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ON THE MODULUS OF CONTINUITY OF ANALYTIC FUNCTIONS

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1. Introduction and results

We shall assume throughout the paper that G is an open set in the plane such that ∂G, the boundary of G, contains at least two (finite) points and that f is a function continuous in $\bar{G}$, the closure of G, and analytic in G.

We shall call a non-decreasing continuous function $\mu\colon [0, \infty)\to[0, \infty)$ a *majorant*. The function μ must always satisfy some extra conditions, but these vary.

We consider the following problem. Suppose that

$$|f(z_1)-f(z_2)| \leqq \mu(|z_1-z_2|) \tag{1.1}$$

for all $z_1, z_2\in\partial G$. When does (1.1) remain valid for all $z_1, z_2\in\bar{G}$? We prove the following result.

Theorem 1. *Let G and f be as above, and let μ be a majorant such that $\log \mu(e^t)$ is a concave function of t for real t and that*

$$B = \lim_{t\to 0+} \frac{\log \mu(t)}{\log t} \leqq 1. \tag{1.2}$$

We set

$$A = \lim_{t\to \infty} \frac{\log \mu(t)}{\log t} \leqq B \tag{1.3}$$

and assume that

$$f(z) = o(|z|) \tag{1.4}$$

if $A<1$, and that

$$f(z) = o(|z|^2) \tag{1.5}$$

if $A=1$, as $z\to\infty$ in any unbounded component of G. If (1.1) *holds for all $z_1, z_2\in\partial G$, then* (1.1) *remains valid for all $z_1, z_2\in\bar{G}$. If* (1.1) *holds for a fixed $z_1\in\partial G$ and all $z_2\in\partial G$, then* (1.1) *remains valid for this z_1 and all $z_2\in\bar{G}$.*

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If instead of (1.4) *or* (1.5) *we only have*

$$f(z) = o(|z|^q) \tag{1.6}$$

for some $q>0$, *and if the conclusion of the theorem fails, then* G *contains a neighbourhood of infinity, and* f *has a pole at infinity.*

We may allow the case $\mu\equiv 0$. For if Theorem 1 has been proved in all other cases, we may apply it with $\mu\equiv\varepsilon$ for an arbitrary positive ε to deduce that it remains valid if $\mu\equiv 0$. From now on we assume that μ does not vanish identically. Hence $A\geqq 0$.

The condition (1.2) is natural, for $\mu(t)=t^{B+o(1)}$ as $t\to 0$, and if $B>1$, then (1.1) cannot be true for all $z_1, z_2\in G$ unless f is constant.

Tamrazov [2, Theorem 9.3, p. 167] showed that if μ is a majorant satisfying a growth condition, e.g.

$$\mu(2t) \leqq 2\mu(t), \quad t > 0, \tag{1.7}$$

and if the conclusion of Theorem 1 for a fixed $z_1\in\partial G$ is correct for all bounded Jordan domains G, then $\log\mu(e^t)$ is concave. Hence this condition is necessary. The reason behind it is that we want the functions $-\log\mu(|z-z_0|)$ to be subharmonic for $z\in G$, for any $z_0\in\partial G$.

Gehring, Hayman and the author [1, Theorem 1] proved Theorem 1 for $\mu(t)=Mt^\alpha$, $M>0$, $0\leqq\alpha\leqq 1$. In this case $\log\mu(e^t)=\alpha t+\log M$ is concave, and $A=B=\alpha$. For $0\leqq A<1$, our Theorem 1 applies to other functions that $\mu(t)=Mt^\alpha$. However, as we shall show in Section 3, for $A=1$ we get only the functions $\mu(t)=Mt$.

Our growth conditions (1.4) and (1.5) and their dependence on A are the same as [1, (1.2), (1.3)] in the case dealt with in [1]. As remarked in [1, p. 243], the functions z and z^2, respectively, with $G=\{|z|>1\}$, show that o cannot be replaced by O in (1.4) and (1.5).

Results like Theorem 1 were obtained by Tamrazov [2] for special open sets and for majorants μ satisfying a growth condition which we take to be (1.7). This is satisfied, for example, by any subadditive μ. Let $\operatorname{cap} E$ denote the logarithmic capacity of the compact set E. Then Tamrazov's results [2, Theorems 4.1., 6.1, 9.1] can be summarized in a slightly simplified form as follows. Suppose that G is bounded or that G contains a neighborhood of infinity, in which case f is required to remain analytic at ∞. Hence ∂G is bounded, and f is also bounded. If (1.1) holds for $z_1, z_2\in\partial G$, and if $z_1\in\partial G$, $z_2\in G$, then

$$|f(z_1)-f(z_2)| \leqq 27\mu(|z_1-z_2|)\,|z_1-z_2|\,(2\operatorname{cap} E(z_1, z_2))^{-1}, \tag{1.8}$$

where

$$E(z_1, z_2) = \left\{|z-z_1| \leqq \tfrac{1}{2}|z_1-z_2|\right\}\setminus G.$$

If, in addition, G is simply connected, then

$$|f(z_1)-f(z_2)| \leqq C\mu(|z_1-z_2|), \quad z_1, z_2\in \bar{G}, \tag{1.9}$$

where $C=108$. If G is suitable, one can use (1.8) together with [2, Lemma 4.1] to deduce that (1.9) holds for some C depending on G but not on z_1 and z_2. If $\log \mu(e^t)$ is concave, we can take $C=1$ in (1.9) if (1.7) holds and ∂G satisfies certain capacity density conditions. Moreover, if (1.1) holds for a fixed $z_1 \in \partial G$ and all $z_2 \in \partial G$, if f is bounded and if $\log \mu(e^t)$ is concave, then (1.9) holds for this z_1 and all $z_2 \in \bar{G}$ with $C=1$, provided that ∂G is thick enough. In this case ∂G need not be bounded.

Theorem 1 generalizes Tamrazov's results when $\log \mu(e^t)$ is concave, since we need no assumption on the capacity density of ∂G.

References to other earlier works related to this subject can be found in [1, p. 244] and in [2, p. 141—143].

2. Lemmas

Let u be a subharmonic function in the plane. We set

$$M(r, u) = \sup \{u(z) \mid |z| = r\}.$$

For all functions u that we shall consider, we have

$$M(r, u) = O(\log r) \tag{2.1}$$

as $r \to \infty$.

To prove Theorem 1, we need two lemmas. The first lemma follows from [1, Theorem 2].

Lemma 1. *Suppose that u is subharmonic, non-negative and not constant in the plane, that* (2.1) *holds as $r \to \infty$, and that $u(z)=0$ for some z. Then the limit*

$$\beta = \lim_{r \to \infty} M(r, u)/\log r \tag{2.2}$$

exists and $0<\beta<\infty$. Suppose further that there is a component D of the set $\{z \mid u(z)>0\}$ such that u is harmonic in D and possesses there a local conjugate v, and that for some α, $0<\alpha\leq 1$, and some positive R, the function

$$F(z) = z^{1-\alpha} \exp(u+iv)$$

remains single-valued in $D \cap \{|z|>R\}$. Then D contains the set $\{|z|>R_0\}$ for some R_0, and F has a pole of order $\beta+1-\alpha$ at infinity.

Note that any such component D is unbounded.

Our second lemma is a generalization of Tamrazov's result [2, Lemma 4.1, p. 156].

Lemma 2. *Let G be an open set with at least one (finite) boundary point, and suppose that*

$$f(z) = o(|z|^2) \tag{2.3}$$

as $z\to\infty$ *in any unbounded component of* G. *Then for every positive* t *we have*

$$\begin{aligned}(2.4)\qquad &\sup\{|f(z_1)-f(z_2)|\,\big|\,|z_1-z_2|\leqq t,\ z_1,z_2\in\bar{G}\}\\ &=\sup\{|f(z_1)-f(z_2)|\,\big|\,|z_1-z_2|\leqq t,\quad z_1\in\partial G,\ z_2\in\bar{G}\}.\end{aligned}$$

Tamrazov proved Lemma 2 with the additional hypothesis that ∂G is bounded and f is bounded (cf. Section 1).

To prove Lemma 2, we consider a fixed positive t, denote the left and right hand sides of (2.4) by L and R, respectively, and note that $R\leqq L$. To prove that $L\leqq R$, we may assume that $L>0$, $R<\infty$, so that f is not constant.

Pick $z_1, z_2\in G$ such that with $h=z_2-z_1$ we have $|h|\leqq t$. It suffices to show that with

$$g(z)=f(z)-f(z+h)$$

we have $|g(z_1)|\leqq R$. The function g is defined and continuous in the closure of the non-empty open set

$$(2.5)\qquad G_1=\{z\,|\,z\in G,\ z+h\in G\}\subset G$$

and analytic in G_1. Further, g is bounded in any compact subset of $\bar{G}_1$, and $z_1\in G_1$. If $z\in\partial G_1$, then $z\in\partial G$ or $z+h\in\partial G$, so that $|g(z)|\leqq R$. If g is bounded in G_1, it follows from the maximum principle that $|g(z_1)|\leqq R$.

Suppose that g is unbounded, and let D be a component of the open set $\{z\in G_1\big|\,|g(z)|>R\}$, so that D is unbounded. We set

$$(2.6)\qquad \begin{aligned}u(z)&=\log|g(z)|-\log R,\quad z\in D,\\ u(z)&=0,\quad z\notin D,\end{aligned}$$

and note that u is subharmonic and non-constant in the plane, and harmonic and positive in D, and possesses a local conjugate in D, namely, $\arg g(z)$. By (2.3), the condition (2.1) holds, and we may apply Lemma 1 to u with $\alpha=1$. It follows that D and thus G contains a neighborhood of ∞, so that by (2.3), f remains analytic or has a pole of order one at ∞. But in both cases g remains bounded at ∞, which is against our assumption. Hence $L\leqq R$, and Lemma 2 is proved.

3. Proof of Theorem 1

Suppose that the assumptions of Theorem 1 are satisfied and take a fixed $z_1\in\partial G$. The function

$$(3.1)\qquad u_1(z)=\log|f(z)-f(z_1)|-\log\mu(|z-z_1|)$$

is subharmonic in G and satisfies by (1.2),

$$\limsup_{z\to\zeta}u_1(z)\leqq 0$$

for all $\zeta\in\partial G$, $\zeta\neq z_1$. Further, u_1 is bounded above in any compact subset of $\bar{G}$ not containing z_1.

First we show that u_1 is bounded above in a neighbourhood of z_1 in $\bar{G}$. Suppose that this is not true, and let D_1 be a component of the open set $\{z\,|\,|z-z_1|<1,\ z\in G,\ u_1(z)>M\}$, where $M=\max(0, \sup\{u_1(z)|z\in\bar{G},\ |z-z_1|=1\})$. Hence $z_1\in\partial D_1$.

We set $\varphi(z)=(z-z_1)^{-1}$, $\varphi(G)=G_1$, $\varphi(D_1)=D_2$, and note that $\varphi(z_1)=\infty$, and $0\notin G_1$. We set

$$u(z) = u_1(\varphi^{-1}(z))-M,\quad z\in D_2,$$
$$u(z) = 0,\quad z\notin D_2,$$

so that u is non-negative, non-constant and subharmonic in the plane and positive in D_2. Also,

$$u(z) = \log|f(z_1+1/z)-f(z_1)|-\log\mu(1/|z|)-M,\quad z\in D_2.$$

By (1.2),

$$-\log\mu(1/|z|)\leqq(B+o(1))\log|z|\leqq(1+o(1))\log|z| \tag{3.2}$$

as $z\to\infty$. Hence (2.1) holds. Now we apply Lemma 1 to u to obtain $0<\beta\leqq B\leqq 1$. In particular, $B>0$.

Choose ε such that $0<\varepsilon<\beta\leqq B$. Consider the function

$$u_2(z) = \log|f(z_1+1/z)-f(z_1)|+(B-\varepsilon)\log|z|-M,\quad z\in D_2.$$

Since $M(r, u)\sim\beta\log r$, since $-\log\mu(1/|z|)\sim B\log r$, $r=|z|$, and since

$$u(z)-u_2(z) = -\log\mu(1/|z|)-(B-\varepsilon)\log|z|,\quad z\in D_2,$$

we have $M(r, u_2)\sim(\beta-\varepsilon)\log r\to\infty$ as $r\to\infty$. Further,

$$(B-\varepsilon)\log|z| < -\log\mu(1/|z|)$$

when $|z|$ is large enough, so that $u_2(z)<0$ if $z\in\partial D_2$ and $|z|>R$. We have $u_2(z)\leqq M_2$ for $|z|=R$, for some positive M_2, so that

$$\limsup_{z\to\zeta} u_2(z)-M_2\leqq 0$$

for every finite boundary point ζ of $D_3=D_2\cap\{|z|>R\}$. Since u_2 is unbounded in D_3, we can find an unbounded component D_4 of the set $\{z\in D_3|u_2(z)>M_2\}$. Hence the function

$$u_3(z) = u_2(z)-M_2,\quad z\in D_4,$$
$$u_3(z) = 0,\quad z\notin D_4,$$

is non-negative, non-constant and subharmonic in the plane. In D_4, u_3 is positive and harmonic, and possesses there a local conjugate v such that with $\alpha=\beta-\varepsilon$, the function

$$F(z) = z^{1-\alpha}\exp(u_3+iv)$$

is single-valued in D_4. Hence we can apply Lemma 1 to u_3 to deduce that D_4 contains a neighbourhood of infinity. But since $D_4 \subset G_1$, it follows that G contains a punctured neighbourhood of z_1, so that z_1 is a regular point of f and so there exists a positive integer k and a_k unequal to zero such that

$$f(\zeta)-f(z_1) \sim a_k(\zeta-z_1)^k \quad \text{as} \quad \zeta \to z_1,$$
$$f(z_1+1/z)-f(z_1) \sim a_k z^{-k} \quad \text{as} \quad z \to \infty.$$

This together with (1.2) and (3.2) implies that

$$u(z) = -k \log |z| - \log \mu(1/|z|) + O(1) \leqq o(\log |z|)$$

as $z \to \infty$, which contradicts the result $\beta > 0$. So in any case u_1 is bounded above in a neighbourhood of z_1 in $\bar{G}$.

We want to show that $u_1(z) \leqq 0$ in G, since in view of Lemma 2 this clearly completes the proof of Theorem 1. If u_1 is bounded above in G, this follows from the maximum principle. Suppose then that u_1 is not bounded above in G, and let D be a component of $\{z | u_1(z) > 0\}$. Since u_1 is bounded above in any compact subset of $\bar{G}$, as we proved above, the set D must be unbounded.

We set $u(z) = u_1(z)$ in D, $u(z) = 0$ outside D. To make u subharmonic, we may have to redefine u at z_1, cf. [1, p. 248]. Then we apply Lemma 1 to u. We assume that (1.4) or (1.5) holds. Without loss of generality we may assume that $z_1 = 0$. Thus we obtain

$$\sup_{|z|=r,\, z \in D} \log |f(z)-f(0)| \sim (\beta + A) \log r$$

as $r \to \infty$. Suppose now that $A < 1$. We choose ε such that $0 < \varepsilon < \beta$ and $A + \varepsilon < 1$, and R such that

$$\log \mu(|z|) < (A+\varepsilon) \log |z|, \quad |z| \geqq R,$$

and set

$$u_2(z) = \log |f(z)-f(0)| - (A+\varepsilon) \log |z| - M_3, \quad z \in D,$$

where $M_3 \geqq 0$ and M_3 is so large that $u_2(z) \leqq 0$ if $z \in D$ and $|z| = R$. Hence u_2 has negative boundary values in $D \cap \{|z| > R\}$, but u_2 is unbounded above in $D \cap \{|z| > R\}$. Therefore we can find an unbounded component D_1 of the set $\{z \in D | |z| > R,\ u_2(z) > 0\}$, and we define $u_3(z) = u_2(z)$ in D_1, $u_3(z) = 0$ outside D_1.

The function u_3 is subharmonic in the plane, harmonic in D_1, and possesses there a local conjugate. We may apply Lemma 1 to u_3 with $\alpha = 1 - A - \varepsilon$ to deduce that D_1 and so G contains a neighbourhood of ∞. Hence by (1.4), f remains bounded at ∞, so that u_1 is bounded above, which is contrary to our assumption. This proves Theorem 1 if $A < 1$.

Suppose that $A = 1$. We shall soon show that then $\mu(t) = Mt$, $M > 0$. Since (1.5) holds, Theorem 1 follows now from [1, Theorem 1].

If $A = 1$, then $B = 1$. Since

$$\eta(t) = \log \mu(e^t) - \log \mu(1)$$

is concave with $\eta(0)=0$, we have

$$0 = 2\eta(0) \geqq \eta(t)+\eta(-t), \quad t>0,$$
$$\eta(t)/t \leqq \eta(-t)/(-t), \quad t>0.$$

Further $\eta(t)/t$ decreases to 1 and $\eta(-t)/(-t)$ increases to 1 as t increases from 0 to ∞. Hence $\eta(t)\equiv t$, $\mu(t)\equiv t\mu(1)$. This proves Theorem 1 if $A=1$.

Suppose finally that (1.6) holds for some $q>0$ but that the conclusion of Theorem 1 fails. Suppose that (1.1) fails for some $z_1\in\partial G$, $z_2\in G$. We can still deduce that u_1, given by (3.1), is bounded above in a neighbourhood of z_1 in $\bar{G}$ and that (2.1) is satisfied with $u=u_1$. Since u_1 must be unbounded above in G, we obtain as before from Lemma 1 that G contains a neighbourhood of infinity. Now (1.6) implies that f has a pole at infinity.

If (1.1) fails for some $z_1, z_2\in G$ but not for any $z_1\in\partial G$, $z_2\in\bar{G}$, then we take these $z_1, z_2\in G$, set $h=z_2-z_1$ and $g(z)=f(z)-f(z+h)$. If R is defined as in the proof of Lemma 2 for $t=|h|$, then the function u given by (2.6) must be unbounded in G_1, given by (2.5). Since u satisfies (2.1), we deduce from Lemma 1 that G contains a neighbourhood of infinity. Now (1.6) implies that f has a pole at infinity. Theorem 1 is proved.

Remark. After this paper had been written, I was informed that there is a recent preprint of Tamrazov containing results similar to those in this paper.

References

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