Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 10, 1985, 283—288

Commentationes in honorem
Olli Lehto
LX annos nato

ON CERTAIN ESTIMATIONS FOR THE FIFTH AND SIXTH COEFFICIENT OF BOUNDED REAL UNIVALENT FUNCTIONS

RONALD KORTRAM and OLLI TAMMI

1. Introduction

The class of normalized bounded univalent functions f, defined in $U=\{z\in C \mid |z|<1\}$, is

$$S(b) = \{f \mid f(z) = b(z+a_2z^2+\ldots),\ |f(z)| < 1,\ 0 < b \leqq 1\}.$$

The wish to estimate the coefficients a_n for lower indexes n and at least for certain intervals of b has led to a sequence of more or less successful attempts and methods. In a set of papers the authors have specialized in the estimation technique, applied mainly to a_4, which is based on an inequality obtainable by area integration and called here the Power inequality [2], [3], [4], [7]. In the real subclass $S_R(b)\subset S(b)$ the estimation of a_4 has recently been completed [1]. Therefore, testing the technique available for somewhat higher indexes is relevant just now.

In the estimations mentioned the knowledge of the lower coefficient bodies, denoted here by (a_2, a_3), (a_2, a_3, a_4) etc., appeared to be of primary importance. The range of a_5 in a_2, a_3, a_4 was estimated by the aid of the Power inequality in [6]. Combining this with the knowledge of (a_2, a_3, a_4) one can expect some improvement of the following result [3]: a_5 is maximized in $S(b)$ on $0.685\leqq b\leqq 1$ by the mapping of the type 4:4 (we refer here to the notation $\alpha:\beta$ of the slit domains consisting of U minus a collection of slits with α starting points and $\beta\geqq\alpha\geqq 1$ endpoints; [6], [7]). It appears that even in the class $S_R(b)$ the whole interval $e^{-2/3}\leqq b\leqq 1$, expected to be the best possible, cannot be reached. This clearly stresses what follows. Inequalities sharp for new types of algebraic equality function are needed. — The success in [1] was based on a process of this kind. Similarly, the importance and difficulty of governing the lower coefficient bodies increases with the index.

The above conclusions are confirmed further when the Power inequality for the coefficient a_6 in $S_R(b)$ is tested.

doi:10.5186/aasfm.1985.1031

2. Maximizing a_5 in terms of a_2

According to [6] we have for a_5 (the abbreviations Δ and γ_2 are taken from [6])

(1)
$$a_5-2a_2a_4-\frac{3}{2}a_3^2+4a_2^2a_3-\frac{3}{2}a_2^4 \leqq \frac{1}{2}(1-b^4)-b^2a_2^2$$
$$-(\gamma_2+b^2a_2)\Delta^{-1}\left[(\gamma_2+b^2a_2)\log b-\left(a_3-\tfrac{1}{2}a_2^2\right)a_2\right]$$
$$-\left(a_3-\tfrac{1}{2}a_2^2\right)\Delta^{-1}\left[\left(a_3-\tfrac{1}{2}a_2^2\right)(a_3-a_2^2-1+b^2)-a_2(\gamma_2+b^2a_2)\right];$$
$$\gamma_2 = a_4-2a_2a_3+a_2^3,$$
$$\Delta = -\log b\left[a_3-1+b^2-\left(1+\frac{1}{\log b}\right)a_2^2\right] < 0.$$

Rewrite this by the aid of a perfect square term which includes a_4:

(2)
$$a_5-\tfrac{1}{2}(1-b^4) \leqq -\Delta^{-1}\log b\,(\gamma_2+b^2a_2-K)^2+\Delta^{-1}\log b\,K^2$$
$$-\Delta^{-1}\left(a_3-\frac{1}{2}a_2^2\right)^2(a_3-a_2^2-1+b^2)+\frac{3}{2}a_3^2-3b^2a_2^2-\frac{1}{2}a_2^4;$$
$$K = (\log b)^{-1}a_2\left(a_3-\tfrac{1}{2}a_2^2+\Delta\right).$$

Disregarding the limits of a_4 we maximize the right side in γ_2:

(3)
$$a_5-\tfrac{1}{2}(1-b^4) \leqq Aa_3^2+2Ba_3+C;$$
$$A = \frac{3}{2}+\frac{1}{\log b} < 0 \quad \text{for} \quad e^{-2/3} < b < 1,$$
$$B = \left(\frac{1}{2\log b}-\frac{1}{2}\right)a_2^2,$$
$$C = (1-4b^2)a_2^2+\left(\frac{1}{2}+\frac{1}{4\log b}\right)a_2^4.$$

For $e^{-2/3}<b<1$, by maximizing the right side of (3) in terms of a_3, we are similarly led to the estimation

(4)
$$a_5-\frac{1}{2}(1-b^4) \leqq a_2^2\left[1-4b^2+\frac{1}{4}\,\frac{11+4\log b}{2+3\log b}\,a_2^2\right].$$

The equality here is reached on the parabola

(5)
$$a_3 = \frac{\log b-1}{3\log b+2}\,a_2^2.$$

The parabola (5) lies in the coefficient body (a_2, a_3) (cf. e.g. [8], p. 49) so far as

$$a_2^2 \leqq (1-b^2)\frac{3\log b+2}{2\log b+3}.$$

For the complementary values of a_2 the maximum of the right side of (3) is attained on the lower boundary arc $a_3=a_2^2-1+b^2$ of (a_2, a_3).

Numerical evaluation yields the interval $0.66 \leqq b \leqq 1$ for which the right side of (3) is non-positive. Thus on this interval

$$a_4 \leqq \tfrac{1}{2}(1-b^4), \tag{6}$$

and the equality is reached, as before, by the radial slit mapping 4:4.

3. The use of (a_2, a_3, a_4)

Some improvement can be expected when the limits of a_4 in terms of a_2 and a_3 are taken into consideration. From [7], p. 285, we read out for γ_2:

$$-\frac{2}{3}(1-b^3)+\frac{b}{2}a_2^2-\frac{a_2^3}{12}+\frac{\left(a_3-\frac{3}{4}a_2^2-ba_2\right)^2}{2(1-b)+a_2} \leqq \gamma_2$$

$$\leqq \frac{2}{3}(1-b^3)-\frac{b}{2}a_2^2-\frac{a_2^3}{12}-\frac{\left(a_3-\frac{3}{4}a_2^2+ba_2\right)^2}{2(1-b)-a_2}.$$

Applying this to evaluating the maximum of the right side of (2) in (a_2, a_3) we can push the estimation (6) to the interval

$$0.625 \leqq b \leqq 1.$$

The expected endpoint $e^{-2/3}=0.51$ is clearly outside the scope of the Power inequality. A substantial improvement necessitates an inequality sharp for the algebraic extremal functions of the more specific type 3:4 or 2:4.

4. On the coefficient a_6

Let us start from the general form of the Power inequality in $S_R(b)$ ([7], (6), p. 177, $x_0=0$):

$$\sum_{-N}^{N} k y_k^2 \leqq \sum_{-N}^{N} k x_k^2.$$

Apply this, instead of $f(z)$, to $\sqrt{f(z^2)}$ and choose $N=5$. The numbers y_ν $(\nu=-5, \ldots, 5)$ are polynomials of the a_ν-coefficients which include the parameters x_ν $(\nu=-5, \ldots, 5)$. Introduce the new parameters $u_k=-ky_{-k}$ $(k=1, \ldots, 5)$. From the lower cases we know to expect a maximizing inequality if the parameters are chosen to satisfy $y_\nu = -y_{-\nu}$, $x_\nu = -x_{-\nu}$. Moreover, for the even coefficient a_6 we may choose $u_2 = u_4 = 0$, $u_5 = 1$. This leaves us the inequality

$$\sum_1^5 u_\nu y_\nu \leqq \sum_1^5 \frac{|u_\nu|^2}{\nu} \quad (u_2 = u_4 = 0)$$

with

$$y_1 = \frac{1}{2} a_4 - \frac{3}{4} a_2 a_3 + \frac{5}{16} a_2^3 + u_3 \left(\frac{1}{2} a_3 - \frac{3}{8} a_2^2\right) + \frac{1}{2} u_1 a_2 + x_1 b^{1/2},$$

$$y_3 = \frac{1}{2} a_5 - a_2 a_4 - \frac{5}{8} a_3^2 + \frac{29}{16} a_2^2 a_3 - \frac{85}{128} a_2^4 + u_3 \left(\frac{1}{2} a_4 - a_2 a_3 + \frac{13}{24} a_2^3\right)$$

$$+ u_1 \left(\frac{1}{2} a_3 - \frac{3}{8} a_2^2\right) + \frac{1}{2} x_1 b^{1/2} a_2 + x_3 b^{3/2},$$

$$y_5 = \frac{1}{2} a_6 - a_2 a_5 - \frac{3}{2} a_3 a_4 + 2 a_2^2 a_4 + \frac{21}{8} a_2 a_3^2 - \frac{59}{16} a_2^3 a_3 + \frac{689}{640} a_2$$

$$+ u_3 \left(\frac{1}{2} a_5 - a_2 a_4 - \frac{5}{8} a_3^2 + \frac{29}{16} a_2^2 a_3 - \frac{85}{128} a_2^4\right) + u_1 \left(\frac{1}{2} a_4 - \frac{3}{4} a_2 a_3 + \frac{5}{16} a_2^3\right)$$

$$+ x_1 b^{1/2} \left(\frac{1}{2} a_3 - \frac{1}{8} a_2^2\right) + \frac{3}{2} x_3 b^{3/2} a_2 + x_5 b^{5/2};$$

$$x_1 = \left(2 a_3 - \frac{1}{8} a_2^2\right) b^{1/2},$$

$$x_3 = \frac{5}{6} a_2 b^{3/2},$$

$$x_5 = \frac{1}{5} b^{5/2}.$$

There are left two free parameters u_1 and u_3. From the experience gained from lower coefficients we know that the choice of u_1 and u_3 which eliminates a_4 and a_5 yields the maximum for a_6 in a_2 and a_3. Hence, take

$$u_3 = a_2, \quad u_1 = \frac{3}{2} a_3 - \frac{a_2^2}{2}$$

and obtain for a_6

(7)
$$a_6 - \frac{2}{5}(1 - b^5)$$

$$\leqq a_2^2 \left[\frac{2}{3} - \frac{25}{6} b^3 + \left(\frac{1}{2} - \frac{b}{32}\right) a_2^2 - \frac{917}{960} a_2^3 + \left(\frac{7}{2} a_2 + b - 3\right) a_3\right] + \left(\frac{9}{2} - 8b - \frac{7}{2} a_2\right) a_3^2.$$

Again, we have to estimate the right side under the restriction given by the coefficient body (a_2, a_3).

To the point (a_2, a_3) one can find a restriction which is simpler than that implied in the coefficient body. The coefficients a_ν are limited, according to [5], (50), so that

$$b^2 \sum_1^N \nu |a_\nu|^2 \leqq 1 - \frac{N}{1-b^{2N}} b^{2(N+1)} |a_N|^2.$$

Setting $N=6$ yields

$$2|a_2|^2 + 3|a_3|^2 \leqq b^{-2} - 1 - b^{-2} \frac{6|a_6|^2}{1-b^{12}}.$$

According to the idea presented in [3] we may restrict ourselves to the cases $|a_6| \geqq \frac{2}{5}(1-b^5)$, which implies for the point (a_2, a_3) the general restriction in $S_R(b)$:

$$\frac{a_2^2}{p^2} + \frac{a_3^2}{q^2} \leqq 1;$$

$$p^2 = \frac{1}{2}\left[b^{-2} - 1 - b^{-2} \frac{24}{25} \frac{(1-b^5)^2}{1-b^{12}}\right],$$

$$q^2 = \frac{2}{3} p^2.$$

Under this condition the right side of (8) remains non-positive for

$$0.774 \leqq b \leqq 1,$$

where thus

$$a_6 \leqq \frac{2}{5}(1-b)^5.$$

The equality is attained by the radial slit mapping 5:5.

References

[1] Jokinen, O.: On the use of Löwner identities for bounded univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 41, 1982, 1—52.

[2] Kortram, R., and O. Tammi: On the first coefficient regions of bounded univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 592, 1974, 1—27.

[3] Kortram, R., and O. Tammi: On the range of validity of certain inequalities of the fourth coefficient of a bounded univalent function. - Bull. Soc. Sci. Lett. Łódź, 24, 1, 1974, 1—6.

[4] Kortram, R., and O. Tammi: On the second coefficient region of bounded univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 1, 1975, 155—175.

[5] SCHIFFER, M., and O. TAMMI: A Green's inequality for the power matrix. - Ann. Acad. Sci. Fenn. Ser. A I Math. 501, 1971, 1—15.

[6] TAMMI, O.: Inequalities for the fifth coefficient for bounded real univalent functions. - J. Analyse Math. 30, 1976, 481—497.

[7] TAMMI, O.: Extremum problems for bounded univalent functions. — Lecture Notes in Mathematics 646. Springer-Verlag, Berlin—Heidelberg—New York, 1978.

[8] TAMMI, O.: Extremum problems for bounded univalent functions II. - Ibidem 913, 1982.

University of Nijmegen
Institute of Mathematics
Nijmegen
The Netherlands

University of Helsinki
Department of Mathematics
SF-00100 Helsinki
Finland

Received 3 October 1983