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ON GAP SERIES AND THE LEHTO–VIRTANEN MAXIMUM PRINCIPLE

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1. Hadamard gap series

Let $\boldsymbol{D}$ denote the unit disk. We consider first functions

$$f(z)=\sum_{k=0}^{\infty} a_k z^{n_k} \quad (z\in \boldsymbol{D}) \tag{1.1}$$

with Hadamard gaps, i.e.,

$$\frac{n_{k+1}}{n_k} \geqq \lambda > 1 \quad \text{for} \quad k=1,2,\ldots. \tag{1.2}$$

We assume that the coefficients are unbounded. Then the maximal term satisfies

$$\mu(r) \equiv \max_k |a_k| r^{n_k} \to \infty \quad \text{as} \quad r\to 1-0. \tag{1.3}$$

T. Murai [6] has shown that f has the asymptotic value ∞ at every point of $\partial\boldsymbol{D}$. This problem had been raised and partially answered by G. R. MacLane [5, p. 46]; compare also [2]. A quantitative version of Murai's result was recently proved by D. Gnuschke and the author:

Theorem 1 [3]. *Let f have Hadamard gaps and unbounded coefficients. Then, for every $\zeta\in\partial\boldsymbol{D}$, there is a Jordan arc C ending at ζ such that*

$$\frac{|f(z)|}{\mu(|z|)} > \alpha \quad \textit{for} \quad z\in C \tag{1.4}$$

where α is a positive constant depending only on λ.

The proof uses the Lehto–Virtanen maximum principle [4]. We shall prove a variant of this principle and deduce a partial converse of Theorem 1:

Theorem 2. *Let f have Hadamard gaps and let*

$$\mu\left(\frac{1+r}{2}\right) \leqq M_1\mu(r) \quad \left(\frac{1}{2}\leqq r<1\right), \quad \mu(r)\to\infty \quad (r\to 1-0). \tag{1.5}$$

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Then there is a constant M_2 *such that, for almost all* $\zeta\in\partial\mathbf{D}$,

$$\inf_{z\in C}\frac{|f(z)|}{\mu(|z|)}\leqq M_2 \tag{1.6}$$

for every Jordan arc C *ending at* ζ.

It follows that Theorem 1 is best possible for *every* Hadamard function f satisfying (1.5). Note that this refers only to the approach along an arc. The average size of $|f(z)|/\mu(r)$ on $|z|=r$ is much larger if

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{i\theta})}{\mu(r)}\right|^2 d\theta=\sum_{k=0}^{\infty}\left(\frac{|a_k|\,r^{n_k}}{\mu(r)}\right)^2\to+\infty \quad\text{as}\quad r\to 1-0.$$

The central limit theorem for Hadamard gap series [8, p. 264] gives more precise information about the average size.

We use the assumption of lacunarity only in connection with the following lemma, a variant of a standard result [1, Lemma 2.1].

Lemma 1. *If* f *has Hadamard gaps and if*

$$\mu\left(\frac{1+r}{2}\right)\leqq M_1\mu(r)\quad\textit{for}\quad \frac{1}{2}\leqq r<1 \tag{1.7}$$

for some constant M_1, *then there exists* M *such that*

$$(1-|z|^2)\,|f'(z)|\leqq M\mu(|z|)\quad\textit{for}\quad z\in\mathbf{D}. \tag{1.8}$$

Proof. Let $|z|=r\geqq 1/2$. Then, by (1.3) and (1.7),

$$|a_k|\,r^{n_k/2}\leqq\mu(\sqrt{r})\leqq M_1\mu(r)\quad(k=0,1,\ldots).$$

Hence we see from (1.1) that

$$r|f'(z)|\leqq\sum_{k=0}^{\infty}n_k|a_k|\,r^{n_k}\leqq M_1\mu(r)\sum_{k=0}^{\infty}n_k r^{n_k/2}.$$

Using the fact that $n_k/n_j\leqq\lambda^{-(j-k)}$ for $k\leqq j$, we deduce that

$$\frac{r|f'(z)|}{1-\sqrt{r}}\leqq M_1\mu(r)\sum_{k=0}^{\infty}\Big(\sum_{n_k\leqq m}n_k\Big)r^{m/2}$$

$$\leqq\frac{M_1\lambda\mu(r)}{\lambda-1}\sum_{k=0}^{\infty}(m+1)r^{m/2}=\frac{M_1\lambda\mu(r)}{(\lambda-1)(1-\sqrt{r})^2},$$

and this implies (1.8).

2. A Bloch-type condition

We drop now the assumption that f has Hadamard gaps and write

(2.1) $$f(z)=\sum_{n=0}^{\infty}a_n z^n, \quad \mu(r)=\max_n |a_n| r^n.$$

We consider the condition

(2.2) $$(1-|z|^2)|f'(z)|\leqq M\mu(|z|) \quad (z\in\boldsymbol{D})$$

where M is a constant. It holds for all Bloch functions and also for all Hadamard series satisfying (1.7) as Lemma 1 states.

Lemma 2. *If* (2.2) *holds, then*

(2.3) $$(1-r^2)\mu'(r)\leqq M\mu(r) \quad (0\leqq r<1)$$

where μ' denotes the right-hand derivative.

Proof. There exists v such that $\mu(x)=|a_v|x^v$ in some interval to the right of r. Hence (2.2) shows that

$$\mu'(r)=v|a_v|r^{v-1}\leqq \max_{|z|=r}|f'(z)|\leqq \frac{M\mu(r)}{1-r^2}.$$

Remarks. 1. It follows from Lemma 2 that, conversely, condition (1.7) is a consequence of (1.8).

2. It is not difficult to show that

(2.4) $$\mu(r)\leqq \frac{e}{2}\max(1-|z|^2)|f'(z)| \quad (0\leqq r<1).$$

Hence the constant in (2.2) satisfies $M\geqq 2e^{-1}$.

The next result is a variant of a result of O. Lehto and K. I. Virtanen [4, Theorem 7]. For $\zeta\in\partial\boldsymbol{D}$, we consider the Stolz angle

(2.5) $$\Delta_\varrho(\zeta)=\left\{|\arg(1-\bar{\zeta}z)|<\frac{\pi}{4}, \quad |z-\zeta|<\varrho\right\} \quad (\varrho>0)$$

of opening $\pi/2$ (which could be replaced by any number $<\pi$).

Theorem 3. *Let f be analytic in $\boldsymbol{D}$ and let* (2.2) *be statisfied. Let C be a Jordan arc ending at $\zeta\in\partial\boldsymbol{D}$. If*

(2.6) $$|f(z)|\geqq M_2\mu(|z|) \quad \textit{for } z\in C$$

where M_2 is a constant depending only on M, then

(2.7) $$|f(z)|\geqq \mu(|z|) \quad \textit{for} \quad z\in\Delta_\varrho(\zeta)$$

if ϱ is sufficiently small.

Proof of Theorem 2. Let E be the set of all $\zeta\in\partial\boldsymbol{D}$ such that (2.6) holds for some curve C ending at ζ. It follows from (1.5) and Lemma 1 that (2.2) is satisfied. Hence Theorem 3 shows that (2.7) holds for $\zeta\in E$, $\varrho=\varrho(\zeta)>0$.

Since $\mu(r)\to\infty$ as $r\to 1-0$, we deduce from (2.7) that $f(z)\to\infty$ as $z\to\zeta$ in a Stolz angle at $\zeta\in E$. Hence it follows from Plessner's theorem (e.g. [7, p. 324]) that $\operatorname{mes} E=0$. This proves Theorem 2 because (1.6) holds for every $\zeta\in\partial\boldsymbol{D}\setminus E$.

In order to prove Theorem 3 we introduce

$$\psi(z)=\frac{\mu(|z|)}{|f(z)|}\quad (z\in\boldsymbol{D}). \tag{2.8}$$

The function (see (2.1))

$$\log\psi(z)=\max_n(\log|a_n|+n\log|z|)-\log|f(z)| \tag{2.9}$$

is subharmonic in $\boldsymbol{D}$ except for logarithmic poles at the zeros of f.

Furthermore, we see that, with $z=re^{i\theta}$,

$$\left|\frac{\partial\psi}{\partial r}\right|=\left|\frac{\mu'}{|f|}-\frac{\mu}{|f|}\operatorname{Re}\left[e^{i\theta}\frac{f'}{f}\right]\right|\leqq\frac{\mu'}{|f|}+\frac{\mu|f'|}{|f|^2},$$

$$\left|\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|=\frac{\mu}{|f|}\left|\operatorname{Re}\left[ie^{i\theta}\frac{f'}{f}\right]\right|\leqq\frac{\mu|f'|}{|f|^2},$$

where μ' denotes the right-hand derivative. Hence

$$\frac{|\operatorname{grad}\psi|}{1+\psi^2}\leqq\frac{\mu'|f|+2\mu|f'|}{\mu^2+|f|^2}\leqq\frac{\mu'}{2\mu}+\frac{2|f'|}{\mu},$$

and it follows from (2.2) and (2.3) that

$$\frac{|\operatorname{grad}\psi(z)|}{1+\psi(z)^2}\leqq\frac{5}{2}\frac{M}{1-|z|^2}\quad\text{for}\quad z\in\boldsymbol{D}. \tag{2.10}$$

Lemma 3. *Let* $0<\psi(z)\leqq+\infty$ $(z\in\boldsymbol{D})$, *let* $\log\psi$ *be subharmonic in* $\boldsymbol{D}$ *except for logarithmic poles and let* (2.10) *be satisfied. Let* G *be a domain in* $\boldsymbol{D}$ *bounded by an arc of* $\partial\boldsymbol{D}$ *and a circular arc* A *forming an angle* $4\pi/5$ *with* $\partial\boldsymbol{D}$. *If*

$$\psi(z)\leqq e^{-K}\quad(z\in\partial G\setminus A),\quad K=2\pi M/\sin\frac{\pi}{5} \tag{2.11}$$

then $\psi(z)\leqq 1$ *for* $z\in G$.

This lemma is closely related to the Lehto–Virtanen maximum principle [4, Theorem 7] [7, Theorem 9.1]. We have specialized the parameters for an easier statement. Note that ψ need not be the modulus of a meromorphic function.

Proof (compare [7, pp. 264/265]). Suppose $\psi(z)\leqq 1$ $(z\in G)$ does not hold. Then there exists a circular arc B with the same endpoints as A forming an angle $\beta<4\pi/5$ with $\partial\boldsymbol{D}$ such that $\psi(z)\leqq 1$ in the part H of $G\setminus B$ with $\partial H\subset\partial G\cup B$ and furthermore $\psi(z_0)=1$ for some $z_0\in B$. If γ is a Möbius transformation of $\boldsymbol{D}$ onto $\boldsymbol{D}$ then $\psi\circ\gamma$ satisfies the same assumptions. Hence we may assume that A (and thus B) ends at ± 1 and that $z_0=iy_0$.

The subharmonic function

$$u(z)=\log\psi(z)+\frac{K}{\beta}\left(\arg\frac{1+z}{1-z}-\frac{\pi}{2}\right)\quad(z\in H) \tag{2.12}$$

is bounded by $-K$ both on $B\cap\partial H$ (because $\psi(z)\leqq 1$) and on $\partial G\setminus B$ (because of (2.11)). Hence the maximum principle shows that $u(z)\leqq -K$ for $z\in H$.

We conclude that

$$K+\frac{K}{\beta}\left(2\arctan y-\frac{\pi}{2}\right)\leqq -\log\psi(z)\quad(iy\in H).$$

Since both sides vanish for $y=y_0$ we see that

$$\frac{2K}{\beta(1+y_0^2)}\leqq -\frac{\partial}{\partial y}\log\psi(iy)|_{y=y_0}\leqq\frac{5M}{1-y_0^2},$$

by (2.10) and because $\psi(iy_0)=1$. By (2.11), this is equivalent to $\sin\beta/\beta\leqq(5/4\pi)\sin(4\pi/5)$, and this contradicts $\beta<4\pi/5$.

Proof of Theorem 3. We have seen that the function ψ defined by (2.8) satisfies the assumptions of Lemma 3. Let $C'\subset\boldsymbol{D}$ be the subarc of C obtained by deleting a small part near $\partial\boldsymbol{D}$. Let A be the circular arc through the endpoints of C' that forms the angle $4\pi/5$ with $\partial\boldsymbol{D}$ and let G' be the domain between C' and A; if G is not connected, we apply the argument to the components.

We choose $M_2=\exp[-2\pi M/\sin(\pi/5)]$. If (2.6) holds, then $\psi(z)\leqq 1/M_2$ for $z\in C$. Hence (2.11) is satisfied and we conclude that $\psi(z)\leqq 1$ for $z\in G'$. Letting $C'\to C$ we see that $\psi(z)\leqq 1$ in the corresponding domain G. We apply the same argument to the arc A^* with the angle $4\pi/5$ in the opposite direction and obtain $\psi(z)\leqq 1$ for $z\in G^*$. This proves (2.7) because $\Delta_\varrho(\zeta)\subset G\cup G^*$ if ϱ is sufficiently small.

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