ON THE STABILITY OF IDENTIFICATION PATTERNS
FOR DIRICHLET REGIONS

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1. Introduction

Let $G$ be a purely hyperbolic Fuchsian group of the first kind with genus $g \geq 1$ and let $D(z)$ be the Dirichlet region for $G$ with center $z$. It was proved by Beardon [2, Theorem 9.4.5] that for a.e. $z$ $D(z)$ has all cycles of length 3 and, hence, $12g - 6$ sides. Let $p(g)$ give the number of possible identification patterns. In [3] we showed that $p(2) = 8$. For $i = 1, \ldots, p(g)$ let

$$D_i = \{z | D(z) \text{ has } 12g - 6 \text{ sides and pattern } i\}.$$ 

It is obvious that the sets $D_i$ are invariant under $G$. Here it is shown that the sets $D_i$ are open and properties for their boundary curves are proved. Chapters 3 and 4 deal with the case $g = 2$. It is shown that the group of the regular octagon, introduced in [4, Theorem 5.1], only has pattern 6 and its degenerates as patterns for $D(z)$. Groups which change their pattern according to elementary moves [3, Chapter 5] are constructed in Chapter 4.

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2. The sets $D_i$ and $\partial D_i$

In [2, Theorem 9.4.5] it is shown that if $D(z)$ has less than $12g - 6$ sides, then there exist distinct $f_i \in G \setminus I$, $i = 1, 2, 3$, such that

$$(2.1) \quad \text{Im} [z, f_1(z), f_2(z), f_3(z)] = 0,$$

where the cross-ratio cannot be constant. Since $G$ has a countable number of transformations, the "exceptional" centers lie on a countable number of curves defined by (2.1). Each curve is algebraic, since (2.1) gives a polynomial of degree 8 in two real variables.

Definition 2.1. Let $f_i \in G \setminus I$ be distinct, $i = 1, 2, 3$, and let $\gamma$ be the algebraic curve corresponding to the condition (2.1). We define $\Gamma$ as the set of all curves $\gamma$. 

Remark 2.1. By the invariance of the cross-ratio
\[\{z | [z, f_1(z), f_2(z), f_3(z)] \in R\} = \{z | [h(z), h f_1(z), h f_2(z), h f_3(z)] \in R\}\]
for each Möbius transformation \(h\). Hence the set \(\Gamma\) could also be defined by replacing conditions of type (2.1) by conditions
\[\text{Im} [g_1(z), g_2(z), g_3(z), g_4(z)] = 0,\]
where \(g_i \in G\) are distinct, \(i=1, 2, 3, 4\).

Theorem 2.1. Each cycle of length 4 in \(D(z)\) corresponds to \(z\) lying on a curve \(\gamma \in \Gamma\).

Proof. Let \(P\) be a vertex of \(D(z)\) with the other vertices in its cycle \(g_1(P), g_2g_1(P), g_3g_2g_1(P)\). Then \(z, g_1^{-1}(z), (g_2g_1)^{-1}(z)\) and \((g_3g_2g_1)^{-1}(z)\) are concyclic with center \(P\). Because \(g_i\) is a Möbius transformation, this is equivalent to the points \(g_1(z), z, g_2^{-1}(z)\) and \((g_2g_1)^{-1}(z)\) being concyclic with center \(g_1(P)\). The algebraic curves given by
\[\text{Im} [z, g_1^{-1}(z), (g_2g_1)^{-1}(z), (g_3g_2g_1)^{-1}(z)] = 0\]
and
\[\text{Im} [z, g_2^{-1}(z), (g_3g_2)^{-1}(z), g_1(z)] = 0\]
are the same. A similar argument applies to the other vertices in the cycle of \(P\).

Theorem 2.2. If \(D(z)\) has one cycle of length 4 and the others of length 3, then in a neighbourhood \(U\) of \(z\), \(\bigcup_{i=1}^{4} \partial D_i \cap U = U \cap \gamma\), where \(\gamma \in \Gamma\), and for \(w \in U \cap \gamma\) the identification pattern of \(D(w)\) is the same as that of \(D(z)\), while when \(w\) crosses \(\gamma\) in \(U \setminus \{z\}\) a side-pairing transformation is changed in \(D(w)\).

Proof. If \(P\) is a vertex with cycle length 3, then by continuity and discreteness \(z\) has a neighbourhood in which \(D(w)\) has a vertex in a neighbourhood of \(P\) with the same cycle length and adjacent sides labelled by the same mappings as \(P\) has for \(D(z)\).

The condition for \(Q\) being a vertex of \(D(z)\) with cycle length 4 and with adjacent sides labelled by \(f_1, f_3\), the side \(f_2\) being degenerate at \(Q\), is, in hyperbolic distances,
\[\varrho(Q, z) = \varrho(Q, f_1(z)) = \varrho(Q, f_2(z)) = \varrho(Q, f_3(z)) < \varrho(Q, g(z))\]
for all \(g \in G \setminus \{I, f_1, f_2, f_3\}\) and \(f_2(z)\) lies on the circle with center \(Q\) on the part between \(f_1(z)\) and \(f_3(z)\) not containing \(z\). Then also the side \(f_1^{-1}f_3\) is degenerate. Because of continuity and discreteness there exists a neighbourhood \(U\) of \(z\) such that for \(w \in U \cap \gamma\), where \(\gamma\) is given by (2.1), the pattern of \(D(w)\) is the same as that of \(D(z)\), while for \(w \in U \setminus \gamma\), \(D(w)\) has 12g−6 sides since all cycles are of length 3.

We claim the side-pairing mapping added to \(D(z)\) is changed between \(f_2\) and \(f_1^{-1}f_3\) as the curve \(\gamma\) is crossed in \(U \setminus \{z\}\). Denote \(R(w)=[w, f_1(w), f_2(w), f_3(w)]\). Then \(R\) is not a constant. Let \(g_w\) be the Möbius transformation which maps \(w, f_1(w), f_2(w)\) to 0, 1, \(\infty\), respectively. Then \(R(w)=g_w(f_2(w))\). Let \(C(w)\) be the open
disc with \( w, f_1(w), f_2(w) \) on \( \partial C(w) \). Then \( g_w(\partial C(w)) = R \) and \( g_w(C(w)) \) is either the upper or lower half-plane. Assume it is the upper one. Then, if \( \text{Im } R(w) > 0 \), it follows that \( f_2(w) \in C(w) \) and the side labelled by \( f_2 \) appears in \( D(w) \) between the sides labelled by \( f_1 \) and \( f_3 \). If \( \text{Im } R(w) < 0 \), then \( f_2(w) \) is not in the closure of \( C(w) \) and there is no side between the sides \( f_1 \) and \( f_3 \) in \( D(w) \). Since \( R \) is a rational function, there is a neighbourhood \( N \) of \( z \) such that \( R(N) \) covers a neighbourhood of \( R(z) \) at least once and for each \( w \in N \setminus \{z\} \) \( R \) is a homeomorphism when restricted to a neighbourhood \( N' \) of \( w \). If \( w \in \gamma \setminus \{z\} \), then \( R(w) \in R \).

Denote \( U_1 = \{w \in N' \mid \text{Im } R(w) > 0\} \), \( U_2 = \{w \in N' \mid \text{Im } R(w) < 0\} \). Then, when \( w \in U_1 \), \( D(w) \) has a side labelled by \( f_2 \) and, when \( w \in U_2 \), \( f_2 \) is not in \( D(w) \), and the added side is labelled by \( f_1^{-1} f_3 \). The common boundary of \( U_1 \) and \( U_2 \) in \( N' \) is \( \gamma \), hence when \( w \) crosses \( \gamma \) a side-pairing mapping is changed in \( D(w) \).

**Theorem 2.3.** Let \( z \in \bigcup_{i=1}^{p(g)} \partial D_i \) and let \( l \) be the number of the cycles in \( D(z) \) having length \( n_k \equiv 4 \), \( k = 1, \ldots, l \). Then in a neighbourhood \( U \) of \( z \), \( \bigcup_{i=1}^{p(g)} \partial D_i \) lies on at most \( \sum_{k=1}^{l} \left( \frac{n_k}{4} \right) \) curves \( \gamma \in \Gamma \). For each \( \gamma \), \( \gamma \cap U \) is the union of finitely many arcs which intersect only at \( z \), each homeomorphic to an open interval of real numbers.

**Proof.** For \( k = 1, \ldots, l \) let \( P_k \) be a vertex of \( D(z) \) with length of the cycle \( n_k \equiv 4 \). Then there are exactly \( n_k \) points of the set \( G(z) \) on a circle with center \( P_k \). Each choice of 4 of them determines a curve \( \gamma \) through \( z \) by the condition

\[
\text{Im } \left[ g_1(z), g_2(z), g_3(z), g_4(z) \right] = 0.
\]

As in Theorem 2.1, by mapping \( P_k \) to another vertex in the cycle, we see that each vertex in the same cycle gives rise to the same curves \( \gamma \in \Gamma \).

Each curve \( \gamma \) is the preimage of \( R \) under the non-constant rational mapping \( R \) of Theorem 2.2. Hence the last assertion holds in a suitably chosen neighbourhood \( U \) of \( z \in \gamma \).

**Remark 2.2.** Let \( \text{Cl } D(0) \) denote the closure of \( D(0) \). It suffices to consider points in \( \text{Cl } D(0) \) to investigate the sets \( D_i \), \( i = 1, \ldots, p(g) \), for a group \( G \), since the sets \( D_i \) are invariant under \( G \). By Theorem 2.3 and the compactness of \( \text{Cl } D(0) \), \( \bigcup_{i=1}^{p(g)} \partial D_i \cap \text{Cl } D(0) \) is on a finite number of curves \( \gamma \in \Gamma \).

**Theorem 2.4.** Let the group \( G \) act on the open unit disc \( U \), let \( z \in D_i \), \( i = 1, \ldots, p(g) \), and let \( B \) be the union of subsets of \( U \) fulfilling conditions of type (2.1), where \( f_1, f_2, f_3 \) label adjacent sides of \( D(z) \). Let \( W \) be the component of \( U \setminus \text{Cl } B \) containing \( z \). Then \( W \subset D_i \) and \( \partial W \subset \partial D_i \).

**Proof.** Since \( z \in D_i \), \( i = 1, \ldots, p(g) \), \( D(z) \) has a maximal number of sides. Hence, because of continuity, for \( w \in W \), \( D(w) \) has the same pattern as \( D(z) \), since the pattern of \( D(w) \) can change from that of \( D(z) \) only if some of the sides of \( D(z) \).
degenerate to vertices, i.e., if the center \( w \) fulfils at least one condition of type (2.1) for mappings labelling adjacent sides of \( D(z) \). Hence \( W \subset D_i \).

For \( w \in \partial W \), a condition of type (2.1) is valid for mappings labelling three adjacent sides of \( D(z) \) and, hence, at least one side-pair of \( D(z) \) is degenerate in \( D(w) \). Thus \( \partial W \subset \partial D_i \).

**Corollary 2.1.** The set \( D_i, i=1, \ldots, p(g) \), is open.

### 3. Application to the group of the regular octagon

In [4] we studied the group with \( D(0) \) the regular octagon with diametrically opposite pairings. We now derive the curves \( \gamma \in \Gamma \) of Theorem 2.3 through 0. In this case \( \bigcup_{i=1}^{8} \partial D_i \) lies on at most \( \left(\frac{8}{4}\right)=70 \) curves in a neighbourhood of 0. In Theorem 3.1, the dependence of \( D(z) \) on \( z \) is studied for this group.

The generators are \( f_1, f_2, f_3, f_4 \) with the relation \( f_1 f_2 f_3 f_4 f_1^{-1} f_2^{-1} f_3^{-1} f_4^{-1}=I \) and \( f_{k+1}=g^{-k}f_1 g^k, k=1, 2, 3 \), where \( g(z)=(\exp(i5\pi/4))z \).

The cycle of a vertex of \( D(0) \) has length \( S \) and \( \cup_{i=1}^{8} \partial D_i \) is in a neighbourhood of 0 on the curves \( \gamma \) obtained from mappings \( f_1, f_2, f_3, f_4 \) with the relation \( f_1 f_2 f_3 f_4 f_1^{-1} f_2^{-1} f_3^{-1} f_4^{-1}=I \) and \( f_{k+1}=g^{-k}f_1 g^k, k=1, 2, 3 \), where \( g(z)=(\exp(i5\pi/4))z \).

Hence the set of points satisfying the latter equation is obtained by rotating by \( g^{-1} \) the set of points satisfying the former equation. Hence it suffices to consider the cyclic choices of four of the mappings labelling the Dirichlet regions around \( P \) and to take rotations by the powers of \( g \) of the algebraic curves we have obtained to get all curves through 0 on which \( \cup_{i=1}^{8} \partial D_i \) can lie in a neighbourhood of 0. These are shown in Figure 1, calculated by a computer.

**Theorem 3.1.** The group of the regular octagon only has pattern 6 [3] and its degenerates as patterns for \( D(z) \).

**Proof.** Step 1. To get the patterns for \( D(z) \), \( z \in \text{C} D(0) \), it suffices to consider the sector with angle \( [\pi, 5\pi/4] \) in the origin: if we denote \( f_{i+4}=f_i^{-1}, i=1, \ldots, 4 \), then \( f_i=g^{-k}f_j g^k \) holds for \( i,j,k=1, \ldots, 8 \), \( j+k=i \) (mod 8). Choose \( i=1, \ldots, 7 \). We claim that the pattern for \( D(z') \), where \( z'=g^i(z) \), is obtained from \( D(z) \) by replacing the generator \( f_j, j=1, \ldots, 4 \), by \( f_{j-i} \) (mod 8). This follows from the fact that \( g^i \) is a Möbius transformation and \( g^i f_j(z)=g^i f_j g^{-i}(z')=f_{j-i}(z') \).

**Step 2.** Let \( r \) and \( R \) be the radii of the inscribed and circumscribed circles for \( D(0) \). We claim it suffices to consider \( z \) in the triangle \( T \) (Figure 2) with vertices in...
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the hyperbolic polar coordinates $Q = (r, \pi)$, $S = (r, 5\pi/4)$. Let $P = (R, 9\pi/8)$. The set $D(P)$ is a regular octagon with diagonal pairings by $f_4 f_3 f_2, f_1 f_2 f_3, f_1 f_2 f_4^{-1}, f_1 f_3^{-1} f_4^{-1}$. Let $h$ be the reflection in the line through $Q$ and $S$. Then $h$ maps $D(P)$ to $D(0)$ and $h^{-1} f_i h = (f_4 f_3 f_2)^{-1}, h^{-1} f_2 h = f_4 f_3 f_1^{-1}, h^{-1} f_3 h = f_1 f_2 f_4^{-1}, h^{-1} f_4 h = (f_1 f_3 f_4)^{-1}$. We deduce that $D(h(z))$ is obtained from $D(z)$ by replacing each $f_i$, $i = 1, \ldots, 4$, by its conjugate (and reversing the order). Also, the reflections of the curves in $T$ indicating exceptional centers are curves indicating exceptional centers in $h(T)$ belonging to the conjugate mappings.

**Step 3.** We finish by calculating $D(z)$ for $z \in T$. By continuity, in a neighbourhood of 0, $D(z)$ only has sides labelled by $g$, where $D(g(0))$ is adjacent to $D(0)$. We want to find a point $z_0 \neq 0$ in this neighbourhood and calculate $D(z_0)$. 

![Figure 1](image-url)
The set $D(z)$ is in a neighbourhood of 0 obtained from $D(0)$ so that some of the sides degenerated to points open up. The obtained pattern can only change if some of the sides of $D(z)$ degenerate to points. Let $g_1, g_2, g_3$ be adjacent sides of $D(z)$. The side $g_2$ degenerates to a point on the curve $\Im [z, g_{1}(z), g_{2}(z), g_{3}(z)] = 0$. There are two cases to consider depending on whether $g_2$ labels a side or a "degenerate side" of $D(0)$.

Case 1. The regions $D(g_1(0)), D(g_2(0)), D(g_3(0)), D(0)$ have a common vertex, the vertices are adjacent, and $g_2$ labels the side of $D(0)$ between them. The corresponding curves are obtained by rotation by powers of $g$ from the following special case: Take $f_1$ for $g_2$. There are six choices for $g_1$ and six for $g_3$. The number of curves to be drawn by a computer can be diminished by means of symmetry. All obtained curves have euclidean distance at least 0.4 from 0.

Case 2. The regions $D(g_1(0)), D(g_2(0)), D(g_3(0))$ and $D(0)$ have a common vertex. The corresponding curves are in Figure 1.

Hence for $D(-0.3, -0.04)$ it suffices to consider the mappings labelling Dirichlet regions adjacent to $D(0)$; it will have 18 sides. The sides are: $f_1, f_1 f_3^{-1}, (f_4 f_5 f_6)^{-1}, f_2^{-1}, (f_1 f_3)^{-1}, f_3, f_4^{-1}, f_1^{-1}, f_2, (f_4 f_5)^{-1}, f_3^{-1}, (f_1 f_3 f_6)^{-1}, f_4 f_3^{-1}, f_4, f_1 f_3 f_6, f_1 f_3 f_6$, $f_1 f_3 f_6$. The pattern is 6. A side, say $f_1$, disappears by becoming a vertex in $D(z)$ if $\Im [z, f_1 f_3(z), f_1(z), f_1 f_3^{-1}(z)] = 0$. The curves for all sides are already calculated and the result is that the pattern above is preserved in $T$, with the sides $f_4 f_5, f_1 f_3 f_6$ degenerating on $0Q$, $f_1 f_3, f_4 f_5 f_6$ on $0S$, $f_3$ on $QS$ and $f_1 f_3^{-1}$ at 0, $f_1$ at $S$, $f_4$ at $Q$.

The conclusion is that the group of the regular octagon has for patterns for $D(z)$, $z \in \operatorname{Cl}D(0)$, pattern 6, its degenerates with 14 sides on the lines drawn in Figure 2 (and their rotations by powers of $g$), and the regular octagon with diagonal identifications at 0, at the vertices of $D(0)$ and at the midpoints of sides of $D(0)$. The patterns for $z, z \in \operatorname{Cl}D(f(0))$, where $f$ is in the group, are obtained by conjugation.
Remark 3.1. Figure 2 is obtained by using a computer, but exact calculations can also be made. For example, the cross-ratio \([z, f_a(z), f_b(z), f_a f_b(z)]\) can be represented as a quotient of two polynomials of degree four in \(z\), with real coefficients. Hence the side \(f_a f_b\) degenerates on the real axis.

4. Examples of groups with at least two patterns

We construct a group which has points in \(D_1\) and \(D_2\). Other groups corresponding to the transition pattern [3, Chapter 5] can be constructed in a similar way.

The group is obtained by constructing \(D(0)\): Let \(P\) be a sixteengon with every fourth vertex at an equal distance from 0 and with angles \(\pi/2\), all others at an equal distance from 0 and with angles \(2\pi/3\), constructed by using [1, Theorem 1]. Then the angle bisectors are concurrent at 0 and the sides of \(P\) can be identified by using the third sequence in [3, Chapter 6]. Because of the conditions used, \(P\) is \(D(0)\) for the group generated by its side-pairing transformations.

The point 0 is on the curve \(\gamma \in \Gamma\) where the cross-ratio corresponding to the cycle of length 4 is real, and it follows from Theorem 2.2 that there exists a neighbourhood \(N\) of 0 such that in each component of \(N \setminus \gamma\) one of the degenerated side-pairs of \(D(0)\) opens up and the pair opening up is changed when crossing \(\gamma\). The change of the sides changes the pattern between 1 and 2.

References


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