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ON THE STABILITY OF IDENTIFICATION PATTERNS FOR DIRICHLET REGIONS

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1. Introduction

Let G be a purely hyperbolic Fuchsian group of the first kind with genus $g \geqq 1$ and let $D(z)$ be the Dirichlet region for G with center z. It was proved by Beardon [2, Theorem 9.4.5] that for a.e. z $D(z)$ has all cycles of length 3 and, hence, $12g-6$ sides. Let $p(g)$ give the number of possible identification patterns. In [3] we showed that $p(2)=8$. For $i=1, \ldots, p(g)$ let

$$D_i = \{z \mid D(z) \quad \text{has} \quad 12g-6 \quad \text{sides and pattern } i\}.$$

It is obvious that the sets D_i are invariant under G. Here it is shown that the sets D_i are open and properties for their boundary curves are proved. Chapters 3 and 4 deal with the case $g=2$. It is shown that the group of the regular octagon, introduced in [4, Theorem 5.1], only has pattern 6 and its degenerates as patterns for $D(z)$. Groups which change their pattern according to elementary moves [3, Chapter 5] are constructed in Chapter 4.

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2. The sets D_i and ∂D_i

In [2, Theorem 9.4.5] it is shown that if $D(z)$ has less than $12g-6$ sides, then there exist distinct $f_i \in G \setminus I$, $i=1, 2, 3$, such that

$$\operatorname{Im}\,[z, f_1(z), f_2(z), f_3(z)] = 0, \tag{2.1}$$

where the cross-ratio cannot be constant. Since G has a countable number of transformations, the "exceptional" centers lie on a countable number of curves defined by (2.1). Each curve is algebraic, since (2.1) gives a polynomial of degree 8 in two real variables.

Definition 2.1. *Let $f_i \in G \setminus I$ be distinct, $i=1, 2, 3$, and let γ be the algebraic curve corresponding to the condition* (2.1). *We define Γ as the set of all curves γ.*

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Remark 2.1. By the invariance of the cross-ratio

$$\{z|[z, f_1(z), f_2(z), f_3(z)]\in \boldsymbol{R}\} = \{z|[h(z), hf_1(z), hf_2(z), hf_3(z)]\in \boldsymbol{R}\}$$

for each Möbius transformation h. Hence the set Γ could also be defined by replacing conditions of type (2.1) by conditions

$$\operatorname{Im}[g_1(z), g_2(z), g_3(z), g_4(z)] = 0,$$

where $g_i \in G$ are distinct, $i=1, 2, 3, 4$.

Theorem 2.1. *Each cycle of length* 4 *in* $D(z)$ *corresponds to* z *lying on a curve* $\gamma\in\Gamma$.

Proof. Let P be a vertex of $D(z)$ with the other vertices in its cycle $g_1(P), g_2g_1(P), g_3g_2g_1(P)$. Then $z, g_1^{-1}(z), (g_2g_1)^{-1}(z)$ and $(g_3g_2g_1)^{-1}(z)$ are concyclic with center P. Because g_1 is a Möbius transformation, this is equivalent to the points $g_1(z), z, g_2^{-1}(z)$ and $(g_3g_2)^{-1}(z)$ being concyclic with center $g_1(P)$. The algebraic curves given by

$$\operatorname{Im}[z, g_1^{-1}(z), (g_2g_1)^{-1}(z), (g_3g_2g_1)^{-1}(z)] = 0$$

and

$$\operatorname{Im}[z, g_2^{-1}(z), (g_3g_2)^{-1}(z), g_1(z)] = 0$$

are the same. A similar argument applies to the other vertices in the cycle of P.

Theorem 2.2. *If* $D(z)$ *has one cycle of length* 4 *and the others of length* 3, *then in a neighbourhood* U *of* z, $\bigcup_{i=1}^{p(g)} \partial D_i \cap U = U\cap\gamma$, *where* $\gamma\in\Gamma$, *and for* $w\in U\cap\gamma$ *the identification pattern of* $D(w)$ *is the same as that of* $D(z)$, *while when* w *crosses* γ *in* $U\setminus\{z\}$ *a side-pairing transformation is changed in* $D(w)$.

Proof. If P is a vertex with cycle length 3, then by continuity and discreteness z has a neighbourhood in which $D(w)$ has a vertex in a neighbourhood of P with the same cycle length and adjacent sides labelled by the same mappings as P has for $D(z)$.

The condition for Q being a vertex of $D(z)$ with cycle length 4 and with adjacent sides labelled by f_1, f_3, the side f_2 being degenerate at Q, is, in hyperbolic distances,

$$\varrho(Q, z) = \varrho(Q, f_1(z)) = \varrho(Q, f_2(z)) = \varrho(Q, f_3(z)) < \varrho(Q, g(z))$$

for all $g\in G\setminus\{I, f_1, f_2, f_3\}$ and $f_2(z)$ lies on the circle with center Q on the part between $f_1(z)$ and $f_3(z)$ not containing z. Then also the side $f_1^{-1}f_3$ is degenerate. Because of continuity and discreteness there exists a neighbourhood U of z such that for $w\in U\cap\gamma$, where γ is given by (2.1), the pattern of $D(w)$ is the same as that of $D(z)$, while for $w\in U\setminus\gamma$, $D(w)$ has $12g-6$ sides since all cycles are of length 3.

We claim the side-pairing mapping added to $D(z)$ is changed between f_2 and $f_1^{-1}f_3$ as the curve γ is crossed in $U\setminus\{z\}$. Denote $R(w)=[w, f_1(w), f_2(w), f_3(w)]$. Then R is not a constant. Let g_w be the Möbius transformation which maps w, $f_1(w), f_3(w)$ to $0, 1, \infty$, respectively. Then $R(w)=g_w(f_2(w))$. Let $C(w)$ be the open disc with $w, f_1(w), f_3(w)$ on $\partial C(w)$. Then $g_w(\partial C(w)) = \boldsymbol{R}$ and $g_w(C(w))$ is either the upper or lower half-plane. Assume it is the upper one. Then, if $\operatorname{Im} R(w) > 0$, it follows that $f_2(w) \in C(w)$ and the side labelled by f_2 appears in $D(w)$ between the sides labelled by f_1 and f_3. If $\operatorname{Im} R(w) < 0$, then $f_2(w)$ is not in the closure of $C(w)$ and there is no side between the sides f_1 and f_3 in $D(w)$. Since R is a rational function, there is a neighbourhood N of z such that $R(N)$ covers a neighbourhood of $R(z)$ at least once and for each $w \in N \setminus \{z\}$ R is a homeomorphism when restricted to a neighbourhood N' of w. If $w \in \gamma \setminus \{z\}$, then $R(w) \in \boldsymbol{R}$.

Denote $U_1 = \{w \in N' \mid \operatorname{Im} R(w) > 0\}$, $U_2 = \{w \in N' \mid \operatorname{Im} R(w) < 0\}$. Then, when $w \in U_1$, $D(w)$ has a side labelled by f_2 and, when $w \in U_2$, f_2 is not in $D(w)$, and the added side is labelled by $f_1^{-1} f_3$. The common boundary of U_1 and U_2 in N' is γ, hence when w crosses γ a side-pairing mapping is changed in $D(w)$.

Theorem 2.3. *Let $z \in \bigcup_{i=1}^{p(g)} \partial D_i$ and let l be the number of the cycles in $D(z)$ having length $n_k \geqq 4$, $k = 1, \ldots, l$. Then in a neighbourhood U of z, $\bigcup_{i=1}^{p(g)} \partial D_i$ lies on at most $\sum_{k=1}^{l} \binom{n_k}{4}$ curves $\gamma \in \Gamma$. For each γ, $\gamma \cap U$ is the union of finitely many arcs which intersect only at z, each homeomorphic to an open interval of real numbers.*

Proof. For $k = 1, \ldots, l$ let P_k be a vertex of $D(z)$ with length of the cycle $n_k \geqq 4$. Then there are exactly n_k points of the set $G(z)$ on a circle with center P_k. Each choice of 4 of them determines a curve γ through z by the condition

$$\operatorname{Im}\,[g_1(z), g_2(z), g_3(z), g_4(z)] = 0. \tag{2.2}$$

As in Theorem 2.1, by mapping P_k to another vertex in the cycle, we see that each vertex in the same cycle gives rise to the same curves $\gamma \in \Gamma$.

Each curve γ is the preimage of $\boldsymbol{R}$ under the non-constant rational mapping R of Theorem 2.2. Hence the last assertion holds in a suitably chosen neighbourhood U of $z \in \gamma$.

Remark 2.2. Let $\mathrm{Cl}D(0)$ denote the closure of $D(0)$. It suffices to consider points in $\mathrm{Cl}D(0)$ to investigate the sets D_i, $i = 1, \ldots, p(g)$, for a group G, since the sets D_i are invariant under G. By Theorem 2.3 and the compactness of $\mathrm{Cl}D(0)$, $\bigcup_{i=1}^{p(g)} \partial D_i \cap \mathrm{Cl}D(0)$ is on a finite number of curves $\gamma \in \Gamma$.

Theorem 2.4. *Let the group G act on the open unit disc U, let $z \in D_i$, $i = 1, \ldots, p(g)$, and let B be the union of subsets of U fulfilling conditions of type (2.1), where f_1, f_2, f_3 label adjacent sides of $D(z)$. Let W be the component of $U \setminus B$ containing z. Then $W \subset D_i$ and $\partial W \subset \partial D_i$.*

Proof. Since $z \in D_i$, $i = 1, \ldots, p(g)$, $D(z)$ has a maximal number of sides. Hence, because of continuity, for $w \in W$, $D(w)$ has the same pattern as $D(z)$, since the pattern of $D(w)$ can change from that of $D(z)$ only if some of the sides of $D(z)$

degenerate to vertices, i.e., if the center w fulfils at least one condition of type (2.1) for mappings labelling adjacent sides of $D(z)$. Hence $W\subset D_i$.

For $w\in\partial W$, a condition of type (2.1) is valid for mappings labelling three adjacent sides of $D(z)$ and, hence, at least one side-pair of $D(z)$ is degenerate in $D(w)$. Thus $\partial W\subset\partial D_i$.

Corollary 2.1. *The set* D_i, $i=1, \ldots, p(g)$, *is open.*

3. Application to the group of the regular octagon

In [4] we studied the group with $D(0)$ the regular octagon with diametrically opposite pairings. We now derive the curves $\gamma\in\Gamma$ of Theorem 2.3 through 0. In this case $\bigcup_{i=1}^{8}\partial D_i$ lies on at most $\binom{8}{4}=70$ curves in a neighbourhood of 0. In Theorem 3.1, the dependence of $D(z)$ on z is studied for this group.

The generators are f_1, f_2, f_3, f_4 with the relation $f_1f_2f_3f_4f_1^{-1}f_2^{-1}f_3^{-1}f_4^{-1}=I$ and $f_{k+1}=g^{-k}f_1g^k$, $k=1, 2, 3$, where $g(z)=(\exp(i5\pi/4))z$.

The cycle of a vertex of $D(0)$ has length 8 and $\bigcup_{i=1}^{8}\partial D_i$ is in a neighbourhood of 0 on the curves γ obtained from mappings $I, f_1, f_1f_2, f_1f_2f_3, f_1f_2f_3f_4, f_4f_3f_2$, f_4f_3, f_4, labelling adjacent Dirichlet regions. Since the mappings are Möbius transformations and the generators are conjugates, we obtain for example: if we denote $w=g^{-1}(z)$, then $\operatorname{Im}[z, f_1(z), f_1f_2(z), f_1f_2f_3(z)]=0$ is equivalent to $\operatorname{Im}[f_1(w)$, $f_1f_2(w), f_1f_2f_3(w), f_1f_2f_3f_4(w)]=0$. Hence the set of points satisfying the latter equation is obtained by rotating by g^{-1} the set of points satisfying the former equation. Hence it suffices to consider the cyclic choices of four of the mappings labelling the Dirichlet regions around P and to take rotations by the powers of g of the algebraic curves we have obtained to get all curves through 0 on which $\bigcup_{i=1}^{8}\partial D_i$ can lie in a neighbourhood of 0. These are shown in Figure 1, calculated by a computer.

Theorem 3.1. *The group of the regular octagon only has pattern* 6 [3] *and its degenerates as patterns for* $D(z)$.

Proof. Step 1. To get the patterns for $D(z)$, $z\in\operatorname{Cl}D(0)$, it suffices to consider the sector with angle $[\pi, 5\pi/4]$ in the origin: if we denote $f_{i+4}=f_i^{-1}$, $i=1, \ldots, 4$, then $f_i=g^{-k}f_jg^k$ holds for $i, j, k=1, \ldots, 8$, $j+k=i \pmod 8$. Choose $i=1, \ldots, 7$. We claim that the pattern for $D(z')$, where $z'=g^i(z)$, is obtained from $D(z)$ by replacing the generator $f_j, j=1, \ldots, 4$, by $f_{j-i} \pmod 8$. This follows from the fact that g^i is a Möbius transformation and $g^if_j(z)=g^if_jg^{-i}(z')=f_{j-i}(z')$.

Step 2. Let r and R be the radii of the inscribed and circumscribed circles for $D(0)$. We claim it suffices to consider z in the triangle T (Figure 2) with vertices in the hyperbolic polar coordinates 0, $Q=(r, \pi)$, $S=(r, 5\pi/4)$. Let $P=(R, 9\pi/8)$. The set $D(P)$ is a regular octagon with diagonal pairings by $f_4f_3f_2, f_1f_2f_3, f_1f_2f_4^{-1}, f_1f_3^{-1}f_4^{-1}$. Let h be the reflection in the line through Q and S. Then h maps $D(P)$ to $D(0)$ and $h^{-1}f_1h=(f_4f_3f_2)^{-1}, h^{-1}f_2h=f_4f_3f_1^{-1}, h^{-1}f_3h=f_1f_2f_4^{-1}, h^{-1}f_4h=(f_1f_2f_3)^{-1}$. We deduce that $D(h(z))$ is obtained from $D(z)$ by replacing each f_i, $i=1, \ldots, 4$, by its conjugate (and reversing the order). Also, the reflections of the curves in T indicating exceptional centers are curves indicating exceptional centers in $h(T)$ belonging to the conjugate mappings.

Step 3. We finish by calculating $D(z)$ for $z \in T$. By continuity, in a neighbourhood of 0, $D(z)$ only has sides labelled by g, where $D(g(0))$ is adjacent to $D(0)$. We want to find a point $z_0 \neq 0$ in this neighbourhood and calculate $D(z_0)$.

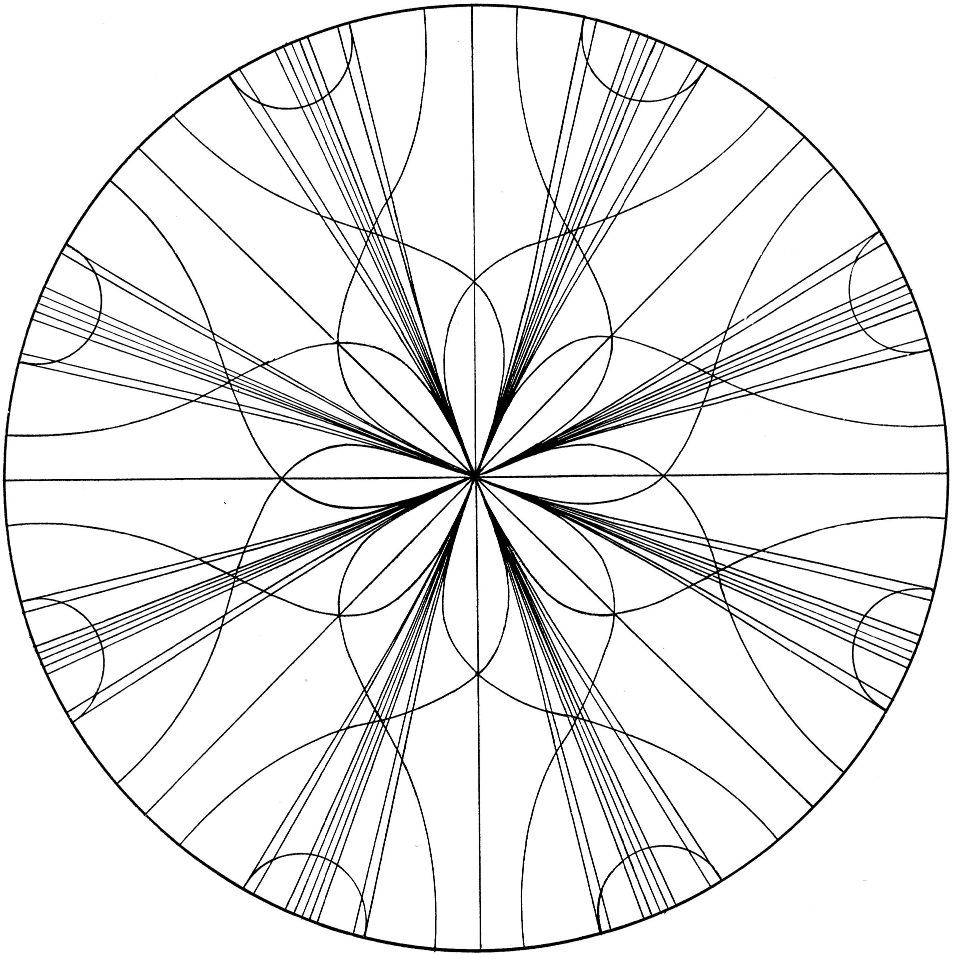

Figure 1

The set $D(z)$ is in a neighbourhood of 0 obtained from $D(0)$ so that some of the sides degenerated to points open up. The obtained pattern can only change if some of the sides of $D(z)$ degenerate to points. Let g_1, g_2, g_3 be adjacent sides of $D(z)$. The side g_2 degenerates to a point on the curve $\operatorname{Im}[z, g_1(z), g_2(z), g_3(z)]=0$. There are two cases to consider depending on whether g_2 labels a side or a "degenerate side" of $D(0)$.

Case 1. The regions $D(g_1(0)), D(g_2(0)), D(0)$ and, respectively, $D(g_2(0)), D(g_3(0)), D(0)$ have a common vertex, the vertices are adjacent, and g_2 labels the side of $D(0)$ between them. The corresponding curves are obtained by rotation by powers of g from the following special case: Take f_1 for g_2. There are six choices for g_1 and six for g_3. The number of curves to be drawn by a computer can be diminished by means of symmetry. All obtained curves have euclidean distance at least 0.4 from 0.

Case 2. The regions $D(g_1(0)), D(g_2(0)), D(g_3(0))$ and $D(0)$ have a common vertex. The corresponding curves are in Figure 1.

Hence for $D(-0.3, -0.04)$ it suffices to consider the mappings labelling Dirichlet regions adjacent to $D(0)$; it will have 18 sides. The sides are: f_1, $f_1f_4^{-1}$, $(f_4f_3f_2)^{-1}$, f_2^{-1}, $(f_1f_2)^{-1}$, f_3, f_4^{-1}, f_1^{-1}, f_2, $(f_4f_3)^{-1}$, f_3^{-1}, $(f_1f_2f_3)^{-1}$, $f_4f_1^{-1}$, f_4, f_4f_3, $f_4f_3f_2$, $f_1f_2f_3$, f_1f_2. The pattern is 6. A side, say f_1, disappears by becoming a vertex in $D(z)$ if $\operatorname{Im}[z, f_1f_2(z), f_1(z), f_1f_4^{-1}(z)]=0$. The curves for all sides are already calculated and the result is that the pattern above is preserved in T, with the sides f_4f_3, $f_1f_2f_3$ degenerating on $0Q$, f_1f_2, $f_4f_3f_2$ on $0S$, f_2, f_3 on QS and $f_1f_4^{-1}$ at 0, f_1 at S, f_4 at Q.

The conclusion is that the group of the regular octagon has for patterns for $D(z)$, $z\in \operatorname{Cl}D(0)$, pattern 6, its degenerates with 14 sides on the lines drawn in Figure 2 (and their rotations by powers of g), and the regular octagon with diagonal identifications at 0, at the vertices of $D(0)$ and at the midpoints of sides of $D(0)$. The patterns for z, $z\in \operatorname{Cl}D(f(0))$, where f is in the group, are obtained by conjugation.

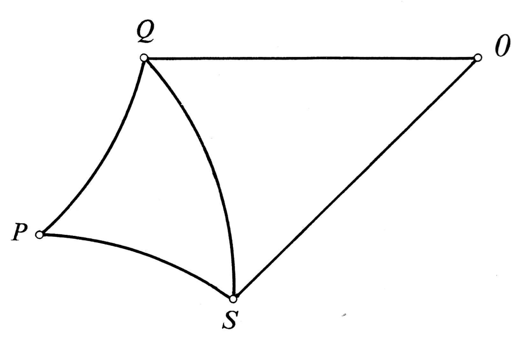

Figure 2

Remark 3.1. Figure 2 is obtained by using a computer, but exact calculations can also be made. For example, the cross-ratio $[z, f_4(z), f_4 f_2(z), f_4 f_3 f_2(z)]$ can be represented as a quotient of two polynomials of degree four in z, with real coefficients. Hence the side $f_4 f_3$ degenerates on the real axis.

4. Examples of groups with at least two patterns

We construct a group which has points in D_1 and D_2. Other groups corresponding to the transition pattern [3, Chapter 5] can be constructed in a similar way.

The group is obtained by constructing $D(0)$: Let P be a sixteengon with every fourth vertex at an equal distance from 0 and with angles $\pi/2$, all others at an equal distance from 0 and with angles $2\pi/3$, constructed by using [1, Theorem 1]. Then the angle bisectors are concurrent at 0 and the sides of P can be identified by using the third sequence in [3, Chapter 6]. Because of the conditions used, P is $D(0)$ for the group generated by its side-pairing transformations.

The point 0 is on the curve $\gamma \in \Gamma$ where the cross-ratio corresponding to the cycle of length 4 is real, and it follows from Theorem 2.2 that there exists a neighbourhood N of 0 such that in each component of $N \setminus \gamma$ one of the degenerated side-pairs of $D(0)$ opens up and the pair opening up is changed when crossing γ. The change of the sides changes the pattern between 1 and 2.

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