_i|<1,\ i=1, \ldots, n-1\}$. Then it is easy to see that D satisfies the condition (2.3) only for $\alpha=1$ and hence D is a Lip_α-extension domain only for $\alpha=1$.

(b) A domain in $\bar{R}^n$ is said to be a K-quasiball if it is the image of a ball under a K-quasiconformal mapping $f: \bar{R}^n \to \bar{R}^n$. It follows from [MS, Theorem 2.15] that every quasiball in R^n is a uniform domain. (See also [GO, Corollary 4].) Hence a quasiball is also a Lip_α-extension domain for $0<\alpha\leqq 1$.

(c) There are Lip_α-extension domains which are not uniform. To construct such a Jordan domain in R^2 let G_0 denote the open triangle bounded by the lines

$$x = 0, \quad y = 0, \quad x-y = 1$$

and for $j=1, 2, \ldots$ let G_j be the open triangle bounded by

$$x = 2^{-2j}, \quad y = 2^{-2j}-2^{-4j}, \quad x+y = 2^{-2j}-2^{-4j}.$$

By Theorem 2.24, each triangle G_j satisfies (2.3) with a constant M independent of j and it is not difficult to show that $D=\bigcup_{j=0}^{\infty} G_j$ is a Jordan domain which satisfies the hypotheses of Theorem 2.25 with $m<16$ and $k=3$. Thus D is a Lip_α-extension domain for $0<\alpha\leqq 1$. However, D is not a quasidisk and hence not uniform. A similar construction in R^n, $n\geqq 3$, yields a Jordan domain D which is a Lip_α-extension domain for $0<\alpha\leqq 1$ but is not a quasiball. The next theorem clarifies the plane case.

2.27. Theorem. *Suppose that* $D\subset R^2$ *is a simply connected* Lip_α*-extension domain with* $\infty\in\partial D$. *If* $D'=\bar{R}^2\setminus\bar{D}$ *is a* Lip_β*-extension domain, then* ∂D *is a quasicircle in* $\bar{R}^2$.

Proof. By Theorem 2.2 and Theorem 2.15 there is a constant c such that the conclusion of Theorem 2.15 holds for D and for D'. To prove the theorem it suffices to show, see [G, Lemma 4], that D is c'-locally connected for some $c'>c$. Theorem 2.15 takes care of the first condition for the c'-local connectivity and we can now argue exactly as in the proof of Theorem 4.2 in [GM] to obtain the second.

2.28. Remark. If J is a quasicircle in $\bar{R}^2$ which contains ∞, then its residual domains D and D' are quasidisks and hence uniform domains, cf. Example 2.26 (b). By Theorem 2.24, D and D' are Lip_α-extension domains for all $0<\alpha\leqq 1$. This together with Theorem 2.27 gives a characterization of quasicircles: *A Jordan curve* J *in* $\bar{R}^2$ *which contains* ∞ *is a quasicircle if and only if its residual domains are* Lip_α*-extension domains for some* α, $0<\alpha\leqq 1$.

3. Quasiconformal mappings and Lip_α-classes

It is well-known that a K-quasiconformal mapping $f\colon D\to R^n$ is $K^{1/(1-n)}$-Lipschitzian on compact subsets of D. However, this does not imply that $f\in \mathrm{loc\,Lip}_\alpha(D)$ for some $0<\alpha\leqq K^{1/(1-n)}$. On the other hand, in Lip_α-extension domains D, $f\in \mathrm{loc\,Lip}_\alpha(D)$ implies $f\in \mathrm{Lip}_\alpha(D)$. In this section we shall give two necessary and sufficient conditions for a quasiconformal mapping f of D to be in $\mathrm{loc\,Lip}_\alpha(D)$. The above result can then be used to conclude $f\in \mathrm{Lip}_\alpha(D)$ and, in particular, we employ this method to study quasiconformal mappings on uniform and John domains.

As we noted in Section 1, for f analytic in a plane domain D the condition

$$|f'(z)| \leqq m d(z, \partial D)^{\alpha-1} \tag{3.1}$$

can be used on some domains D, e.g. on uniform domains, to conclude $f\in \mathrm{Lip}_\alpha(D)$; see [GM]. In this section we have replaced analytic functions and the property (3.1) by quasiconformal mappings and by $f\in \mathrm{loc\,Lip}_\alpha(D)$, respectively. Hence this section is, in a sense, a quasiconformal counterpart of [GM].

3.2. *Quasiconformal maps and* $\mathrm{loc\,Lip}_\alpha$. If $D\subset R^n$ is a domain and $f\colon D\to R^p$, then the boundary cluster set for f is given by

$$C(f,\partial D) = \bigcap \overline{f(U\cap D)}$$

where the intersection is taken over all neighborhoods U of ∂D.

3.3. Lemma. *Suppose that* $f\colon D\to R^p$ *is in* $\mathrm{loc\,Lip}_\alpha(D)$. *Then*

$$d\big(f(x),\, C(f, \partial D)\big) \leqq m d(x, \partial D)^\alpha$$

for $x\in D$ *where* $m=\|f\|_\alpha^{\mathrm{loc}}$.

Proof. Fix $x_1\in D$, choose $x_2\in\partial D$ such that

$$|x_1-x_2| = d(x_1, \partial D) = d$$

and let $U=B^n(x_1, d)\subset D$. Now

$$|f(y_1)-f(y_2)| \leqq m|y_1-y_2|^\alpha$$

for $y_1, y_2\in U$ where $m=\|f\|_\alpha^{\mathrm{loc}}$. Thus f has a continuous extension to $\overline{U}$, $f(x_2)\in C(f, \partial D)$ and

$$\begin{aligned} d\big(f(x_1),\, C(f, \partial D)\big) &\leqq |f(x_1)-f(x_2)| \leqq m|x_1-x_2|^\alpha \\ &= m d(x_1, \partial D)^\alpha. \end{aligned}$$

The next theorem gives a condition in terms of the distances $d(x, \partial D)$ and $d(f(x), \partial D')$ for a quasiconformal mapping $f\colon D\to D'$ to be in $\mathrm{loc\,Lip}_\alpha(D)$.

3.4. Theorem. *Suppose that f is a K-quasiconformal mapping of D onto D' and that $0<\alpha\leqq K^{1/(1-n)}$. Then $f\in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ if and only if there exists a constant $M<\infty$ with*

$$d(f(x), \partial D') \leqq Md(x, \partial D)^\alpha \tag{3.5}$$

for all $x\in D$.

Proof. First suppose that f belongs to $\operatorname{loc}\operatorname{Lip}_\alpha(D)$. Since f is a homeomorphism

$$C(f, \partial D) \subset \overline{f(D)}\setminus f(D) = \bar{D}'\setminus D' = \partial D'$$

and hence by Lemma 3.3

$$d(f(x), \partial D') \leqq d(f(x), C(f, \partial D)) \leqq Md(x, \partial D)^\alpha$$

where $M=\|f\|_\alpha^{\mathrm{loc}}$.

Next suppose that (3.5) holds. Since $0<\alpha\leqq K^{1/(1-n)}$, [GO, Lemma 2] implies that

$$\frac{|f(x_1)-f(x_2)|}{d(f(x_1), \partial D')} \leqq 2\lambda_n^2\left(\frac{|x_1-x_2|}{d(x_1, \partial D)}\right)^\alpha$$

whenever $x_1\in D$ and $|x_1-x_2|\leqq c\, d(x_1, \partial D)$. Here $c=(2\lambda_n^2)^{-1/\alpha}$ and λ_n depends only on n. Thus (3.5) yields

$$\begin{aligned}|f(x_1)-f(x_2)| &\leqq c^{-\alpha}d(f(x_1), \partial D')d(x_1, \partial D)^{-\alpha}|x_1-x_2|^\alpha\\ &\leqq c^{-\alpha}M|x_1-x_2|^\alpha\end{aligned}$$

for $x_1\in D$ and $|x_1-x_2|\leqq c\, d(x_1, \partial D)$. By Theorem 2.13, $f\in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ as desired.

3.6. *Quasihyperbolic boundary condition.* A domain D in R^n satisfies a *quasi-hyperbolic boundary condition* if for some $x_0\in D$ there exist constants a and b_0 such that

$$k_D(x, x_0) \leqq a\log\frac{1}{d(x, \partial D)}+b_0 \tag{3.7}$$

for all $x\in D$. (Cf. [BP, Theorem 1].) Here k_D denotes the quasihyperbolic metric in D, i.e.

$$k_D(x_1, x_2) = \inf_\gamma\int_\gamma d(x, \partial D)^{-1}\,ds, \tag{3.8}$$

where the infimum is taken over all rectifiable curves joining x_1 and x_2 in D. For the basic properties of this metric see [GP] and [GO]. Note that if D satisfies (3.7) for some point $x_0\in D$, then for any point $x_1\in D$

$$k_D(x, x_1) \leqq k_D(x, x_0)+k_D(x_0,x_1) \leqq a\log\frac{1}{d(x, \partial D)}+b_1$$

for all $x\in D$ where $b_1=b_0+k_D(x_0, x_1)$. Thus whether or not D satisfies (3.7) is independent of the choice of the point x_0.

We first study some properties of domains satisfying a quasihyperbolic boundary condition and relate these domains to the domains mentioned in Section 2.

3.9. Lemma. *If D satisfies* (3.7), *then D is bounded with*

$$\operatorname{dia}(D) \leqq 2ae^{b_0/a}.$$

Proof. Let $x_1 \in D$ and let γ be a quasihyperbolic geodesic joining x_0 and x_1 in D, cf. [GO, Lemma 1]. Suppose that γ is parametrized by arc length measured from x_0. For each s write $f(s) = k_D(\gamma(s), x_0)$. Then

$$f(s) = \int_0^s d(\gamma(t), \partial D)^{-1}\, dt$$

and thus $f'(s) = d(\gamma(s), \partial D)^{-1}$ for each s. Now by (3.7)

$$f(s) \leqq a \log \frac{1}{d(\gamma(s), \partial D)} + b_0 = a \log f'(s) + b_0$$

and hence

$$\log f'(s) \geqq \frac{1}{a}(f(s) - b_0).$$

This yields a differential inequality

$$f'(s)e^{\frac{1}{a}(b_0 - f(s))} \geqq 1 \tag{3.10}$$

for $0 < s < l$ where l is the length of γ. The inequality (3.10) can be written in the form

$$1 \leqq -a\frac{d}{ds}\left(e^{\frac{1}{a}(b_0 - f(s))}\right)$$

and by integration we obtain

$$l = \int_0^l ds \leqq -a\int_0^l \frac{d}{ds}\left(e^{\frac{1}{a}(b_0 - f(s))}\right) ds$$

$$= a\left(e^{\frac{1}{a}(b_0 - f(0))} - e^{\frac{1}{a}(b_0 - f(l))}\right) \leqq ae^{b_0/a}$$

since $f(0) = 0$. This holds for each $x_1 \in D$; hence

$$\operatorname{dia}(D) \leqq 2ae^{b_0/a}$$

as desired.

3.11. Lemma. *If D is a John domain, then D satisfies a quasihyperbolic boundary condition.*

Proof. Let D be a John domain with constants a, b and let x_0 be a John center for D. Next given $x_1 \in D$ let γ denote a rectifiable curve joining x_1 to x_0 in D which satisfies (2.16). We consider two cases.

Suppose first that

$$d(x_1, \partial D) \geqq \frac{a+b}{a} l(\gamma). \tag{3.12}$$

Then by (3.12)

$$d(x, \partial D) \geqq d(x_1, \partial D) - |x_1 - x| \geqq \frac{b}{a} l(\gamma)$$

for all $x \in \gamma$ and hence

$$k_D(x_1, x_0) \leqq \int_\gamma \frac{ds}{d(x, \partial D)} \leqq \frac{a}{b}.$$

On the other hand, $d(x, \partial D) \leqq a$ for all $x \in D$ by (2.16) and thus

$$k_D(x_1, x_0) \leqq \frac{a}{b} + \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} \leqq \frac{a}{b} \log \frac{1}{d(x_1, \partial D)} + \frac{a}{b} (\log a + 1).$$

Suppose next that (3.12) does not hold. Then we can choose a proper subarc γ_1 of γ with endpoints x_1 and x_2 so that

$$d(x_1, \partial D) = \frac{a+b}{a} l(\gamma_1).$$

Then

$$k_D(x_1, x_2) \leqq \frac{a}{b}$$

by what was proved above. If $x \in \gamma \setminus \gamma_1 = \gamma_2$, then by (2.16)

$$d(x, \partial D) \geqq b \frac{s}{l(\gamma)} \geqq \frac{b}{a} s$$

where $x = \gamma(s)$. Thus

$$k_D(x_2, x_0) \leqq \int_{\gamma_2} \frac{ds}{d(x, \partial D)} \leqq \frac{a}{b} \int_{l(\gamma_1)}^{l(\gamma)} \frac{ds}{s}$$

$$= \frac{a}{b} \log \left(\frac{l(\gamma)}{d(x_1, \partial D)} \frac{a+b}{a} \right) \leqq \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} + 1$$

and we obtain

$$k_D(x_1, x_0) \leqq k_D(x_1, x_2) + k_D(x_2, x_0)$$

$$\leqq \frac{a}{b} \log \frac{a}{d(x_1, \partial D)} + 1 + \frac{a}{b}.$$

Combining these two cases yields

$$k_D(x_1, x_0) \leqq a' \log \frac{1}{d(x_1, \partial D)} + b_0'$$

with

$$a' = \frac{a}{b}, \quad b_0' = \frac{a}{b} (\log a + 1) + 1.$$

3.13. Remark. The converse of Lemma 3.11 is false, i.e. there are domains which satisfy a quasihyperbolic boundary condition but which are not John domains. In fact, the plane domain D constructed in Example 2.26 (c) has this property. Since D is not a John domain, it remains to show that D satisfies a quasihyperbolic boundary condition. To see this let z_j denote the center of the maximum disk inscribed in G_j for $j=0, 1, \ldots$ and let w_j denote the midpoint of the line segment $\partial G_0 \cap G_j$ for $j=1, 2, \ldots$. Fix $z \in D \cap \bar{G}_j$, let γ be the segment joining z_j to z and let s denote arc length measured along γ from z. Then by elementary geometry

$$d(\gamma(s), \partial D) \geqq s/3$$

for $0 \leqq s \leqq |z-z_j|$ and

$$d(\gamma(s), \partial D) \geqq d(z, \partial D)/2$$

for $0 \leqq s \leqq d(z, \partial D)/2$; these estimates imply that

$$k_D(z, z_j) \leqq 3 \log \frac{1}{d(z, \partial D)} + 3. \tag{3.14}$$

Thus (3.7) holds for points $z \in \bar{G}_0$. If $z \in G_j \setminus \bar{G}_0$, then (3.14) and the triangle inequality yield

$$k_D(z, z_0) \leqq k_D(z, z_j) + k_D(z_j, w_j) + k_D(w_j, z_0)$$

$$\leqq 3 \log \frac{1}{d(z, \partial D)} + 6 \log \frac{1}{d(w_j, \partial D)} + 9 \leqq 15 \log \frac{1}{d(z, \partial D)} + 9$$

since

$$d(z, \partial D) \leqq \tfrac{1}{2} 2^{-2j} < \left(\tfrac{1}{2} 2^{-4j}\right)^{1/2} = d(w_j, \partial D)^{1/2}.$$

Hence D satisfies a quasihyperbolic boundary condition.

3.15. Remark. By Lemma 2.18 a bounded uniform domain is a John domain. Hence Lemma 3.11 holds for bounded uniform domains as well.

3.16. *Quasihyperbolic boundary condition and quasiconformal mappings.* For a quasiconformal mapping $f\colon D \to D'$ we shall relate the condition $f \in \operatorname{loc} \operatorname{Lip}_\alpha(D)$ with quasihyperbolic boundary conditions on D and D'.

3.17. Theorem. *Suppose that f is a K-quasiconformal mapping of D onto D'. If D' satisfies a quasihyperbolic boundary condition, then $f \in \operatorname{loc} \operatorname{Lip}_\alpha(D)$ for some $0 < \alpha \leqq K^{1/(1-n)}$.*

Proof. By Theorem 3.4 it suffices to show that there exist constants α and M such that $0 < \alpha \leqq K^{1/(1-n)}$ and

$$d(f(x), \partial D') \leqq M d(x, \partial D)^\alpha \tag{3.18}$$

for all $x \in D$. Fix $x_0 \in D$ such that D' satisfies the quasihyperbolic boundary condition (3.7) at $f(x_0)$. Now [GP, Lemma 2.1] and [GO, Theorem 3] yield

$$\log \frac{d(x_0, \partial D)}{d(x, \partial D)} \leqq k_D(x, x_0) \leqq c(k_{D'}(f(x), f(x_0))+1)$$

$$\leqq c\left(a \log \frac{1}{d(f(x), \partial D')} + b_0 + 1\right)$$

$$= ac\left(\log \frac{1}{d(f(x), \partial D')} + \log e^{(b_0+1)/a}\right),$$

where $c=c(n, K)$. Thus

$$\frac{d(x_0, \partial D)}{d(x, \partial D)} \leqq \left(e^{(b_0+1)/a} \frac{1}{d(f(x), \partial D')}\right)^{ac}$$

and hence

$$d(f(x), \partial D') \leqq N d(x, \partial D)^{\beta},$$

where

$$N = e^{(b_0+1)/a} d(x_0, \partial D)^{-\beta}, \quad \beta = 1/(ac).$$

Set $\alpha = \min(\beta, K^{1/(1-n)})$ and $M = \max(N, \mathrm{dia}(D'))$. Observe that D' is bounded by Lemma 3.9. If now $d(x, \partial D) \leqq 1$, then by the previous inequality

$$d(f(x), \partial D') \leqq N d(x, \partial D)^{\beta} \leqq M d(x, \partial D)^{\alpha}$$

while if $d(x, \partial D) \geqq 1$, then

$$d(f(x), \partial D') \leqq \mathrm{dia}\,(D') \leqq M d(x, \partial D)^{\alpha}.$$

Thus (3.18) holds as desired.

Lemma 3.11 gives the following corollary of Theorem 3.17.

3.19. Corollary. *Suppose that f is a K-quasiconformal mapping of D onto a John domain D'. Then $f \in \mathrm{loc\,Lip}_{\alpha}(D)$ for some $0 < \alpha \leqq K^{1/(1-n)}$.*

Next we shall study the converse of Theorem 3.17.

3.20. Lemma. *Suppose that f is a K-quasiconformal mapping of D onto D' and that D satisfies a quasihyperbolic boundary condition. If $f \in \mathrm{loc\,Lip}_{\alpha}(D)$ for some $0 < \alpha \leqq K^{1/(1-n)}$, then D' satisfies a quasihyperbolic boundary condition.*

Proof. By Theorem 3.4 there is M such that

$$d(f(x), \partial D') \leqq M d(x, \partial D)^{\alpha} \tag{3.21}$$

for all $x \in D$. Fix $x_0 \in D$ so that D satisfies (3.7) at x_0. By [GO, Theorem 3]

$$k_{D'}(f(x), f(x_0)) \leqq c(k_D(x, x_0)+1) \tag{3.22}$$

where $c=c(n, K)$ and the inequality (3.21) yields

$$\frac{1}{d(x, \partial D)} \leqq \left(\frac{M}{d(f(x), \partial D')}\right)^{1/\alpha}; \tag{3.23}$$

hence by (3.22), (3.7) and (3.23)

$$\begin{aligned} k_{D'}(f(x), f(x_0)) &\leqq c\left(a \log \frac{1}{d(x, \partial D)} + b_0 + 1\right) \\ &\leqq a' \log \frac{1}{d(f(x), \partial D')} + b_0' \end{aligned}$$

where $a'=a\,c/\alpha$ and $b_0'=a' \log M + c(b_0+1)$. Thus D' satisfies a quasihyperbolic boundary condition at $f(x_0)$.

Theorem 3.17 and Lemma 3.20 give a characterization for $f \in \operatorname{loc} \operatorname{Lip}_\alpha(D)$ in terms of D and D'.

3.24. Theorem. *Suppose that f is a K-quasiconformal mapping of D onto D' and that D satisfies a quasihyperbolic boundary condition. Then $f \in \operatorname{loc} \operatorname{Lip}_\alpha(D)$ for some $0<\alpha \leqq K^{1/(1-n)}$ if and only if D' satisfies a quasihyperbolic boundary condition.*

3.25. *Quasiconformal mappings and* $\operatorname{Lip}_\alpha$. For a quasiconformal mapping $f\colon D \to D'$ we combine the results above with Section 2 to conclude that $f \in \operatorname{Lip}_\alpha(D)$. The first theorem follows directly from Theorem 3.17.

3.26. Theorem. *Suppose that f is a K-quasiconformal mapping of D onto D' and that D' satisfies a quasihyperbolic boundary condition. Then there is an α, $0<\alpha \leqq K^{1/(1-n)}$, such that $f \in \operatorname{Lip}_\alpha(D)$ whenever D is a $\operatorname{Lip}_\alpha$-extension domain.*

3.27. Corollary. *Suppose that f is a K-quasiconformal mapping of D onto D', that D satisfies* (2.3) *for all $0<\alpha \leqq K^{1/(1-n)}$ and that D' satisfies a quasihyperbolic boundary condition. Then $f \in \operatorname{Lip}_\alpha(D)$ for some $0<\alpha \leqq K^{1/(1-n)}$.*

Proof. The corollary follows from Theorem 3.26 and Theorem 2.2.

Theorem 2.24, Lemma 3.11 and Corollary 3.27 yield

3.28. Theorem. *Suppose that f is a K-quasiconformal mapping of a uniform domain D onto a John domain D'. Then $f \in \operatorname{Lip}_\alpha(D)$ for some $0<\alpha \leqq K^{1/(1-n)}$.*

3.29. Remark. An inspection of the proofs leading to Theorem 3.28 shows that α depends only on n, K and the constants for D' and that $\|f\|_\alpha$ depends only on n, K, the constants for D and D' and $d(f^{-1}(y_0), \partial D)$ where y_0 is a John center of D'.

Finally we obtain the following far reaching extension of a well known theorem due to A. Mori [LV, Theorem II.3.2] from Theorem 3.28 and Lemma 2.18.

3.30. Corollary. *Suppose that D and D' are bounded uniform domains in R^n and that f is a K-quasiconformal mapping of D onto D'. Then $f \in \mathrm{Lip}_\alpha(D)$ and $f^{-1} \in \mathrm{Lip}_\alpha(D')$ for some $0 < \alpha \leqq K^{1/(1-n)}$, where α depends only on n, K, D and D'.*

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