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ON PARABOLICITY OF A RIEMANN SURFACE

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In connection with [2] Professor Grunsky asked the author personally the following question at the Colloquium held at Joensuu in August, 1978.

Let $a_1, a_2, \ldots$ be a sequence of positive numbers decreasing to zero, and denote the points $-1+ia_k$ and $1+ia_k$ in the w-plane by α_k and β_k respectively. Let s_k be the segment $\alpha_k\beta_k$. Let $\alpha=-1$ and $\beta=1$, and s the segment $\alpha\beta$. Consider an extended plane P slit along $s, s_1, s_2, \ldots,$ and an extended plane P_k slit only along s_k for $k=1, 2, \ldots$. Connect P_k crosswise with P through s_k. Let $Q_1, Q_2, \ldots$ be extended planes slit along s, and identify the upper shore of Q_1 with the lower shore of P, the upper shore of Q_2 with the lower shore of Q_1, and so on. Denote by R the resulting simply connected Riemann surface $P\cup P_1\cup P_2\cup\ldots\cup Q_1\cup Q_2\cup\ldots$. The question is as to whether R is of parabolic type.

In this paper we shall prove

Theorem. *The surface R is of parabolic type.*

Proof. In view of Corollaire 2 of [4, p. 201], it will be sufficient to show that the family Γ of curves starting from a closed disk Δ in P and tending to the ideal boundary of R has infinite extremal length. We may assume that Δ lies above s_1. Denote by P^+ (respectively P^-) the upper (respectively lower) half of P; let P^- include $(-\infty, -1)\cup(1, \infty)$.

Divide Γ into four families. The first family Γ_1 consists of curves c of Γ such that some terminal part of c is contained in $P^-\cup Q_1\cup Q_2\cup\ldots$. The second (respectively third) family Γ_2 (respectively Γ_3) consists of curves of Γ each of which contains a sequence of points of P^+ converging to α (respectively β). The fourth family Γ_4 consists of curves of $\Gamma-\Gamma_2-\Gamma_3$ each of which contains a sequence of points of P^+ converging to a point of $s-\{\alpha\}-\{\beta\}$. Then $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$.

Since $Q_1\cup Q_2\cup\ldots$ forms a "half" of the Riemann surface defined by the logarithmic function, the extremal length $\lambda(\Gamma_1)=\infty$. To prove $\lambda(\Gamma_2)=\infty$, map the part R' of R lying above the left half plane $\mathrm{Re}\, w<0$ conformally onto the left half plane $\mathrm{Re}\, z<0$, and denote by g the mapping function. Let $w=f(z)$ be the composition of g^{-1} and the projection of R' into the w-plane. Take any $c\in\Gamma_2$ and let $\{w_n\}$ be a sequence of points of $c\cap P^+$ which converges to α and whose image by g converges to z_0 on the imaginary axis. For any $c'\in\Gamma_2$ we can find a sequence $\{\gamma_n\}$ of arcs in P^+ such that γ_n connects w_n and c' for each n and its length tends to

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0 as $n\to\infty$. Suppose $f^{-1}(\gamma_n)$ does not tend to z_0 as $n\to\infty$. Then there exist a subsequence $\{f^{-1}(\gamma_{n_k})\}$ and a sequence $\{c_k\}$ of arcs in $\operatorname{Re} z<0$ such that c_k is a subarc of $f^{-1}(\gamma_{n_k})$ for each k, one end of c_k converges to z_0 and the other end of c_k converges to a point $z_0'\neq z_0$ on the imaginary z-axis. By applying Koebe's theorem (cf. Hilfssatz on p. 19 of [1]) we conclude that f is constantly equal to α. This is impossible. Thus the image by g of $c'\cap R'$ contains a sequence of points tending to z_0. By symmetry R is mapped conformally outside a point or a segment on the imaginary axis. The image by g of c' and hence the image of every curve of Γ_2 contains a sequence of points tending to z_0. Therefore it converges to z_0 or is not rectifiable. It follows that $\lambda(\Gamma_2)=\infty$; see pp. 134—135 of [3] for a proof, for instance. Similarly $\lambda(\Gamma_3)=\infty$.

Finally let Λ_n $(n\geqq 2)$ be the subfamily of Γ_4 such that the cluster set of the part in P of each curve of Λ_n is contained in $(-1+1/n,\ 1-1/n)\subset s$. Evidently $\Gamma_4=U_n\Lambda_n$. To prove $\lambda(\Lambda_n)=\infty$, denote by $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ the points $-1+1/n+ia_k$ and $1-1/n+ia_k$ respectively, and map P_k conformally onto a rectangle D_k of height one so that the end segments $\alpha_k\alpha_k^{(n)}$ and $\beta_k^{(n)}\beta_k$ correspond to the sides of length one; observe that $D_1, D_2, \ldots$ have the same shape. Given a curve of Λ_n, its image in D_k connects opposite sides if k is large. Define a density ϱ_k in P_k by means of the constant density $1/k$ in D_k, and let ϱ be the density on R equal to ϱ_k in P_k for $k=1, 2, \ldots$ and to 0 elsewhere. Then $\int_c \varrho\, ds=\infty$ for every $c\in\Lambda_n$ and $\iint \varrho^2\, dx\, dy<\infty$. Hence $\lambda(\Lambda_n)=\infty$ for every n so that $\lambda(\Gamma_4)=\infty$. Thus $\lambda(\Gamma)=\infty$.

Remark. The proof was presented at the Colloquium held at Kyoto on the occasion of Professor Lehto's visit to Japan; see [5].

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