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Olli Lehto
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QUASICONFORMAL MAPPINGS WITHOUT BOUNDARY EXTENSIONS

T. KUUSALO

1. According to the classical theorem of C. Carathéodory [1] and of W. F. Osgood and E. H. Taylor [6], a conformal mapping $f\colon D\to D'$ between two Jordan domains D and D' in the complex plane $\boldsymbol{C}$ always has a *boundary extension,* i.e. there is a homeomorphism $f^*\colon \bar{D}\to\bar{D}'$ of the respective closed domains such that $f^*|_D=f$. As it is well known, the theorem holds for planar quasiconformal mappings as well. In fact, G. Faber's proof for conformal mappings carries over to the quasiconformal case with minor modifications only ([3], cf. also R. Courant [2]; Lehto–Virtanen [5], pp. 44—46). J. Väisälä proved in [7] the existence of a boundary extension for all quasiconformal mappings $f\colon D\to D'$ between n-dimensional Jordan domains $D, D'\subset\boldsymbol{R}^n$ quasiconformally equivalent to the unit ball $B^n\subset\boldsymbol{R}^n$, $n\geqq 2$ (also in Väisälä [8], pp. 51—67). But unlike the planar case, even a Jordan domain $D\subset\boldsymbol{R}^n$ homeomorphic to the unit ball B^n is not necessarily quasiconformally equivalent to B^n when $n\geqq 3$ (Gehring–Väisälä [4]). We shall give below an example which shows that *for arbitrary Jordan domains* $D, D'\subset\boldsymbol{R}^n$, $n\geqq 3$, *a quasiconformal mapping* $f\colon D\to D'$ *does not always have a boundary extension.*

I wish to express here my gratitude to Vesa Lappalainen, who has made all the excellent drawings below using the computer facilities of the University of Jyväskylä. Furthermore, I would like to thank the Royal Swedish Academy of Sciences for supporting my stay at the Mittag–Leffler Institute during the academic year 1983—1984.

2. Let Γ be the spiral curve in the closed unit disc $\bar{B}^2\subset\boldsymbol{C}$ described by

$$\Gamma(r) = re^{(i\alpha/2)(r-r^{-1})}$$

for $0\leqq r\leqq 1$, $\alpha>0$ a constant. If $\varrho=\varrho(z)=\varrho(z,0)$ is the *hyperbolic distance* of $z\in\bar{B}^2$ from the origin with respect to the *hyperbolic length element*

$$d\varrho = 2(1-|z|^2)^{-1}|dz|,$$

and $\theta=\arg(z)$ the *argument* of z, the trace of Γ can also be given as

$$\Gamma = \left\{z\in\bar{B}^2\colon \theta = -\frac{\alpha}{\sinh(\varrho)},\quad 0\leqq\varrho\leqq+\infty\right\}.$$

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The curve Γ has *hyperbolic curvature*

$$\chi(\varrho) = \frac{1+\alpha^{-2}\tanh^4(\varrho)}{\tanh(\varrho)[1+\alpha^{-2}\tanh^2(\varrho)]^{3/2}}$$

at the parameter value $r=\tanh(\varrho/2)$. Now the boundary of the domain

$$G = \left\{z\in C\colon 0<|z|<1,\ \left|\theta+\frac{\alpha}{\sinh(\varrho)}\right| < \frac{\pi}{2}\right\}$$

consists of a circular arc together with two spirals $\Gamma_\pm = \pm i\Gamma$ (cf. Figure 1; here as well as in all subsequent drawings $\alpha=2$). Because the closed domain $\bar{G}$ lies to the right of the boundary arc Γ_+, and the hyperbolic curvature of Γ_+ is positive, we can define in $\bar{G}$ a *hyperbolic orthogonal projection* $p\colon \bar{G}\to\Gamma_+$ such that for all $z\in\bar{G}$ the hyperbolic arc $zp(z)\subset\bar{G}$ is perpendicular to Γ_+ and that $\varrho(z, p(z))$ is the hyperbolic distance of z from Γ_+,

$$\varrho(z, p(z)) = \varrho(z, \Gamma_+) = \min_{z'\in\Gamma_+} \varrho(z, z').$$

The projection p is easily seen to be infinitely differentiable in G. Furthermore, the positive curvature of Γ_+ implies that p is *locally strictly contractive* in G,

$$\varrho(p(z), p(z_0)) < \varrho(z, z_0)$$

for all $z\neq z_0$ in a sufficiently small neighbourhood of $z_0\in G$.

If $T(\varrho)$ is a primitive of

$$(1+\alpha^2\tanh^{-2}(\varrho))^{1/2},$$

we can define by $\tau(z)=T(\varrho(z))$, $z\in\Gamma_+$, a *hyperbolic length scale* on Γ_+ such that

$$\varrho_+(z_1, z_2) = |\tau(z_1)-\tau(z_2)|$$

gives the hyperbolic length of the subarc of Γ_+ with endpoints $z_1, z_2\in\Gamma_+$. As $\tau(0)=-\infty$, $\tau(i)=+\infty$, the spiral Γ_+ is infinitely long in both directions. We extend τ to a function $\tau\colon \bar{G}\to\bar{R}$ by setting

$$\tau(z) = \tau(p(z))$$

for arbitrary $z\in\bar{G}$.

3. With respect to the (*quasi*) *hyperbolic length element*

$$d\sigma = \frac{ds}{r}, \quad r = (x_1^2+x_2^2)^{1/2}$$

in $\bar{R}^3=R^3\cup\{\infty\}$ the x_3-axis forms the circle at infinity, and a hyperbolic isometry identifying the closed unit disc $\bar{B}^2\subset C$ with the closed half-plane $\{x\in\bar{R}^3\colon x_2=0, x_1\geqq 0\}$ can be defined by

$$x_1+ix_3 = h(z) = \frac{1-z}{1+z}.$$

If we rotate the image $h(\bar{G})$ of $\bar{G}$ around the x_3-axis, we get a *closed Jordan domain*

(a sectorial cross cut in Figure 2)

$$\bar{D} = \left\{ x \in \boldsymbol{R}^3 \colon \left| \psi + \frac{\alpha}{\sinh(\sigma)} \right| \leqq \frac{\pi}{2}, \quad 0 \leqq \sigma \leqq +\infty \right\},$$

where

$$\psi = \psi(x) = \arg\left(\frac{1-r-ix_3}{1+r+ix_3} \right),$$

and

$$\sigma = \sigma(x) = \sigma(x, S^1)$$

is the *hyperbolic distance* of $x \in \boldsymbol{R}^3$ from the unit circle S^1 of the plane $\boldsymbol{R}^2 = \{x \in \boldsymbol{R}^3 \colon x_3 = 0\}$. The boundary of $\bar{D}$ consists of two topological discs

$$E_{\pm} = \left\{ x \in \boldsymbol{R}^3 \colon \psi + \frac{\alpha}{\sinh(\sigma)} = \pm\frac{\pi}{2}, \quad 0 \leqq \sigma \leqq +\infty \right\}$$

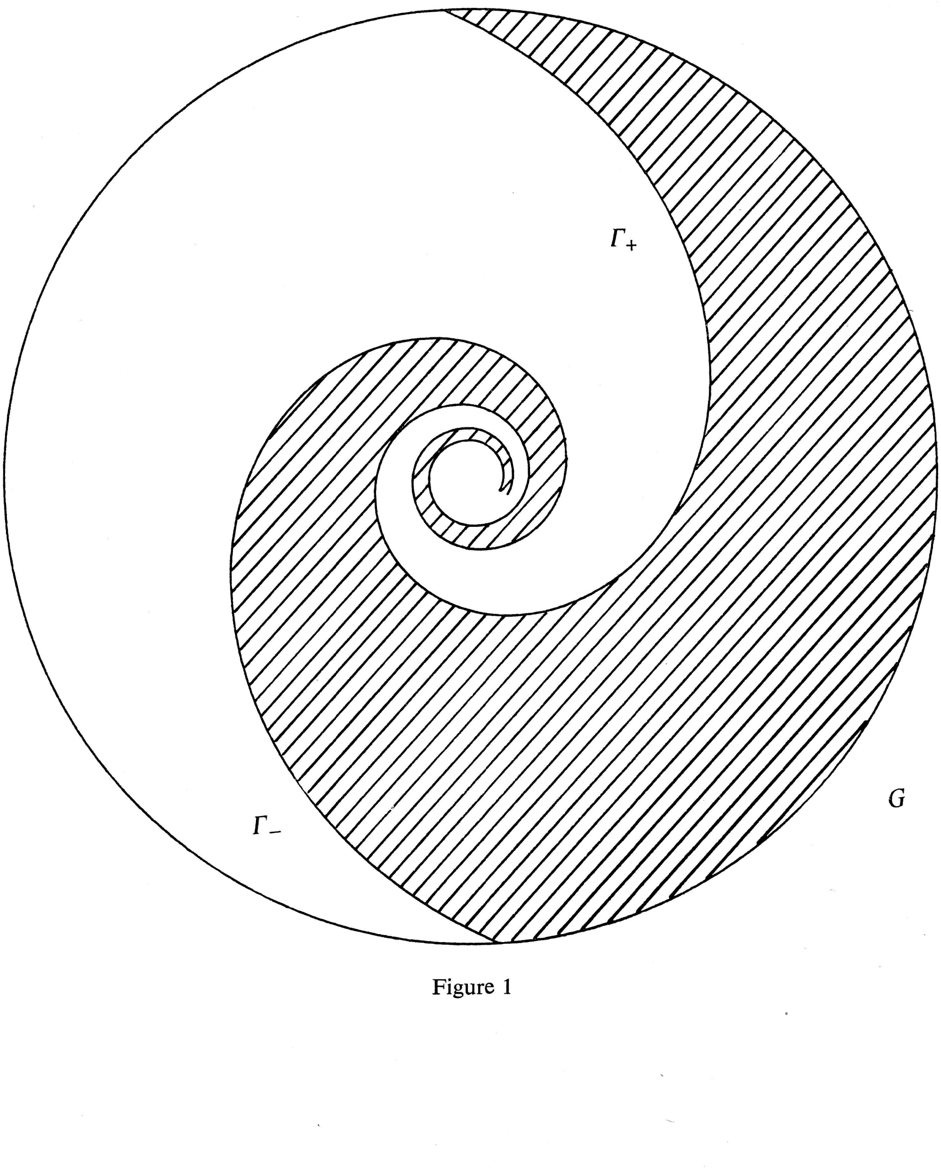

Figure 1

glued together along the circle $S^1 = E_+ \cap E_-$. The function τ extends by rotation to a *parameter function* $\tau\colon \bar{D} \to \bar{R}$ such that $\tau^{-1}(-\infty) = S^1$ and $\tau^{-1}(+\infty) = \{(0, 0, -1)\}$. As in the preceding paragraph, we can define in the whole domain $\bar{D}$ outside the x_3-axis a *locally contractive hyperbolic orthogonal projection* p into the lower half-boundary E_+ such that $\tau(x) = \tau(p(x))$ for all x in $\bar{D}$ outside the x_3-axis.

If we let $S^1 \subset E_+$ collapse to a point and set

$$\varphi(x) = \arg(x_1 + ix_2)$$

for $x \in \mathbf{R}^3$, we see that

$$\theta(x) = e^{\tau(x) + i\varphi(x)}$$

defines an *isometry* and thus a *conformal mapping* of E_+ onto the Riemann sphere $\bar{\mathbf{C}}$ with respect to the *hyperbolic length element* $d\sigma$ in E_+ and the *logarithmic length element*

$$dv = |z|^{-1}|dz|$$

in $\bar{\mathbf{C}}$, so that $\theta(S^1) = 0$ and $\theta(0, 0, -1) = \infty$.

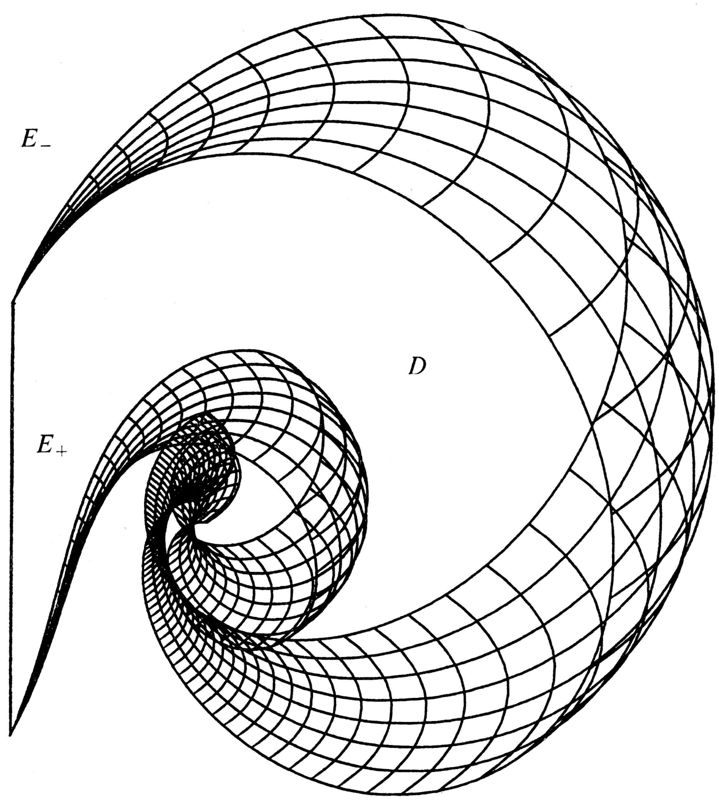

Figure 2

Using the complex notation

$$x = (\zeta, x_3) = (x_1 + ix_2, x_3) \in \boldsymbol{R}^3,$$
$$ax = (a\zeta, |a| x_3) \in \boldsymbol{R}^3$$

for $x \in \boldsymbol{R}^3$, $a \in \boldsymbol{C}$, we define for any real λ the mapping $f_\lambda \colon \bar{D} \setminus S^1 \to \bar{D} \setminus S^1$ by

$$f_\lambda(x) = \exp\big(i\lambda\tau(x)\big)x.$$

The *restriction* $f_\lambda \colon E_+ \setminus S^1 \to E_+ \setminus S^1$ is *quasiconformal* for all $\lambda \in \boldsymbol{R}$, its *symmetrized derivative* having the eigenvalues

$$c_1(\lambda) = \big(1 + \lambda^2/2 + (\lambda^2 + \lambda^4/4)^{1/2}\big)^{1/2},$$
$$c_2(\lambda) = c_1(\lambda)^{-1}$$

at all nonsingular points of E_+. Now f_λ preserves the equidistance surfaces $S_t = \{x \in D \colon \sigma(x, E_+) = t\}$, $t > 0$, in D, and because the projection p is *locally contractive,* we see, using appropriate local coordinates, that the eigenvalues $c_1(x, \lambda) \geqq c_2(x, \lambda) \geqq c_3(x, \lambda) > 0$ of the *symmetrized derivative* of f_λ satisfy

$$c_1(x, \lambda) = c_3(x, \lambda)^{-1} \leqq c_1(\lambda),$$
$$c_2(x, \lambda) = 1$$

at all points x of D outside the x_3-axis. Thus $f_\lambda \colon D \to D$ is quasiconformal with dilatations

$$K(f_\lambda) = K_o(f_\lambda) = K_I(f_\lambda) = c_1(\lambda)^3.$$

Because $\tau(x) \to -\infty$ as $x \to S^1$, we have

$$C(f_\lambda, b) = S^1$$

for the cluster set at any boundary point $b \in S^1 \subset \partial D$ when $\lambda \neq 0$. Thus in this case *the quasiconformal mapping $f_\lambda \colon D \to D$ does not have any boundary extension.* To visualize the situation we have depicted in Figure 3 the image of $\Gamma' = h(\Gamma) \cap D$ when $\lambda = 0.25$.

We could also choose a function $\lambda(\tau)$ with a uniformly bounded derivative $\lambda'(\tau)$ converging to 0 when $\tau \to -\infty$ but such that $\lambda(\tau) \to -\infty$ at the same time, and define $f \colon D \to D$ by setting

$$f(x) = \exp\,[i(\lambda \circ \tau)(x)]x$$

for all $x \in D$. Then also $f \colon D \to D$ would be a quasiconformal mapping having $C(f, b) = S^1$ as cluster set at all $b \in S^1$ but with the local maximal dilatation $K(x, f)$ approaching one when x approaches S^1.

4. Remarks. (i) The example above was chosen because it allows a fairly simple and explicit representation. The Jordan domain D has also quite nice symmetry properties, e.g. the Möbius transformation generated by the successive reflections in the plane $\boldsymbol{R}^2$ and in the unit sphere $S^2 \subset \boldsymbol{R}^3$ maps D conformally onto its exterior. The boundary $\partial D = E_+ \cup E_-$ is quasiconformally collared with the exception of the

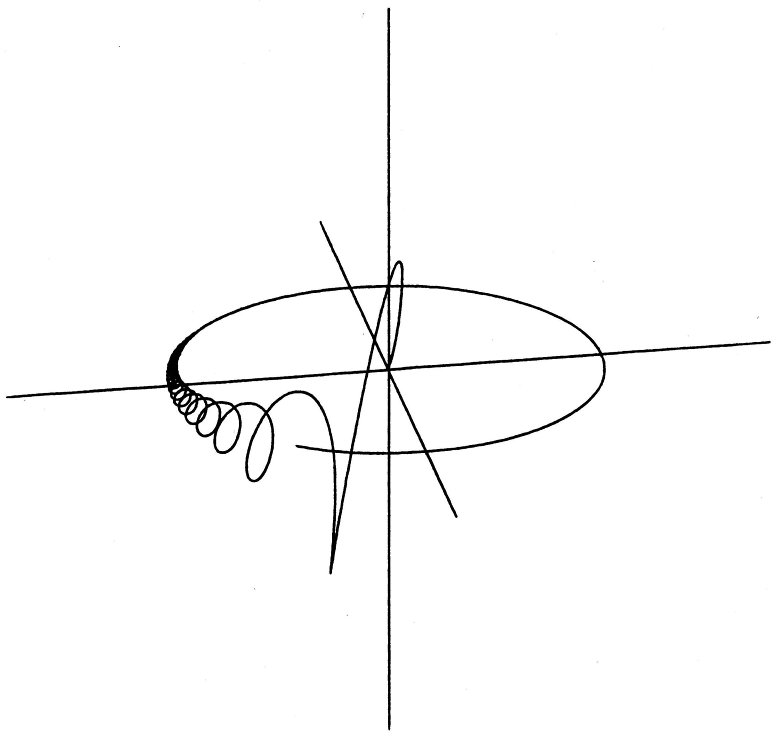

Figure 3

circle $S^1 = E_+ \cap E_-$. In fact, the mapping Ψ defined by

$$\Psi(x) = \left(\frac{e^{\tau(x)} \zeta}{|\zeta| \cosh \sigma(x, E_+)}, -e^{\tau(x)} \tanh \sigma(x, E_+) \right) \in \boldsymbol{R}^3$$

for $x = (\zeta, x_3) \in \bar{D}$ maps D homeomorphically onto a domain D^* in the lower half-space $x_3 < 0$ bounded by the *plane* $\bar{\boldsymbol{R}}^2 = \Psi(E_+)$ and by the *quasiconformal sphere* $E_-^* = \Psi(E_-)$, the singular part $S^1 = E_+ \cap E_-$ of the boundary collapsing onto the origin, $\Psi(S^1) = \{0\}$ (cf. Figure 4). The mapping Ψ preserves $\varphi(x) = \arg(\zeta)$, and $\sigma(\Psi(x), \boldsymbol{R}^2) = \sigma(x, E_+)$ for all $x \in D$. In the direction of the line $\varphi(x) = \varphi(x_0)$, $\sigma(x, E_+) = \sigma(x_0, E_+)$ the mapping Ψ has at $x_0 \in D$ the contraction coefficient

$$\varkappa(x_0) = \left[1 + \chi(\sigma(p(x_0))) \tanh \sigma(x_0, E_+)\right]^{-1}.$$

Because we have $\sigma(x, E_-) \sim \alpha^{-1} \pi \sigma^2$ for the width of D and $\chi(\sigma) \sim \sigma^{-1}$ for the hyperbolic curvature when x approaches S^1 on E_+, we see that $\varkappa$ is uniformly bounded away from 0 in D, so that also the mapping $\Psi: D \to D^*$ is quasiconformal. When $\alpha = 2$, $\varkappa(x) > 1/3$ in the whole domain D, so that in this case $K(\Psi) < 9$.

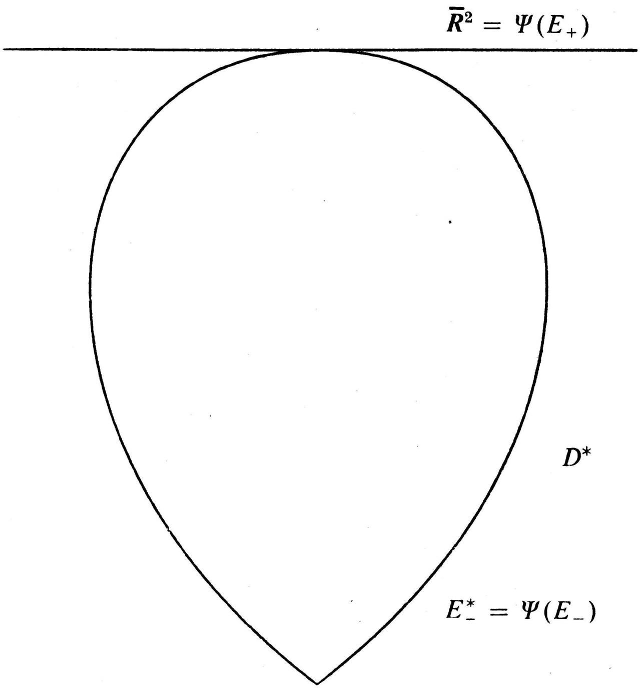

Figure 4

(ii) We can make the same construction in $\boldsymbol{R}^n$, $n>3$, using the *(quasi)hyperbolic length element*

$$d\sigma = (x_1^2+\ldots+x_{n-1}^2)^{-1/2}\, ds$$

and defining as above an *n-dimensional Jordan domain*

$$D_n = \left\{x \in \boldsymbol{R}^n \colon \left|\psi + \frac{\alpha}{\sinh(\sigma)}\right| < \frac{\pi}{2},\quad 0 < \sigma \leqq +\infty\right\}$$

such that the boundary of D_n contains the *unit sphere* S^{n-2} of the hyperplane $\boldsymbol{R}^{n-1} = \{x \in \boldsymbol{R}^n \colon x_n = 0\} \subset \boldsymbol{R}^n$. Only the rotations require a somewhat more careful consideration than before. To this purpose we choose antisymmetric linear endomorphisms $A_1, \ldots, A_q$ of $\boldsymbol{R}^{n-1}$ such that the orbit of

$$O(t) = \prod_{j=1}^{q} \exp(tA_j) \in SO(n-1) \subset SO(n)$$

for $t \to \pm\infty$ is dense in the orthogonal group $SO(n-1)$ of $\boldsymbol{R}^{n-1}$. Let $\tau \colon D_n \to \boldsymbol{R}$ be the *parameter function* for D_n. For all real λ we can now define by

$$f_\lambda(x) = O(\lambda\tau(x))x$$

a quasiconformal mapping $f_\lambda\colon D_n \to D_n$ with

$$K(f_\lambda) \leqq c_1(\lambda a)^n,$$

where

$$a = \sum_{j=1}^{q} \|A_j\|.$$

Furthermore, for all $b \in S^{n-2} \subset \partial D_n$

$$C(f_\lambda, b) = S^{n-2}$$

when only $\lambda \neq 0$.

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Institut Mittag-Leffler
Auravägen 17
S—182 62 Djursholm
Sweden

University of Jyväskylä
Department of Mathematics
SF—40100 Jyväskylä
Finland

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