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ANALYTIC SURFACES IN C^2 AND THEIR LOCAL HULL OF HOLOMORPHY

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1. Introduction

a) We consider real analytic surfaces M (i.e. $\dim_R M=2$) in C^2. Two such surfaces M, $\tilde{M}$ are called equivalent if they can be mapped into each other by a mapping which is biholomorphic in the complex structure of C^2. Actually, we are concerned only with local equivalence which refers to the neighbourhood of a point p of M.

One has to distinguish points p for which the tangent space $T_pM=V$ is a complex line (i.e. $V=iV$) and those for which V is totally real (i.e. $V\cap iV=(0)$). Points of the second type are uninteresting since any two surfaces near such points are locally equivalent. Interesting are only the so-called "exceptional points" which were considered already by E. Bishop [1]. They are characterized as points p with a complex tangent while for every $q\in M$ in a neighborhood of p T_qM is totally real. Near such points one finds holomorphic invariants. For example, in the non-degenerate case, where also the quadratic approximation at p has an exceptional point, one can introduce local coordinates $(z, w)\in C^2$ so that the surface is given by

$$w = F(z,\bar{z}) = z\bar{z}+\gamma(z^2+\bar{z}^2)+E(z,\bar{z}) \tag{1.1}$$

where p corresponds to $z=w=0$, $\gamma\geqq 0$ and where E is a complex valued analytic function vanishing of at least third order at the origin. Then γ is the holomorphic invariant introduced by Bishop. One speaks of the elliptic, parabolic and hyperbolic case according to $\gamma\in[0, 1/2)$, $\gamma=1/2$ of $\gamma>1/2$.

The elliptic case has been studied in detail in [5] where, however, the case $\gamma=0$ had to be excluded. We showed in particular, that for $0<\gamma<1/2$ the surface (1.1) is locally equivalent to an algebraic surface

$$w = z\bar{z}+\Gamma(w)(z^2+\bar{z}^2) = G(z,\bar{z}),$$

$$\Gamma = \gamma+\varepsilon w^s \quad \text{with} \quad \varepsilon = \pm 1 \quad \text{or} \quad 0.$$

This surface lies in the three-dimensional hyperplane $\operatorname{Im} w=0$ and forms a surface with elliptical cross sections. It is easily seen that the local hull of holomorphy of

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this surface is given by the interior of this surface, i.e.

$$\{z, w, \operatorname{Im} w = 0, \quad \operatorname{Re} w \leqq G(z, \bar{z})\}$$

and therefore is an analytic three-dimensional surface with boundary.

b) In this paper we study the case $\gamma=0$ which seems particularly simple since its quadratic approximation $w=z\bar{z}$ is given by the rotation-symmetrical paraboloid in $\operatorname{Im} w=0$. But one meets with unexpected difficulties and it is not yet known whether also in this case the local hull of holomorphy is real analytic at p.

According to the work of Kenig and Webster [4] it is established that this hull of holomorphy is smooth but its analytic character is still not known. To this question this paper makes a contribution which we now formulate.

By formal considerations, carried out in Section 2, one sees that in the case $\gamma=0$ the surface (1.1) can be brought in the form

$$w = z\bar{z} + z^s + \bar{z}^s + O_{s+1} \tag{1.2}$$

where $s \geqq 3$ and O_{s+1} denotes a function vanishing of order $\geqq s+1$ at the origin, or into the form

$$w = z\bar{z} + O_n \tag{1.3}$$

for any n. In the second case we set $s=\infty$; in this case (1.1) can *formally* be transformed into $w=z\bar{z}$ where, however, the convergence of this transformation still remains in question. We point out that s is also a holomorphic invariant and therefore the distinction between the cases (1.2), (1.3) has invariant meaning.

It was our goal to decide about the analytic character of the hull of holomorphy in these cases but so far we have not been able to settle the case $s<\infty$. But since birthdays can not be postponed we can only present our partial result for $s=\infty$ and then formulate the open question as a rather specific problem.

Theorem. *A surface M of the form* (1.1) *with $\gamma=0$ and $s=\infty$ is holomorphically equivalent to*

$$w = z\bar{z}. \tag{1.4}$$

In other words, if (1.1) can formally be transformed into the form (1.4) then also holomorphically. As a consequence, also in this case the hull of holomorphy is a locally real analytic three dimensional hypersurface with boundary.

Problem. In the case $s<\infty$ such a statement is not known to us and is possibly false. If the hull of holomorphy is analytic we can by a transformation

$$(z, w) \to (z, w + g(z, w))$$

map it into $\operatorname{Im} w=0$ (see [2]) and this amounts to a solution of the functional equation

$$\operatorname{Im} \{F(z, \bar{z}) + g(z, F)\} = 0. \tag{1.5}$$

As a matter of fact the question whether the hull of holomorphy is analytic at the origin is equivalent to the question whether (1.5) possesses a solution $g=g(z, w)$ holomorphic at the origin, which we state as an open problem.

Another question is whether one can decide from a finite part of the Taylor expansion of F about the equivalence class to which M belongs. For elliptic points with $0<\gamma<1/2$ this is indeed the case since γ, s and $\varepsilon=\pm 1$ are a full set of invariants [5]. However, for $\gamma=0$ this has not been established and may even be false. One knows of other problems of this type from the theory of iteration of conformal mappings $z\to f(z)$ near a fixed point $0=f(0)$. Here two mappings f_1, f_2 are called equivalent if there exists a conformal mapping $z\to u(z)$ with $u(0)=0$, $u'(0)\neq 0$ with $f_1 \circ u = u \circ f_2$. If $|f'(0)|\neq 0, 1$ then $f'(0)=\lambda$ is the only invariant and the mapping is equivalent to a linear mapping. This is also true for most values of λ on the unit circle. However, for roots of unity λ, the equivalence class is not determined by a finite part of the Taylor expansion [3], [6]. Is the equivalence problem for surfaces near an elliptic point with $\gamma=0$ of this nature? These questions of complex analysis lead to nonlinear functional equations with rather startling properties. For example, in the hyperbolic case one has to distinguish between cases where the solution of the quadratic equation $\lambda^2-\gamma^{-1}\lambda+1=0$ is a root of unity or not. One is led to difficult functional analytic problems. Even the proof of the stated theorem seems not quite straight-forward and we used the rapidly convergent iteration technique which seems well adapted to the situation. In the following sections we carry out the detailed proof of this theorem.

2. Formal considerations and a linear problem

We consider a real analytic two-dimensional manifold M in C^2 of the form

$$w = F(z, \bar{z}) = z\bar{z} + \dots \tag{2.1}$$

where F is a complex-valued power series in z, $\bar{z}$ without constant or linear part. At first we ignore questions of convergence and subject (2.1) to a transformation

$$\begin{aligned} z' &= z + f(z, w), \\ w' &= w + g(z, w) \end{aligned} \tag{2.2}$$

also given by formal power series in z, w without constant part and whose linear part is the identity transformation. We write these series as power series in z:

$$f = \textstyle\sum_{l=0}^{\infty} z^l f_l(w), \quad g = \sum_{l=0}^{\infty} z^l g_l(w) \tag{2.3}$$

and show the

Proposition 2.1. *There exists a unique formal transformation* (2.2) *satisfying*

$$f_0 = f_1 = 0 \tag{2.4}$$

which transforms (2.1) *into*

$$w' = F'(z', \bar{z}') = z'\bar{z}' + \varphi(z') + \bar{\varphi}(\bar{z}').$$

Proof. We have to solve the equations

$$F + g(z, F) = (z + f(z, F))(\bar{z} + \bar{f}(\bar{z}, \bar{F})) + \varphi(z + F) + \bar{\varphi}(\bar{z} + \bar{F}) \tag{2.5}$$

which is done by comparison of coefficients. Denoting the weight of a term $z^\nu w^\mu$ by $\nu + 2\mu = s$ we write $f^{(s)}(z, w)$, $g^{(s)}(z, w)$ for the terms of weight s in f, g, respectively. Thus we have $f^{(s)}(tz, t^2 w) = t^s f^{(s)}(z, w)$ and similarly for $g^{(s)}$. If we collect terms of degree s in z, $\bar{z}$ in the equation (2.5) we obtain for $s \geqq 3$

$$F^{(s)} + g^{(s)} = \bar{z} f^{(s-1)}(z, z\bar{z}) + z \bar{f}^{(s-1)}(\bar{z}, z\bar{z}) + \varphi^{(s)}(z) + \bar{\varphi}^{(s)}(\bar{z}) + G^{(s)}(z, z)$$

where $F^{(s)}$, $\varphi^{(s)}$ denotes the homogeneous terms of degree s in z, $\bar{z}$ of F, and $G^{(s)}(z, \bar{z})$ is also such a homogeneous polynomial which depends only on $g^{(\sigma)}$, $f^{(\sigma-1)}$, $\varphi^{(\sigma)}$ for $\sigma < s$. Thus we have to solve the equation

$$-g^{(s)} + \bar{z} f^{(s-1)} + z \bar{f}^{(s-1)} + \varphi^{(s)}(z) + \overline{\varphi^{(s)}}(\bar{z}) = \Gamma^{(s)}(z, \bar{z}) \tag{2.6}$$

for g, f, φ if Γ is given. For later purpose we consider this equation right a way for general series g, f, φ and Γ, without constant terms and not only their homogeneous parts.

Replacing $\bar{z}$ by a new independent variable ζ we are led to the equation

$$-g(z, z\zeta) + \zeta f(z, z\zeta) + z \bar{f}(\zeta, z\zeta) + \varphi(z) + \bar{\varphi}(\zeta) = \Gamma(z, \zeta). \tag{2.7}$$

We obtain again (2.6) by considering just the homogeneous terms of degree s in this equation.

Therefore it suffices to prove that (2.7) possesses a unique solution with the normalization (2.4). We write the series for Γ

$$\Gamma(z, \zeta) = \sum \gamma_{\alpha\beta} z^\alpha \zeta^\beta$$

in the form

$$\Gamma = \Gamma_0(z\zeta) + \sum_{l=1}^{\infty} (z^l \Gamma_l(z\zeta) + \zeta^l \Gamma_{-l}(z\zeta))$$

where

$$\Gamma_l(w) \begin{cases} = \sum_{k=0}^{\infty} \gamma_{k+l,k} w^k & \text{for} \quad l \geqq 0, \\ = \sum_{j=0}^{\infty} \gamma_{j,j-l} w^j & \text{for} \quad l \leqq 0. \end{cases}$$

With the representation (2.3) for f, g and with

$$\varphi(z) = \sum_{l=1}^{\infty} \varphi_l z^l, \quad \varphi_l \text{ constants},$$

the equation (2.7) takes the form

$$\begin{aligned} -g_l(w) + w f_{l+1}(w) + \varphi_l &= \Gamma_l, \\ w \bar{f}_{l+1}(w) + \bar{\varphi}_l &= \Gamma_{-l} \end{aligned} \tag{2.8}$$

for $l \geqq 2$ and for $l=0, 1$ we obtain the three equations

$$\begin{aligned} -g_1(w)+wf_2+\bar{f}_0+\varphi_1 &= \Gamma_1, \\ -g_0(w)+wf_1+w\bar{f}_1 &= \Gamma_0, \\ f_0+w\bar{f}_2+\bar{\varphi}_1 &= \Gamma_{-1}. \end{aligned}$$

With the normalization $f_0=f_1=0$ the first and third equation can be incorporated into (2.8) for $l \geqq 1$ and the middle equation takes the form

$$-g_0(w) = \Gamma_0(w). \tag{2.9}$$

This equation is uniquely solved by defining $g_0=-\Gamma_0$. From (2.8) we obtain for $w=0$

$$-g_l(0)+\varphi_l = \Gamma_l(0), \quad \bar{\varphi}_l = \Gamma_{-l}(0) \quad \text{for} \quad l \geqq 1$$

which again is uniquely solvable. Setting

$$G_l(w) = (\Gamma_l(w)-\Gamma_l(0))/w$$

it remains to solve the equations

$$\left.\begin{aligned} g_l(0)-g_l(w)+wf_{l+1} &= wG_l \\ w\bar{f}_{l+1} &= wG_{-l} \end{aligned}\right\} \quad \text{for} \quad l \geqq 1.$$

Hence we get

$$\begin{aligned} \bar{f}_{l+1} &= G_{-l}, \\ g_l(0)-g_l(w) &= w(G_l-\bar{G}_{-l}) \end{aligned}$$

and with $g_l(0)=-\Gamma_l(0)+\bar{\Gamma}_{-l}(0)$ we obtain

$$-g_l(w) = \Gamma_l(w)-\bar{\Gamma}_{-l}(w).$$

This shows indeed that (2.8), (2.9) can be solved uniquely with the normalization (2.4) and Proposition 2.1 is proven.

For later purposes we record the solutions f, g, φ of the equation (2.7) in a different form:

$$\begin{cases} \bar{\varphi}(\zeta) = \Gamma(0, \zeta), \\ \bar{f}(\zeta, w) = \sum_{l \geqq 1} \zeta^{l+1} G_{-l}(w), \\ g(z, w) = -\Gamma_0(w)-\Gamma_+(z, w)+\bar{\Gamma}_-(z, w) \end{cases} \tag{2.10}$$

where

$$\Gamma_+(z, w) = \sum_{l \geqq 1} z^l \Gamma_l(w), \quad \Gamma_-(\zeta, w) = \sum_{l \geqq 1} \zeta^l \Gamma_{-l}(w).$$

There is an arbitrariness in the transformation (2.2) which was fixed by the normalization (2.4). This arbitrariness is due to the fact that the automorphism group of $w=z\bar{z}$ is given by

$$\begin{cases} z' = a(w)\dfrac{z-wb(w)}{1-z\bar{b}(w)}, \quad a(0)=1, \\ w' = a(w)\bar{a}(w)w \end{cases} \tag{2.11}$$

where $a(w)-1$, $b(w)$ are arbitrary series without constant terms. These formulae were given already at the end of [5], where also Proposition 2.1 was stated without proof.

This group still acts on the normal form

$$w = z\bar{z}+\varphi(z)+\bar{\varphi}(\bar{z})$$

provided by Proposition 2.1 and it is not easy to get a full set of invariants out of this representation. Here we are content with the remark that the number $s \geqq 3$ in

$$\varphi(z) = c_s z^s+\dots, \quad c_s \neq 0,$$

is obviously an invariant; if $\varphi \equiv 0$ we set $s=\infty$. Replacing z, w by $(\lambda z, |\lambda|^2 w)$, $\lambda \neq 0$, we can always achieve that the transformed representation is given by

$$w = z\bar{z}+z^s+\bar{z}^s+\psi(z)+\bar{\psi}(\bar{z})$$

with a power series ψ containing terms of order $>s$ only. Otherwise, if $s=\infty$ we can formally achieve

$$w = z\bar{z}. \tag{2.12}$$

This latter case $s=\infty$ may be considered as an exceptional case. But it is this case which we consider now and show that if $s=\infty$ the above normal form (2.12) can not only be achieved by formal series but even by convergent ones. This is the content of the theorem of the introduction.

3. Estimates

a) Before proving the theorem of the introduction we need some estimates concerning the solution of the linear problem solved in Section 2.

We write our surface M as

$$w = F(z, \bar{z}) = z\bar{z}+E(z, \bar{z})$$

where we assume that $E(z, \zeta)$ is holomorphic near $z=\zeta=0$ and vanishing of order $\geqq 3$ there. By a stretching transformation $(z, \zeta, w) \to (az, a\zeta, a^2 w)$, $E(z, \zeta)$ is replaced by $a^{-2}E(az, a\zeta)$ and we may therefore assume that E is holomorphic in the polydisc $|z|, |\zeta|<1$ and satisfies

$$\sup_{|z|, |\zeta|<1} |E(z, \zeta)| < \eta \tag{3.1}$$

for a given $\eta>0$. After this preparation we will construct a biholomorphic transformation to decrease this error term E to make it ultimately equal to zero.

The equations to be solved for the transformation (2.2) are

$$F(z, \bar{z})+g(z, F) = w' = \big(z+f(z, F)\big)\big(\bar{z}+\bar{f}(\bar{z}, \bar{F})\big)$$

or

$$-g(z, F)+\bar{z}f(z, F)+z\bar{f}(\bar{z}, \bar{F}) = E-|f(z, F)|^2. \tag{3.2}$$

We replace these nonlinear difference equations by the simplified linear equations

$$-g(z, z\bar{z})+\bar{z}f(z, z\bar{z})+z\bar{f}(\bar{z}, z\bar{z})+\varphi(z)+\bar{\varphi}(\bar{z}) = E \tag{3.3}$$

which was already considered in the previous Section (see (2.7)). We use (3.3) to define the transformation f, g (with the normalization $f_0=f_1=0$). It will not achieve that the error E is reduced to zero but it will diminish this error. Repeating such a procedure will lead to a convergent transformation.

b) To carry out this procedure we prove some estimates for the solutions of (3.3). We define the domains

$$D_r = \{(z, \zeta)\in C^2, \quad |z| < r, \ |\zeta| < r\},$$

$$\Delta_r = \{(z, w)\in C^2, \quad |z| < r, \ |w| < r^2\}$$

and define the norms

$$\|E\|_r = \sup_{D_r} |E(z, \zeta)|, \quad |f|_r = \sup_{\Delta_r} |f(z, w)|.$$

Proposition 3.1. *If* $E=E(z, \zeta)$ *holomorphic in* D_r, $E(0, 0)=0$ *and* $1/2<\varrho<r\leqq 1$ *then the equation* (3.3) *has a unique normalized solution* f, g, φ *holomorphic in* Δ_r *and satisfying the inequalities*

$$\begin{cases} |f|_\varrho, \ |g|_\varrho < c_1(r-\varrho)^{-1}\|E\|_r, \\ |f_z|_\varrho+|f_w|_\varrho+|g_z|_\varrho+|g_w|_\varrho \leqq c_1(r-\varrho)^{-2}\|E\|_r, \end{cases} \tag{3.4}$$

$$\sup_{|z|<r} |\varphi(z)| \leqq \|E\|_r \tag{3.5}$$

where c_1 *is an absolute constant (which we do not determine). Moreover,* f, g, φ *vanish for* $z=w=0$.

Proof. The solution of the equations (3.3) was already given in (2.10) where Γ must be replaced by $E(z, \zeta)$. From these equations it is evident that if E is holomorph in D_r then $f(z, w)$, $g(z, w)$ are holomorphic in

$$\{(z, w)\,|\,|z| < r, \quad |w| < |z|r\}$$

since $w=z\zeta$ and $|\zeta|<r$. By Hartog's theorem the hull of holomorphy of this domain is Δ_r and so f, g are holomorph in Δ_r and $\varphi(z)$ in $|z|<r$.

Evidently the estimate (3.5) follows from $\bar{\varphi}(\zeta)=E(0, \zeta)$. If we write again

$$E(z, \zeta) = E_0(z\zeta)+\textstyle\sum_{l\geqq 1} (z^l E_l(z\zeta)+\zeta^l E_{-l}(z\zeta))$$

we have

$$z^l E_l(z\zeta) = \frac{1}{2\pi_i}\int_0^{2\pi} E(e^{i\vartheta} z, e^{-i\vartheta}\zeta)e^{-il\vartheta}\, d\vartheta \quad \text{for} \quad l\geqq 0,$$

$$\zeta^l E_l(z\zeta) = \frac{1}{2\pi i}\int_0^{2\pi} E(e^{i\vartheta} z, e^{-i\vartheta}\zeta)e^{-il\vartheta}\, d\vartheta \quad \text{for} \quad l\leqq 0$$

and hence by Schwarz' lemma

$$|E_l(w)| \leqq r^{-|l|}\|E\|_r \quad \text{for} \quad |w| \leqq r^2.$$

Also from Schwarz' lemma we conclude for

$$G_l(w) = (E_l(w) - E_l(0))\, w^{-1}$$

the estimate

$$|G_l(w)| \leqq 2r^{-|l|-2}\|E\|_r \quad \text{for} \quad |w| < r^2.$$

Inserting this into (2.10) we obtain

$$|f|_\varrho \leqq \frac{2}{r}\sum_{l=1}^{\infty}\left(\frac{\varrho}{r}\right)^l \|E\|_r \leqq \frac{2}{r-\varrho}\|E\|_r,$$

$$|g|_\varrho \leqq \left(1+2\sum_{l=1}^{\infty}\left(\frac{\varrho}{r}\right)^l\right)\|E\|_r \leqq \frac{2}{r-\varrho}\|E\|_r.$$

The estimates for the derivatives are easily derived from these with the help of Cauchy estimate: Set $\tau=(r+\varrho)/2$ and use

$$|f|_\tau \leqq \frac{2}{r-\tau}\|E\|_r$$

and by Cauchy's estimate

$$|f_z|_\varrho \leqq \frac{1}{\tau-\varrho}|f|_\tau \leqq \frac{2\|E\|_r}{(\tau-\varrho)(r-\tau)} = \frac{8\|E\|_r}{(r-\varrho)^2}.$$

This proves the inequalities (3.4), and Proposition 3.1.

c) For any formal power series E we define $\operatorname{ord} E$ as the lowest order of the nonvanishing terms. We set $\operatorname{ord} E=\infty$ if $E\equiv 0$.

Proposition 3.2. *If* $w=z\bar{z}+E(z,\bar{z})$ *satisfies* $s=\infty$ *and*

$$\operatorname{ord} E \geqq d$$

then the transformed surface $w'=z'\bar{z}'+E'$ *obtained by the transformation* (2.2), *given by* (3.3), *satisfies*

$$\operatorname{ord} E' \geqq 2d-2.$$

Proof. If we solve (3.3) we obtain solutions f, g for which the weight is $\geqq d-1, d$, respectively. The error term E' as a function of $z, \bar{z}$ consists of the differences between the terms in (3.2) and (3.3), e.g.

$$g(z, F) - g(z, z\bar{z})$$

of order $\geqq (d-2)+d=2d-2$. Similarly

$$\operatorname{ord}\left(\zeta f(z, F) - \zeta f(z, z\zeta)\right) \geqq 1+(d-3)+d = 2d-2$$

and

$$\operatorname{ord} |f|^2 \geqq 2(d-1).$$

From this one concludes that

$$\operatorname{ord}\left(E'-\varphi(z')-\bar{\varphi}(\bar{z}')\right) \geqq 2d-2$$

and since $s=\infty$ also $\operatorname{ord}\varphi \geqq 2d-2$ which yields the statement. In other words the transformation constructed by solving (3.3) has the effect of almost doubling the order of vanishing on E.

d) With the so-constructed transformation (2.2): $\psi:(z', w')\to(z, w)$ we write the transformed surface $M'=\psi^{-1}M=\{(z', w')|\psi(z', w')\in M\}$ in the form

$$w' = F'(z', \bar{z}') = z'\bar{z}'+E'(z', \bar{z}')$$

and estimate E' in $D_{r'}$. To obtain a formula for E' we return to (3.2) but include the term E' and replace $\bar{z}$ by ζ to get $E'(z', \zeta')-g(z, F)+\zeta f(z, F)+z\bar{f}(\zeta, \bar{F})= E(z, \zeta)-f(z, F)\bar{f}(\zeta, \bar{F})$. Subtracting the equation (3.3) which served to define f, g, φ we obtain

(3.6) $$E'(z', \zeta') = Q(z, \zeta)$$

where

(3.7) $$Q(z, \zeta) = \left(g(z, F)-g(z, z\zeta)\right)-\zeta\left(f(z, F)-f(z, z\zeta)\right)$$
$$-z\left(\bar{f}(\zeta, \bar{F})-\bar{f}(\zeta, z\zeta)\right)-f(z, F)\bar{f}(\zeta, \bar{F})+\varphi(z)+\bar{\varphi}(\zeta).$$

The relation between (z, ζ) and (z', ζ') is given by

(3.8) $$\begin{cases} z' = z+f\left(z, F(z, \zeta)\right) \\ \zeta' = \zeta+\bar{f}\left(\zeta, \bar{F}(\zeta, z)\right). \end{cases}$$

In order to restrict the variables to appropriate domains we introduce in addition to r', r in

$$\tfrac{1}{2} < r' < r \leqq 1$$

the intermediate points $\sigma, \varrho\in(r', r)$ by setting

$$\varrho = r-\frac{1}{3}(r-r'), \quad \sigma = r-\frac{2}{3}(r-r')$$

so that

(3.9) $$r-\varrho = \varrho-\sigma = \sigma-r' = \frac{1}{3}(r-r').$$

Proposition 3.3. *There exists an absolute constant $\delta>0$ such that for*

(3.10) $$\frac{\|E\|_r}{(r-r')^2} < \delta$$

the above defined mapping $\psi\colon (z', w') \to (z, w)$ *takes* $\Delta_\sigma \to \Delta_\varrho$ *and takes* M *into* $\psi^{-1}M = M'$ *such that* $E'(z', \zeta')$ *is holomorphic in* $z', \zeta' \in D_{r'}$ *and satisfies*

$$\|E'\|_{r'} \leqq c_2 \|E\|_r \left\{ \frac{\|E\|_r}{(r-r')^2} + \left(\frac{r'}{r}\right)^{d/2} \right\} \tag{3.11}$$

provided that ord $E \geqq d$.

Proof. To verify the first statement we have to determine the inverse mapping $\psi = \Psi^{-1}$ of

$$\Psi\colon (z, w) \to \big(z' = z + f(z, w), \quad w' = w + g(z, w)\big).$$

By (3.4) we obtain for the Jacobian $d\Psi$ the estimate

$$|d\Psi - I|_\varrho < c_1 (r-\varrho)^{-2} \|E\|_r < \tfrac{1}{2}$$

if we require that $\delta < (18c_1)^{-1}$ in (3.10). By the standard iteration procedure for the inverse mapping we obtain for $(z, w) = \Psi^{-1}(z', w')$ the estimate

$$\begin{aligned} |z - z'| + |w - w'| &< 2\big(|f(z', w')| + |g(z', w'|\big) \\ &< 2c_1 (r-\varrho)^{-1} \|E\|_r < r - \varrho = \varrho - \sigma \end{aligned}$$

if $(z, w) \in \Delta_\varrho$. Hence $|z'| < \sigma$, $|w'| < \sigma^2$ implies

$$|z| < \sigma + (\varrho - \sigma) = \varrho, \quad |w| < \sigma^2 + (\varrho - \sigma) < \sigma^2 + (\varrho^2 - \sigma^2) = \varrho^2,$$

i.e. $\psi = \Psi^{-1}$ maps Δ_σ into Δ_ϱ as claimed.

In just the same way we can solve the equations (3.8) for $(z, \zeta) \in D_\sigma$ if (z', ζ') is given in $D_{r'}$. Indeed, for $(z, \zeta) \in D_\sigma$ we have

$$|F(z, \zeta)| \leqq \sigma^2 + \|E\|_r < \varrho^2$$

if $\|E\|_r < \varrho^2 - \sigma^2$ which follows again from (3.10). Therefore we obtain for $f(z, F(z, \zeta))$, $\bar f(\zeta, \bar F(\zeta, z))$, $(z, \zeta) \in D_\sigma$ the same bounds as in the first line of (3.4). Using Cauchy's estimate we find for the first derivatives of these functions a bound

$$c_3 \|E\|_r (r - r')^{-2} < c_3 \delta$$

in D_σ. We conclude that for sufficiently small δ and for $(z', \zeta') \in D_{r'}$ the equations (3.8) possess a solution (z, ζ) in D_σ. Therefore from (3.6) we find

$$\|E'\|_{r'} \leqq \|Q\|_\sigma.$$

To estimate $\|Q\|_\sigma$ we consider the various terms of Q in (3.7). For the first parenthesis we find for $(z, \zeta) \in D_\sigma$

$$|g(z, F) - g(z, z\zeta)| \leqq \sup_{\Delta_\varrho} |g_w| \|E\|_\sigma \leqq c_1 (r-\varrho)^{-2} \|E\|_r^2$$

where we used (3.4). The terms of the second and third parenthesis in (3.7) can be appraised similarly, and the next term gives

$$|f(z, F) \bar f(\zeta, \bar F)| \leqq c_1^2 (r-\varrho)^{-2} \|E\|_r^2.$$

For the last term we use $\varphi(z)=\bar{E}(z, 0)$ so that

$$|\varphi(z)| \leqq \|E\|_r \quad \text{for} \quad |z| < r.$$

Since $\operatorname{ord} E \geqq d$ also φ vanishes of order $\geqq d$ and Schwarz lemma yields

$$|\varphi(z)| \leqq \left(\frac{\sigma}{r}\right)^d \|E\|_r \quad \text{for} \quad |z| < \sigma$$

so that

$$\|Q\|_\sigma \leqq c_4 \left\{(r-\varrho)^{-2}\|E\|_r^2 + \left(\frac{\sigma}{r}\right)^d \|E\|_r\right\}.$$

To obtain the claimed inequality (3.11) we replace $r-\varrho$ by $(r-r')/3$ and use

$$\left(\frac{\sigma}{r}\right)^2 \leqq \frac{r'}{r}$$

which follows from (3.9). Indeed with $\theta = r'/r$ we have

$$\left(\frac{\sigma}{r}\right)^2 \Big/ \frac{r'}{r} = (1+2\theta)^2/9\theta \leqq 1 \quad \text{for} \ \frac{1}{2} < \theta \leqq 1.$$

This completes the proof of Proposition 3.3.

4. Convergence proof

a) We construct the desired transformations of M into the form $w = z\bar{z}$ by repeatedly applying Proposition 3.3. This will be done by the rapidly convergent iteration technique which we now describe. For this purpose we consider a sequence of such surfaces

$$M_\nu\colon\ w = F_\nu(z, \bar{z}) = z\bar{z} + E_\nu$$

where M_0 agrees with the given surface M, i.e. $F = F_0$. Moreover, all surfaces are supposed to be equivalent: $M_{\nu+1} = \psi_\nu^{-1} M_\nu$ where ψ_ν is a biholomorphic mapping taking $\Delta_{\sigma_\nu} \to \Delta_{\varrho_\nu}$. Here $\sigma_\nu < \varrho_\nu$ is a sequence of numbers which will be chosen so as to make Proposition 3.3 applicable: We pick a sequence r_ν such that

(4.1) $$\tfrac{1}{2} < r_{\nu+1} < r_\nu < r_0 = 1$$

and set

(4.2) $$\varrho_\nu = r_\nu - \frac{1}{3}(r_\nu - r_{\nu+1}); \quad \sigma_\nu = r_\nu - \frac{2}{3}(r_\nu - r_{\nu+1})$$

so that r, r', ϱ, σ in Proposition 3.3 correspond to $r_\nu, r_{\nu+1}, \varrho_\nu, \sigma_\nu$. Our goal will be to select the sequence so that

(4.3) $$\|E_\nu\|_{r_\nu} \to 0 \quad \text{for} \quad \nu \to \infty$$

and that

$$\Psi_\nu = \psi_0 \circ \psi_1 \circ \ldots \circ \psi_{\nu-1}\colon\ \Delta_{\sigma_{\nu-1}} \to \Delta_{\varrho_0}$$

converges in $\Delta_{1/2}$ to a biholomorphic map Ψ_*. Then it follows from (4.3) that $\Psi_*^{-1}M = M_*$ agrees with $w = z\bar{z}$.

b) To define the above sequence F_ν, ψ_ν, r_ν we set, for example,

$$r_\nu = \frac{1}{2}\left(1+\frac{1}{\nu+1}\right), \quad \nu = 0, 1, \ldots \tag{4.4}$$

so that (4.1) holds and

$$\begin{cases} (r_\nu - r_{\nu+1})^{-1} = 2(\nu+1)(\nu+2), \\ \dfrac{r_{\nu+1}}{r_\nu} = 1 - \dfrac{1}{(\nu+2)^2}. \end{cases} \tag{4.5}$$

The point is that $r_\nu - r_{\nu+1}$ does not grow too fast. We set $F_0 = F$ and define ψ_0, F_1 by applying Proposition 3.3 to $M\colon w = F_0$ obtaining $M_1 = \psi_0 M_0$, $w = F_1$ and more generally $M_{\nu+1} = \psi_\nu M_\nu$, $w = F_\nu = z\bar{z} + E_\nu$. By Proposition 3.2 and since $s = \infty$ we find that

$$\operatorname{ord} E_\nu \geqq d_\nu = 2^\nu + 2 \quad \text{for} \quad \nu \geqq 0. \tag{4.6}$$

To justify the procedure we have to verify that the assumption (3.10) holds for all $\nu \geqq 0$. For this purpose we set

$$\varepsilon_\nu = (r_\nu - r_{\nu+1})^{-2} \| E \|_{r_\nu} \tag{4.7}$$

and rewrite (3.11) in the form

$$\varepsilon_{\nu+1} \leqq \left(\frac{r_\nu - r_{\nu+1}}{r_{\nu+1} - r_{\nu+2}}\right)^2 c_2 \varepsilon_\nu \left(\varepsilon_\nu + \left(\frac{r_{\nu+1}}{r_\nu}\right)^{d_\nu/2}\right).$$

This implies

$$\varepsilon_{\nu+1} \leqq c\varepsilon_\nu (\varepsilon_\nu + \lambda_\nu) \tag{4.8}$$

with $c = 3^2 c_2$ and where, by (4.6),

$$\lambda_\nu = \left(1 - \frac{1}{(\nu+2)^2}\right)^{d_\nu/2}$$

tends to zero.

From (4.8) one can show that ε_ν tends to zero faster than any exponential $e^{-\alpha\nu}(\alpha > 0)$, if ε_ν is sufficiently small. For our purpose it suffices that

$$\varepsilon_\nu \leqq \varepsilon_0 c_5 2^{-\nu} \quad \text{for} \quad 0 < \varepsilon_0 < c_6^{-1}. \tag{4.9}$$

To verify this inequality we may assume $c > 1$ in (4.8). We choose N so large that $\lambda_\nu < (4c)^{-1}$ for $\nu \geqq N$ and take ε_0 so small that

$$\varepsilon_0 < (4c)^{-1}(2c)^{-N}.$$

Then one has for $\nu \leqq N$, using $\lambda_\nu, \varepsilon_\nu < 1$,

$$\varepsilon_\nu < (2c)^\nu \varepsilon_0 \leqq (2c)^N \varepsilon_0 < 1$$

and for $\nu > N$ one finds, using $\lambda_\nu, \varepsilon_\nu < (4c)^{-1}$,

$$\varepsilon_\nu < 2^{-\nu+N}\varepsilon_N \leqq 2^{-\nu}(4c)^N \varepsilon_0 < \frac{1}{4c}.$$

The last two inequalities imply (4.9) with $c_5 = (4c)^N$, $c_6 = 4c(2c)^N$.

The assumption (3.10) of the Proposition 3.3 follows now from (4.9) if $\varepsilon_0 < \min(\delta c_5^{-1}, c_6^{-1}) = \delta^*$ which, in turn, follows from our original assumption (3.1) if $\eta = \delta^*/16$, since $r_0 - r_1 = 1/4$. Also (4.3) follows readily from (4.7) and (4.9). Finally, the convergence of $\Psi_\nu = \psi_0 \circ \psi_1 \circ \ldots \circ \psi_{\nu-1}$ in $\Delta_{1/2}$ follows from the convergence of the product of the Jacobians

$$\Pi_{\nu=0}^{\infty} |d\psi_\nu|_{\Delta_{\varrho_\nu}} \leqq \Pi_{\nu=0}^{\infty}(1 + c_1\varepsilon_\nu).$$

This finishes the proof of the theorem.

c) The mapping $\Psi = \lim_{\nu\to\infty} \Psi_\nu$ defines a biholomorphic mapping and its inverse $\psi = \Psi^{-1}$

$$\psi\colon \begin{array}{l} z' = z + f(z, w) \\ w' = w + g(z, w) \end{array}$$

takes M into the desired form $w' = z'\bar{z}'$. Of course, ψ need not satisfy the normalization condition and we want to show that also the normalized series converges. For this purpose we replace ψ by $\tilde{\psi} = \psi_{ab} \circ \psi$ where ψ_{ab} denotes the mapping (2.11) of the automorphism group of $w = z\bar{z}$. One can always determine holomorphic functions $a(w)$, $b(w)$ with $a(0) = 1$, $b(w) = 0$ so that $\tilde{\psi}$ is normalized. We first set $a \equiv 1$, and determine $b(w)$ so that

$$\tilde{f}_0(w) = \tilde{f}(0, w) = 0.$$

This equation becomes

$$f(0, w) = Wb(W) \quad \text{where} \quad W = w + g(0, w).$$

Solving the second equations for $w = w(W) = W + \ldots$ we obtain $b(W)$, which is holomorph and vanishes at the origin since $\operatorname{ord} f \geqq 2$. Now assuming $f_0 = 0$ we set $b \equiv 0$ and determine $a = a(w)$ so that

$$\tilde{f}_1(w) = \tilde{f}_z(0, w) = 0.$$

Because of

$$1 + \tilde{f}_z(0, w) = a(W)(1 + f_z(0, w)), \quad W = w + g(0, w)$$

this amounts to

$$a(W) = (1 + f_z(0, w))^{-1}$$

where $w = w(W)$, $a(W)$ are again holomorphic and $a(0) = 1$.

This shows that normalized series f, g which were determined in Section 2 are convergent if $s = \infty$. It is conceivable that the convergence can be established more directly.

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