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ESTIMATES OF THE INNER RADIUS OF UNIVALENCY OF DOMAINS BOUNDED BY CONIC SECTIONS

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1. Introduction

Let A be a simply connected domain in the extended plane, conformally equivalent to the unit disc. Denote by $M(A)$ the set of locally injective meromorphic functions in A and by ϱ_A the density of the Poincaré metric in A, so normalized that

$$\varrho_A(h(z))|h'(z)| = (2 \operatorname{Im} z)^{-1}$$

if h is a conformal map of the upper half-plane H onto A. The set $Q(A)$ of the Schwarzian derivatives S_f of $f \in M(A)$ is a Banach space with the norm

$$\|S_f\|_A = \sup \{|S_f(z)|\varrho_A(z)^{-2}| \, z \in A\}.$$

The size of $\|S_f\|_A$ is connected to the global injectiveness of f. If $\|S_f\|_A=0$, f is a Möbius transformation and hence univalent. If A is Möbius equivalent to a disc, $\|S_f\|_A \leqq 2$ implies that f is univalent and $\|S_f\|_A > 6$ implies that f is not univalent [8, 4]. Results due to Ahlfors [1] and Gehring [2] show that if A is a quasidisc, then there exists a positive number b such that $\|S_f\|_A \leqq b$ implies that f is univalent and has a quasiconformal extension to the plane, and that this property characterizes quasidiscs.

Motivated by these results, Lehto [6] defined the inner radius of univalency $\sigma_I(A)$ of A as the supremum (or maximum) of numbers b such that every $f \in M(A)$ satisfying $\|S_f\|_A \leqq b$ is univalent. If A is a quasidisc and $T(A) \subset Q(A)$ is the universal Teichmüller space of A, i.e., the set of S_f such that f has a quasiconformal extension to the plane, then $\sigma_I(A)$ is the radius of the largest ball centered at the origin of $Q(A)$ and contained in $T(A)$. Equivalently, $\sigma_I(A)$ is the distance of S_h from the boundary of $T(H)$. Relatively little is known about the actual value of $\sigma_I(A)$ for a given domain A. The classical results of Nehari [8] and Hille [3] imply that $\sigma_I(A)=2$ for any domain A Möbius equivalent to a disc, and Lehto [7] and the author [5] have observed that $\sigma_I(A)<2$ for all other simply connected domains. Denote by A_k

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the angular domain $\{z||\arg z|<k\pi/2\}$. If $0<k\leqq 1$, one easily computes $\sigma_I(A_k)=$ $=2k^2$. As pointed out by Lehto [7], the method employed by Ahlfors in proving that a quasidisc has a positive radius of univalency can be used to obtain explicit lower estimates for $\sigma_I(A)$, once a differentiable quasiconformal reflection of A, i.e., a sense-reversing quasiconformal $\lambda\colon A\to A^*$, $A^*=\bar{C}\setminus\bar{A}$, fixing the boundary points of A, is known. More precisely,

$$\sigma_I(A) \geqq 2 \inf_{z\in A} \frac{|\lambda_{\bar{z}}(z)|-|\lambda_z(z)|}{|\lambda(z)-z|^2\varrho_A(z)^2}. \tag{1}$$

For an obtuse angle A_k, $1<k<2$, the natural reflection

$$\lambda(z) = -z(\bar{z}/z)^{1/k}$$

yields

$$\sigma_I(A_k) \geqq 4k-2k^2 \tag{2}$$

[7], where in fact equality holds [5]. Taking

$$E_r = \{z|\,(r\operatorname{Re} z)^2+(\operatorname{Im} z)^2 > 1\}, \quad 0<r\leqq 1,$$

to be the outside of an ellipse with half-axes $1/r$, 1, (1) applied to the reflection

$$\lambda(z) = ((1-r)/w+(1+r)/\bar{w})/(2r)$$

with

$$z = ((1+r)w+(1-r)/w)/(2r)$$

gives

$$\sigma_I(E_r) \geqq 8r^2/(1+r)^2 \tag{3}$$

[7], which is asymptotically sharp for $r\to 0$ and $r\to 1$. (The last assertion is obvious, and if $\lim_{r\to 0}\sigma_I(E_r)=0$ were not true, there would exist a sequence (r_n), $r_n\to 0$, such that each E_{r_n} would be a K-quasidisc with a fixed K [2].)

2. Domains bounded by a branch of a hyperbola

There are simple cases in which a straightforward application of (1) produces no results. Consider, for example, the domain

$$G = \{z|\operatorname{Re} z<0 \quad \text{or} \quad (\operatorname{Re} z)^2-(\operatorname{Im} z)^2<1\}$$

bounded by a rectangular hyperbola. A natural 3-quasiconformal reflection $\lambda\colon G\to G^*$, continuously differentiable outside the asymptote rays of ∂G, is given by the formulas

$$\lambda(z) = (2-\bar{z}^2)^{1/2}$$

for those z which lie in the domain bounded by ∂G and the asymptotes of ∂G, and

$$\lambda(z) = (2-e^{i\pi/3}\bar{z}^{4/3}z^{2/3})^{1/2}$$

for $\pi/4<\arg z<7\pi/4$. For this λ, however, the right-hand side of (1) equals zero.

For $c>0$, set

$$G_c = \{z|\ \mathrm{Re}\, z < 0 \quad \text{or} \quad (c\,\mathrm{Re}\, z)^2-(\mathrm{Im}\, z)^2 < 1\}.$$

Information on $\sigma_I(A_k)$ can be used to compute $\sigma_I(G_c)$ utilizing an idea introduced in [5]:

Theorem 1. *For every positive c,*

$$\sigma_I(G_c) = (8/\pi)\arctan c-(8/\pi^2)(\arctan c)^2.$$

Proof. Let k be an arbitrary number satisfying $2\arctan c<k\pi<\pi$. Draw a circular arc in G_c, with both endpoints on the real axis and meeting the real axis at angle $k\pi/2$, tangent to ∂G_c at a point z_0. The infinite domain T_k bounded by this arc and its mirror image in the real axis is Möbius equivalent to A_{2-k}, and $G_c\subset T_k$. By [5], a conformal map $f\colon T_k\to A_{2-k}\cap\{z|1-z\in A_{2-k}\}$ satisfying $f(z_0)=f(\bar{z}_0)=\infty$ and $\|S_f\|_{T_k}=4(2-k)-2(2-k)^2=4k-2k^2$ exists. Since $\varrho_{G_c}(z)>\varrho_{T_k}(z)$ for all $z\in G_c$, $\|S_f\|_{G_c}\leqq 4k-2k^2$. Also, $f(G_c)$ is not a Jordan domain. Consequently,

$$\sigma_I(G_c) \leqq (8/\pi)\arctan c-(8/\pi^2)(\arctan c)^2. \tag{4}$$

To prove the opposite inequality, observe that for every $\varepsilon>0$, there exists an $f\in M(G_c)$ such that $\|S_f\|_{G_c}<\sigma_I(G_c)+\varepsilon$ and f is not univalent. Assume $f(z_1)=f(z_2)$, $z_1\neq z_2$. Then either z_1 and z_2 are in the closure of a disc or half-plane $U\subset G_c$ or in the closure of an angular domain $U\subset G_c$ Möbius equivalent to A_{2-k} for some k, $2\arctan c<k\pi<\pi$. (U is bounded, for example, by tangents to ∂G_c through z_1 and z_2.) Since f is not univalent in the closure of U, $\|S_f\|_U\geqq\sigma_I(U)\geqq 4k-2k^2$. Again taking into account the monotonicity of the Poincaré metric with respect to domain, one gets $\|S_f\|_{G_c}\geqq\|S_f\|_U$, and the inequality opposite to (4) follows.

We next consider the domain G_c^*, complementary to G_c. Reasoning as above, we obtain

Theorem 2. *For every positive c,*

$$\sigma_I(G_c^*) = (8/\pi^2)(\arctan c)^2.$$

Proof. Let $k\pi=2\arctan c$. Then $G_c\subset A_k$. The function g, $g(z)=\log z$, maps A_k onto an infinite horizontal strip of width $k\pi$, and $g(\partial G_c^*)$ is not a quasicircle. Since $\|S_g\|_{A_k}=2k^2$, $\sigma_I(G_c)\leqq 2k^2$. On the other hand, assume $f\in M(G_c)$ with $\|S_f\|_{G_c}<\sigma_I(G_c)+\varepsilon$ satisfies $f(z_1)=f(z_2)$. Then, considering the rays parallel to the asymptotes of ∂G_c^* which join z_1 and z_2 to ∞ in G_c^* and the segment joining z_1 to z_2, we observe that z_1 and z_2 are on the boundary of a finite domain U bounded by a segment and a circular arc meeting at an angle at least $k\pi$. Since U is Möbius equivalent to some $A_{k'}$, $k' \geqq k$, we obtain $\sigma_I(G_c) \geqq 2k^2$ as in the proof of Theorem 1.

3. The exterior of an ellipse

The method employed above for domains bounded by a branch of a hyperbola does not yield the exact value of $\sigma_I(E_r)$. However, (3) can be improved considerably, and an upper bound can be established, too.

Theorem 3. *For every* r, $0<r\leqq 1$,

$$(16/\pi)\arctan q-(32/\pi^2)(\arctan q)^2 \leqq \sigma_I(E_r) \tag{5}$$

$$\leqq (16/\pi)\arctan r-(32/\pi^2)(\arctan r)^2,$$

where $q=r/(2-r^2)^{1/2}$.

Proof. Denote by T_r the infinite domain bounded by two circular arcs through $1/r$, i, $-1/r$ and $1/r$, $-i$, $-1/r$, respectively. Then $E_r \subset T_r$, $\{i, -i\} \subset \partial T_r \cap \partial E_r$, and T_r is, by elementary geometry, Möbius equivalent to A_k, $k=2-(4/\pi)\arctan r$. By [5], a conformal map of T_r onto a non-quasidisc exists such that $f(i)=f(-i)$ and $\|S_f\|_{T_r}=4k-2k^2$. As in Theorem 1, we deduce $\sigma_I(E_r)\leqq 4k-2k^2$, or the right-hand side of (5).

To prove the left-hand side of (5), assume $f\in M(E_r)$, $\|S_f\|_{E_r}<\sigma_I(E_r)+\varepsilon$, and $f(z_1)=f(z_2)$, $z_1\neq z_2$. Consider discs D_j, $j=1, 2$, such that $E_r^*\subset D_j$, $z_j\in\partial D_j$ and ∂D_j touches ∂E_r either at two points symmetric with respect to the imaginary axis or one of the points i, $-i$. If $z_1\notin D_2$ or $z_2\notin D_1$, both z_1 and z_2 belong to the closure of a domain $U\subset E_r$ such that $\sigma_I(U)=2$. Otherwise, set $U=(D_1\cap D_2)^*$. Then, by an elementary geometric argument, U is Möbius equivalent to an A_k, $1<k<2-(4/\pi)\arctan q$. In each case, f maps U onto a domain not bounded by a Jordan curve or f is not injective. Hence $\|S_f\|_{E_r}\geqq\|S_f\|_U\geqq 4k-2k^2$, and the left-hand side of (5) follows.

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