ON THE MODULUS OF CONTINUITY OF ANALYTIC FUNCTIONS

A. HINKKANEN

1. Introduction and results

We shall assume throughout the paper that $G$ is an open set in the plane such that $\partial G$, the boundary of $G$, contains at least two (finite) points and that $f$ is a function continuous in $\bar{G}$, the closure of $G$, and analytic in $G$.

We shall call a non-decreasing continuous function $\mu: [0, \infty) \rightarrow [0, \infty)$ a majorant. The function $\mu$ must always satisfy some extra conditions, but these vary.

We consider the following problem. Suppose that

$$|f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$$

for all $z_1, z_2 \in \partial G$. When does (1.1) remain valid for all $z_1, z_2 \in \bar{G}$? We prove the following result.

Theorem 1. Let $G$ and $f$ be as above, and let $\mu$ be a majorant such that $\log \mu(e^t)$ is a concave function of $t$ for real $t$ and that

$$B = \lim_{t \to 0^+} \frac{\log \mu(t)}{t} \leq 1.$$ (1.2)

We set

$$A = \lim_{t \to \infty} \frac{\log \mu(t)}{t} \leq B$$ (1.3)

and assume that

$$f(z) = o(|z|)$$ (1.4)

if $A < 1$, and that

$$f(z) = o(|z|^2)$$ (1.5)

if $A = 1$, as $z \to \infty$ in any unbounded component of $G$. If (1.1) holds for all $z_1, z_2 \in \partial G$, then (1.1) remains valid for all $z_1, z_2 \in \bar{G}$. If (1.1) holds for a fixed $z_1 \in \partial G$ and all $z_2 \in \partial G$, then (1.1) remains valid for this $z_1$ and all $z_2 \in \bar{G}$.

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If instead of (1.4) or (1.5) we only have

$$f(z) = o(|z|^q)$$

for some \( q > 0 \), and if the conclusion of the theorem fails, then \( G \) contains a neighbourhood of infinity, and \( f \) has a pole at infinity.

We may allow the case \( \mu \equiv 0 \). For if Theorem 1 has been proved in all other cases, we may apply it with \( \mu \equiv \varepsilon \) for an arbitrary positive \( \varepsilon \) to deduce that it remains valid if \( \mu \equiv 0 \). From now on we assume that \( \mu \) does not vanish identically. Hence \( A \equiv 0 \).

The condition (1.2) is natural, for \( \mu(t) = t^{B+o(1)} \) as \( t \to 0 \), and if \( B > 1 \), then (1.1) cannot be true for all \( z_1, z_2 \in G \) unless \( f \) is constant.

Tamrazov [2, Theorem 9.3, p. 167] showed that if \( \mu \) is a majorant satisfying a growth condition, e.g.

$$\mu(2t) \equiv 2\mu(t), \quad t > 0,$$

and if the conclusion of Theorem 1 for a fixed \( z_1 \in \partial G \) is correct for all bounded Jordan domains \( G \), then \( \log \mu(e^t) \) is concave. Hence this condition is necessary. The reason behind it is that we want the functions \(-\log \mu(|z-z_0|)\) to be subharmonic for \( z \in G \), for any \( z_0 \in \partial G \).

Gehring, Hayman and the author [1, Theorem 1] proved Theorem 1 for \( \mu(t) = Mt^A, \quad M > 0, \quad 0 \leq A \leq 1 \). In this case \( \log \mu(e^t) = at + \log M \) is concave, and \( A = B = a \). For \( 0 \leq A < 1 \), our Theorem 1 applies to other functions that \( \mu(t) = Mt^A \). However, as we shall show in Section 3, for \( A = 1 \) we get only the functions \( \mu(t) = Mt \).

Our growth conditions (1.4) and (1.5) and their dependence on \( A \) are the same as [1, (1.2), (1.3)] in the case dealt with in [1]. As remarked in [1, p. 243], the functions \( z \) and \( z^2 \), respectively, with \( G = \{|z| > 1\} \), show that \( o \) cannot be replaced by \( O \) in (1.4) and (1.5).

Results like Theorem 1 were obtained by Tamrazov [2] for special open sets and for majorants \( \mu \) satisfying a growth condition which we take to be (1.7). This is satisfied, for example, by any subadditive \( \mu \). Let \( \operatorname{cap} E \) denote the logarithmic capacity of the compact set \( E \). Then Tamrazov's results [2, Theorems 4.1., 6.1, 9.1] can be summarized in a slightly simplified form as follows. Suppose that \( G \) is bounded or that \( G \) contains a neighborhood of infinity, in which case \( f \) is required to remain analytic at \( \infty \). Hence \( \partial G \) is bounded, and \( f \) is also bounded. If (1.1) holds for \( z_1, z_2 \in \partial G \), and if \( z_1 \in \partial G, \ z_2 \in G \), then

$$|f(z_1) - f(z_2)| \leq 27\mu(|z_1 - z_2|)|z_1 - z_2| (2 \operatorname{cap} E(z_1, z_2))^{-1},$$

where

$$E(z_1, z_2) = \left\{ |z - z_1| \leq \frac{1}{2} |z_1 - z_2| \right\} \setminus G.$$

If, in addition, \( G \) is simply connected, then

$$|f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|), \quad z_1, z_2 \in \bar{G},$$
where $C=108$. If $G$ is suitable, one can use (1.8) together with [2, Lemma 4.1] to deduce that (1.9) holds for some $C$ depending on $G$ but not on $z_1$ and $z_2$. If $\log \mu(e^t)$ is concave, we can take $C=1$ in (1.9) if (1.7) holds and $\partial G$ satisfies certain capacity density conditions. Moreover, if (1.1) holds for a fixed $z_1 \in \partial G$ and all $z_2 \in \partial G$, if $f$ is bounded and if $\log \mu(e^t)$ is concave, then (1.9) holds for this $z_1$ and all $z_2 \in G$ with $C=1$, provided that $\partial G$ is thick enough. In this case $\partial G$ need not be bounded.

Theorem 1 generalizes Tamrazov's results when $\log \mu(e^t)$ is concave, since we need no assumption on the capacity density of $\partial G$.

References to other earlier works related to this subject can be found in [1, p. 244] and in [2, p. 141—143].

2. Lemmas

Let $u$ be a subharmonic function in the plane. We set

$$M(r, u) = \sup \{u(z) | |z| = r\}.$$  

For all functions $u$ that we shall consider, we have

$$M(r, u) = O(\log r)$$  as $r \to \infty$.

To prove Theorem 1, we need two lemmas. The first lemma follows from [1, Theorem 2].

Lemma 1. Suppose that $u$ is subharmonic, non-negative and not constant in the plane, that (2.1) holds as $r \to \infty$, and that $u(z)=0$ for some $z$. Then the limit

$$\beta = \lim_{r \to \infty} M(r, u)/\log r$$

exists and $0<\beta<\infty$. Suppose further that there is a component $D$ of the set $\{z|u(z)>0\}$ such that $u$ is harmonic in $D$ and possesses there a local conjugate $v$, and that for some $\alpha$, $0<\alpha \leq 1$, and some positive $R$, the function

$$F(z) = z^{1-\alpha} \exp (u + iv)$$

remains single-valued in $D \cap \{|z|>R\}$. Then $D$ contains the set $\{|z|>R_0\}$ for some $R_0$, and $F$ has a pole of order $\beta+1-\alpha$ at infinity.

Note that any such component $D$ is unbounded.

Our second lemma is a generalization of Tamrazov's result [2, Lemma 4.1, p. 156].

Lemma 2. Let $G$ be an open set with at least one (finite) boundary point, and suppose that

$$f(z) = o(|z|^\beta)$$
as \( z \to \infty \) in any unbounded component of \( G \). Then for every positive \( t \) we have

\[
\sup \{ |f(z_1) - f(z_2)| : |z_1 - z_2| \leq \tau, \ z_1, z_2 \in G \} = \sup \{ |f(z_1) - f(z_2)| : |z_1 - z_2| \leq \tau, \ z_1 \in \partial G, \ z_2 \in G \}.
\]

Tamrazov proved Lemma 2 with the additional hypothesis that \( \partial G \) is bounded and \( f \) is bounded (cf. Section 1).

To prove Lemma 2, we consider a fixed positive \( t \), denote the left and right hand sides of (2.4) by \( L \) and \( R \), respectively, and note that \( R \equiv L \). To prove that \( L \equiv R \), we may assume that \( L = 0 \), \( R < \infty \), so that \( f \) is not constant.

Pick \( z_1, z_2 \in G \) such that with \( h = z_2 - z_1 \) we have \( |h| \leq t \). It suffices to show that with

\[
g(z) = f(z) - f(z + h)
\]

we have \( |g(z_1)| \leq R \). The function \( g \) is defined and continuous in the closure of the non-empty open set

\[
G_1 = \{ z \in G, z + h \in G \} \subset G
\]

and analytic in \( G_1 \). Further, \( g \) is bounded in any compact subset of \( G_1 \), and \( z_1 \in G_1 \). If \( z \in \partial G_1 \), then \( z \in \partial G \) or \( z + h \in \partial G \), so that \( |g(z)| \leq R \). If \( g \) is bounded in \( G_1 \), it follows from the maximum principle that \( |g(z_1)| \leq R \).

Suppose that \( g \) is unbounded, and let \( D \) be a component of the open set \( \{ z \in G_1 | |g(z)| > R \} \), so that \( D \) is unbounded. We set

\[
u(z) = \log |g(z)| - \log R, \quad z \in D,
\]

\[
u(z) = 0, \quad z \notin D,
\]

and note that \( u \) is subharmonic and non-constant in the plane, and harmonic and positive in \( D \), and possesses a local conjugate in \( D \), namely, \( \arg g(z) \). By (2.3), the condition (2.1) holds, and we may apply Lemma 1 to \( u \) with \( \alpha = 1 \). It follows that \( D \) and thus \( G \) contains a neighborhood of \( \infty \), so that by (2.3), \( f \) remains analytic or has a pole of order one at \( \infty \). But in both cases \( g \) remains bounded at \( \infty \), which is against our assumption. Hence \( L \equiv R \), and Lemma 2 is proved.

3. Proof of Theorem 1

Suppose that the assumptions of Theorem 1 are satisfied and take a fixed \( z_1 \in \partial G \). The function

\[
u_1(z) = \log |f(z) - f(z_1)| - \log \mu(|z - z_1|)
\]

is subharmonic in \( G \) and satisfies by (1.2),

\[
\lim \sup_{z \to \xi} \nu_1(z) \equiv 0
\]
for all $\zeta \in \partial G$, $\zeta \neq z_1$. Further, $u_1$ is bounded above in any compact subset of $G$ not containing $z_1$.

First we show that $u_1$ is bounded above in a neighbourhood of $z_1$ in $G$. Suppose that this is not true, and let $D_1$ be a component of the open set $\{z \mid |z - z_1| < 1, z \in G, u_1(z) > M\}$, where $M = \max \{0, \sup \{u_1(z) \mid z \in \bar{G}, |z - z_1| = 1\}\}$. Hence $z_1 \in \partial D_1$.

We set $\phi(z) = (z - z_1)^{-1}$, $\phi(G) = G_1$, $\phi(D_1) = D_2$, and note that $\phi(z_1) = \infty$, and $0 \notin G_1$. We set

$$u(z) = u_1(\phi^{-1}(z)) - M, \quad z \in D_2,$$

so that $u$ is non-negative, non-constant and subharmonic in the plane and positive in $D_2$. Also,

$$u(z) = \log |f(z_1 + 1/z) - f(z_1)| - \log \mu(1/|z|) - M, \quad z \in D_2.$$

By (1.2),

$$-\log \mu(1/|z|) \equiv (B + o(1)) \log |z| \equiv (1 + o(1)) \log |z|$$

as $z \to \infty$. Hence (2.1) holds. Now we apply Lemma 1 to $u$ to obtain $0 < \beta \leq B \leq 1$. In particular, $B > 0$.

Choose $\varepsilon$ such that $0 < \varepsilon < \beta \leq B$. Consider the function

$$u_2(z) = \log |f(z_1 + 1/z) - f(z_1)| + (B - \varepsilon) \log |z| - M, \quad z \in D_2.$$ 

Since $M(r, u) \sim \beta \log r$, since $-\log \mu(1/|z|) \sim B \log r$, $r = |z|$, and since

$$u(z) - u_2(z) = -\log \mu(1/|z|) - (B - \varepsilon) \log |z|, \quad z \in D_2,$$

we have $M(r, u_2) \sim (\beta - \varepsilon) \log r \to \infty$ as $r \to \infty$. Further,

$$(B - \varepsilon) \log |z| < -\log \mu(1/|z|)$$

when $|z|$ is large enough, so that $u_2(z) < 0$ if $z \in \partial D_2$ and $|z| > R$. We have $u_2(z) \leq M_2$ for $|z| = R$, for some positive $M_2$, so that

$$\limsup_{z \to \zeta} u_2(z) - M_2 \equiv 0$$

for every finite boundary point $\zeta$ of $D_3 = D_2 \cap \{|z| > R\}$. Since $u_2$ is unbounded in $D_3$, we can find an unbounded component $D_4$ of the set $\{z \in D_3 \mid u_2(z) > M_3\}$. Hence the function

$$u_3(z) = u_2(z) - M_2, \quad z \in D_4,$$

$$u_3(z) = 0, \quad z \notin D_4,$$

is non-negative, non-constant and subharmonic in the plane. In $D_4$, $u_3$ is positive and harmonic, and possesses there a local conjugate $v$ such that with $\alpha = \beta - \varepsilon$, the function

$$F(z) = z^{1-\alpha} \exp (u_3 + iv)$$

is single-valued in $D_4$. Hence we can apply Lemma 1 to $u_3$ to deduce that $D_4$ contains
a neighbourhood of infinity. But since $D_4 \subset G_4$, it follows that $G$ contains a punctured neighbourhood of $z_1$, so that $z_1$ is a regular point of $f$ and so there exists a positive integer $k$ and $a_k$ unequal to zero such that

$$f(\xi) - f(z_1) \sim a_k (\xi - z_1)^k$$

as $\xi \to z_1$,

$$f(z_1 + 1/z) - f(z_1) \sim a_k z^{-k}$$

as $z \to \infty$.

This together with (1.2) and (3.2) implies that

$$u(z) = -k \log |z| - \log \mu(1/|z|) + O(1) \equiv o(\log |z|)$$

as $z \to \infty$, which contradicts the result $\beta > 0$. So in any case $u_1$ is bounded above in a neighbourhood of $z_1$ in $G$.

We want to show that $u_1(z) \equiv 0$ in $G$, since in view of Lemma 2 this clearly completes the proof of Theorem 1. If $u_1$ is bounded above in $G$, this follows from the maximum principle. Suppose then that $u_1$ is not bounded above in $G$, and let $D$ be a component of $\{z|u_1(z) > 0\}$. Since $u_1$ is bounded above in any compact subset of $G$, as we proved above, the set $D$ must be unbounded.

We set $u(z) = u_1(z)$ in $D$, $u(z) = 0$ outside $D$. To make $u$ subharmonic, we may have to redefine $u$ at $z_1$, cf. [1, p. 248]. Then we apply Lemma 1 to $u$. We assume that (1.4) or (1.5) holds. Without loss of generality we may assume that $z_1 = 0$. Thus we obtain

$$\sup_{|z| = r, z \in D} \log |f(z) - f(0)| \sim (\beta + A) \log r$$

as $r \to \infty$. Suppose now that $A < 1$. We choose $\varepsilon$ such that $0 < \varepsilon < \beta$ and $A + \varepsilon < 1$, and $R$ such that

$$\log \mu(|z|) < (A + \varepsilon) \log |z|, \ |z| \geq R,$$

and set

$$u_2(z) = \log |f(z) - f(0)| - (A + \varepsilon) \log |z| - M_3, \ z \in D,$$

where $M_3 \geq 0$ and $M_3$ is so large that $u_2(z) \equiv 0$ if $z \in D$ and $|z| = R$. Hence $u_2$ has negative boundary values in $D \cap \{|z| = R\}$, but $u_2$ is unbounded above in $D \cap \{|z| > R\}$. Therefore we can find an unbounded component $D_1$ of the set $\{z \in D | z > R, u_2(z) > 0\}$, and we define $u_3(z) = u_2(z)$ in $D_1$, $u_3(z) = 0$ outside $D_1$.

The function $u_3$ is subharmonic in the plane, harmonic in $D_1$, and possesses there a local conjugate. We may apply Lemma 1 to $u_3$ with $\varepsilon = 1 - A - \varepsilon$ to deduce that $D_1$ and so $G$ contains a neighbourhood of $\infty$. Hence by (1.4), $f$ remains bounded at $\infty$, so that $u_1$ is bounded above, which is contrary to our assumption. This proves Theorem 1 if $A < 1$.

Suppose that $A = 1$. We shall soon show that then $\mu(t) = Mt, \ M > 0$. Since (1.5) holds, Theorem 1 follows now from [1, Theorem 1].

If $A = 1$, then $B = 1$. Since

$$\eta(t) = \log \mu(e^t) - \log \mu(1)$$
is concave with $\eta(0)=0$, we have
\[
0 = 2\eta(0) \equiv \eta(t) + \eta(-t), \quad t > 0,
\]
\[
\eta(t)/t \leq \eta(-t)/(-t), \quad t > 0.
\]
Further $\eta(t)/t$ decreases to 1 and $\eta(-t)/(-t)$ increases to 1 as $t$ increases from 0 to $\infty$. Hence $\eta(t) \equiv t$, $\mu(t) \equiv t^\mu(1)$. This proves Theorem 1 if $A=1$.

Suppose finally that (1.6) holds for some $q>0$ but that the conclusion of Theorem 1 fails. Suppose that (1.1) fails for some $z_1 \in \partial G$, $z_2 \in G$. We can still deduce that $u_1$, given by (3.1), is bounded above in a neighbourhood of $z_1$ in $\bar{G}$ and that (2.1) is satisfied with $u=u_1$. Since $u_1$ must be unbounded above in $G$, we obtain as before from Lemma 1 that $G$ contains a neighbourhood of infinity. Now (1.6) implies that $f$ has a pole at infinity.

If (1.1) fails for some $z_1, z_2 \in G$ but not for any $z_1 \in \partial G$, $z_2 \in G$, then we take these $z_1, z_2 \in G$, set $h=z_2-z_1$ and $g(z)=f(z)-f(z+h)$. If $R$ is defined as in the proof of Lemma 2 for $t=|h|$, then the function $u$ given by (2.6) must be unbounded in $G_1$, given by (2.5). Since $u$ satisfies (2.1), we deduce from Lemma 1 that $G$ contains a neighbourhood of infinity. Now (1.6) implies that $f$ has a pole at infinity. Theorem 1 is proved.

Remark. After this paper had been written, I was informed that there is a recent preprint of Tamrazov containing results similar to those in this paper.

References


Imperial College
Department of Mathematics
London SW7 2BZ
England

Current address:
University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48109
USA

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