ON THE SPHERICAL DERIVATIVE OF SMOOTHLY GROWING MEROMORPHIC FUNCTIONS WITH A NEVANLINNA DEFICIENT VALUE

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1. Introduction and results

Let \( f \) be meromorphic in the finite complex plane \( \mathbb{C} \). We write

\[
\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}
\]

and

\[
\mu(r, f) = \sup \{ \varrho(f(z)) : |z| = r \}.
\]

We shall use the usual notations of the Nevanlinna theory. Clunie and Hayman [3] proved the following result.

**Theorem A.** If \( \varphi(r) \) is positive and increasing and \( f(z) \) is a transcendental entire function such that

\[
\log M(r, f) = O\left( \frac{(\log r)^2}{\varphi(r)} \right) \quad (r \to \infty),
\]

then

\[
\limsup_{r \to \infty} \frac{r \mu(r, f)}{\varphi(r) \log r} = \infty.
\]

This result was extended in [9] for functions which have a Nevanlinna deficient value in the following form.

**Theorem B.** Let \( 1 < t < 2 \) and let \( f \) be a transcendental meromorphic function such that \( \delta(\infty, f) > 0 \) and that

\[
T(r, f) = O((\log r)^t) \quad (r \to \infty).
\]

Then

\[
\limsup_{r \to \infty} \frac{\log (r \mu(r, f))}{(\log r)^2 - t} = \infty.
\]

If \( f \) satisfies

\[
T(r, f) = O((\log r)^2) \quad (r \to \infty),
\]

then it satisfies
\[(1.4) \quad \lim_{r \to \infty} \frac{T(2r, f)}{T(r, f)} = 1,\]
too. We shall prove the following extension for Theorem B.

**Theorem 1.** Let \( f \) be a transcendental meromorphic function satisfying (1.4) such that \( \delta(\infty, f) > 0 \). Then

\[(1.5) \quad \limsup_{r \to \infty} \frac{r \mu(r, f)}{T(r, f)} = \infty\]
and

\[(1.6) \quad \limsup_{r \to \infty} \frac{n(r, 0, f) \log(r \mu(r, f))}{T(r, f)} \leq \delta(\infty, f).\]

If, further, \( \delta(\infty, f) > 0 \) and \( f \) satisfies (1.1) for some \( t, 1 < t \leq 2 \), then

\[(1.7) \quad \limsup_{r \to \infty} \frac{\log(r \mu(r, f))}{T(r, f)(\log r)^{1-t}} > 0.\]

Clunie and Hayman [3] proved that, given an increasing function \( \varphi(r) \) such that \( \varphi(r) \to \infty \) as \( r \to \infty \), there is a transcendental entire function \( f \) satisfying (1.3) such that

\[r \mu(r, f) = O(\varphi(r) \log r) \quad (r \to \infty).\]

This example shows, since

\[\log r = o(T(r, f)) \quad (r \to \infty),\]
that (1.5) is essentially sharp. The following theorem shows that (1.2), (1.6) and (1.7) are essentially sharp.

**Theorem 2.** Let \( t, 1 < t < 2 \), and \( d, 0 < d \leq 1 \), be given, and let \( \varphi(r) \) be an increasing function of \( r \) such that \( \varphi(r) \to \infty \) as \( r \to \infty \). There exists a transcendental meromorphic function \( f \) satisfying (1.1) such that \( \delta(\infty, f) = d \),

\[(1.8) \quad \log(r \mu(r, f)) = O(\varphi(r)(\log r)^{d-1}) \quad (r \to \infty),\]
\[(1.9) \quad \limsup_{r \to \infty} \frac{n(r, 0, f) \log(r \mu(r, f))}{T(r, f)} = \delta(\infty, f)\]
and

\[(1.10) \quad \limsup_{r \to \infty} \frac{\log(r \mu(r, f))}{T(r, f)(\log r)^{1-t}} < \infty.\]

The following result shows that (1.3) is the best possible growth condition under which (1.5) holds.

**Theorem 3.** Let \( \varphi(r) \) and \( d \) be as in Theorem 2. There exists a transcendental meromorphic function \( f \) such that \( \delta(\infty, f) = d \),

\[(1.11) \quad T(r, f) = O(\varphi(r)(\log r)^d) \quad (r \to \infty)\]
and
\begin{equation}
\limsup_{r \to \infty} \frac{r \mu(r, f)}{T(r, f)} \equiv 5\delta(\infty, f).
\end{equation}

It is proved in [14] that if \( f \) is a transcendental meromorphic function of order \( \lambda \), then
\begin{equation}
\limsup_{r \to \infty} \frac{r \mu(r, f)}{T(r, f)} \equiv A_0(1+\lambda)\delta(\infty, f),
\end{equation}
where \( A_0 > 0 \) is an absolute constant, and in [11] a counter example is given which shows that (1.13) is essentially sharp for \( 0 < \lambda < \infty \) and \( 0 < \delta(\infty, f) \equiv 1 \). Our Theorem 3 shows that (1.13) is essentially sharp for \( \lambda = 0 \) and \( 0 < \delta(\infty, f) \equiv 1 \). The question "which is the best possible value for \( A_0 \) in (1.13)" remains open.

2. Proof of Theorem 1

Let \( f \) be as in Theorem 1. We write
\[ L(r, f) = \min \{|f(z)| : |z| = r\}. \]
It follows from Lemma 1 of [7] that
\begin{equation}
n(r, a, f) = o(T(r, f)) \quad (r \to \infty)
\end{equation}
for all complex values \( a \) and that there exist sequences \( x_k \) and \( r_k \) such that \( 1 < x_k < r_k < 2x_k < x_{k+1}, \) \( L(x_k, f) = 0 \) and \( L(r_k, f) \equiv 2 \) for any \( k \),
\begin{equation}
\lim_{k \to \infty} \frac{r_k}{x_k} = 1,
\end{equation}
and that
\begin{equation}
\log L(r_k, f) \equiv (\delta(\infty, f) + o(1))T(r_k, f) \quad (k \to \infty).
\end{equation}

For any \( k \), we choose \( z_k \) such that \( x_k < |z_k| < r_k \), \( |f(z_k)| = 1 \), and that \( |f(z)| > 1 \) for \( |z| = |z_k| \) or \( |z| = r_k \), then
\begin{equation}
\log |f(z)| \equiv \frac{\log |z/z_k|}{\log r_k/x_k} \log L(r_k, f),
\end{equation}
and since \( \log |f(z)| \) is superharmonic on \( |z_k| \leq |z| \leq r_k \), we deduce that (2.4) holds for all \( z \) lying in \( |z_k| < |z| < r_k \).

Let \( s > 0 \). From (2.4) it follows that
\[ \log |f(z_k(1+s/|z_k|))| \leq \frac{\log L(r_k, f)}{\log r_k/x_k} (s/|z_k| + o(s)) \]
as \( s \to 0 \), and since
\[ \log |f(z_k + s(z_k/|z_k|))| \leq \log (|f(z_k)| + (s + o(s))|f'(z_k)|) \leq (s + o(s))|f'(z_k)| \quad (s \to 0), \]
we deduce that
\[ |z_k| \varrho(f(z_k)) = |z_k/2| |f'(z_k)| \lesssim (2 \log (r_k/x_k))^{-1} \log L(r_k, f). \]
This together with (2.2) and (2.3) shows that
\[ \frac{|z_k| \varrho(f(z_k))}{T(|z_k|, f)} \to \infty \quad \text{as} \quad k \to \infty, \]
which proves (1.5).

Let \( a_\delta \) be the zeros of \( f \). For any \( k \), we choose \( w_k \) such that \( f(w_k) = 0 \), \( x_k \leq |w_k| < r_k \), and that \( f(z) \neq 0 \) for \( |w_k| < |z| \leq r_k \). Since \( L(r_k, f) > 1 \), there exists \( d_k \), \( 0 < d_k < r_k/|w_k| - 1 \), such that \( |f(w_k(1 + d_k))| = 1 \) and that
\[ |f(w_k(1 + d)| < 1 \quad \text{for} \quad 0 < d < d_k. \]
Applying the Poisson–Jensen formula with \( R = r_k \) and \( w = w_k(1 + d_k) = re^{i\varphi} \), we get
\[ 0 = \log |f(w)| \begin{align*}
  &\lesssim (2\pi)^{-1} \int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\varphi - \alpha)} \, dx \\
  &\quad - \sum_{|a_\delta| < k} \log \left| \frac{R^2 - \bar{a}_\delta w}{R(w - a_\delta)} \right| \\
  &\lesssim \log L(r_k, f) - n(|w_k|, 0, f) \log \frac{2r_k}{|w - w_k|},
\end{align*} \]
which together with (2.3) implies that
\[ n(|w_k|, 0, f) \log (4/d_k) \equiv (\delta(\infty, f) + o(1)) T(r_k, f) \]
as \( k \to \infty \). This implies that
\[ \log (4/d_k) \equiv (\delta(\infty, f) + o(1)) \frac{T(r_k, f)}{n(|w_k|, 0, f)} \]
as \( k \to \infty \).

Since \( |f(w)| = 1 \) and \( f(w_k) = 0 \), there exists \( b_k = w_k(1 + d) \) such that \( 0 < d < d_k \)
and that
\[ |f'(b_k)| \equiv |w - w_k|^{-1} = |d_k w_k|^{-1}. \]
This together with (2.7) and (2.1) implies that
\[ \log (|b_k| \varrho(f(b_k))) \equiv \log (|b_k/2||f'(b_k)|) \equiv \log (2d_k)^{-1} \]
\[ \equiv O(1) + (\delta(\infty, f) + o(1)) \frac{T(r_k, f)}{n(|w_k|, 0, f)} \]
\[ \equiv (\delta(\infty, f) + o(1)) \frac{T(|b_k|, f)}{n(|b_k|, 0, f)} \quad (k \to \infty), \]
which proves (1.6).
Let us suppose that \( f \) satisfies (1.1) for some \( t, \ 1 < t \leq 2 \), and let \( r > 1 \). We get
\[
n(r, 0, f) \equiv (\log r)^{-1} \int_r^* n(t, 0, f) t^{-1} dt = O((\log r)^{-1} N(r^2, 0, f)) = O((\log r)^{-1}) \quad (r \to \infty),
\]
which together with (1.6) proves (1.7). This completes the proof of Theorem 1.

Remark. From (2.5) we get the following result slightly stronger than (1.5).

**Theorem 4.** Let \( f \) be as in Theorem 1. Then
\[
\limsup_{z \to \infty} \frac{|z|^q(f(z))}{T(|z|, f)} = \infty,
\]
where \( E(f) = \{ z; |f(z)| = 1 \} \).

### 3. Proof of Theorem 2

Let \( t, d \) and \( \varphi(r) \) be as in Theorem 2. We set \( r_0 = 100 \), and for \( n \geq 1 \) we choose \( r_n \) such that
\[
(3.1) \quad r_n > \exp \exp \exp (r_{n-1})
\]
and
\[
(3.2) \quad \varphi(r_n/2) > r_{n-1}.
\]
We denote by \([x]\) the integral part of a non-negative real number \( x \). We set
\[
(3.3) \quad s_n = \lfloor (\log r_n)^{f^{-1}} \rfloor,
\]
\[
(3.4) \quad q_n = [s_n(1-d)]
\]
and
\[
f(z) = \prod_{n=1}^{\infty} \frac{(1-z/r_n)^{q_n}}{(1+z/r_n)^{q_n}}.
\]
If \( d = 1 \), then \( f \) is an entire function.

Let \( r_n^{1/2} \leq |z| \leq r_n^{1/2} \). We have
\[
(3.5) \quad \log |f(z)| = (1+o(1))(s_{n-1} - q_{n-1}) \log |z| + s_n \log |(z-r_n)/r_n| + q_n \log |r_n/(z+r_n)|.
\]
Let \( 0 < \varepsilon < 1/9 \). We set
\[
D_n = r_n \exp \left( (-1+\varepsilon)s_n^{-1}(s_{n-1} - q_{n-1}) \log r_n \right)
\]
and
\[
d_n = r_n \exp \left( (-1-\varepsilon)s_n^{-1}(s_{n-1} - q_{n-1}) \log r_n \right).
\]
It follows from (3.5) that

\begin{equation}
\log |z^2 f(z)| \equiv (-e + o(1))(s_{n-1} - q_{n-1}) \log |z|
\end{equation}

in \(|z - r_n| < d_n|\) and that

\begin{equation}
\log |z^{-2} f(z)| \equiv (e + o(1)) \log |z|
\end{equation}

as \(z \to \infty\) outside the union of the discs \(|z - r_n| < D_n|\).

From (3.7) we deduce that

\begin{equation}
\phi(f(z)) \equiv \frac{|f'(z)|}{|f(z)|} = \left(2\pi i\right)^{-1} \int_{|w-z|=1} \frac{1}{(w-z)^2} d\mu_w \equiv (1 + o(1))|z|^{-2}
\end{equation}

as \(z \to \infty\) outside the union of the discs \(|z - r_n| < 1 + D_n|\), and, similarly, from (3.6) we get

\begin{equation}
\phi(f(z)) \equiv |f'(z)| \equiv (1 + o(1))|z|^{-2} \quad (n \to \infty)
\end{equation}

in \(|z - r_n| < d_n| - 1|\).

Let \(d_n - 1 \leq |z - r_n| \leq 1 + D_n|\). We have

\begin{equation}
\phi(f(z)) \equiv |f'(z)/f(z)| = \left|\sum_{p=1}^{\infty} \left(\frac{s_p}{z - r_p} - \frac{q_p}{z + r_p}\right)\right|
\end{equation}

\begin{equation}
\leq \frac{(1 + o(1))s_n}{d_n - 1} = (1 + o(1))d_n^{-1}s_n \quad (n \to \infty),
\end{equation}

which implies together with (3.3) and (3.4) that

\begin{equation}
\log \left(|z|\phi(f(z))\right) \equiv \log \left(s_n r_n d_n^{-1}\right) + o(1)
\end{equation}

\begin{equation}
= O(\log \log r_n) + (1 + e)\frac{s_{n-1} - q_{n-1}}{s_n} \log r_n
\end{equation}

\begin{equation}
\equiv (1 + e + o(1))s_n^{-1}d s_n^{-1}\log r_n \quad (n \to \infty).
\end{equation}

Combining the estimates (3.8), (3.9) and (3.10), we deduce that

\begin{equation}
\log (r\mu(r, f)) \equiv -1 + o(1)
\end{equation}

as \(r \to \infty\) outside the union of the intervals \(r_k/2 < r < 2r_k|\) and that

\begin{equation}
\log (r\mu(r, f)) \equiv (1 + e + o(1))\frac{d s_{k-1}\log r_k}{s_k} \quad (k \to \infty)
\end{equation}

for \(r_k/2 < r < 2r_k|.\) Since we get (3.12) for all \(e > 0|\) and

\begin{equation}
n(2r_k, 0, f) = (1 + o(1))s_k \quad (k \to \infty),
\end{equation}

we deduce that

\begin{equation}
\log (r\mu(r, f)) \equiv (d + o(1))\frac{s_{k-1}\log r_k}{n(2r_k, 0, f)} \quad (k \to \infty)
\end{equation}

for \(r_k/2 < r < 2r_k|\).
It follows from the first main theorem of the Nevanlinna theory and (3.7) that

\begin{equation}
T(2r_k, f) = (1 + o(1))N(2r_k, 0, f) = \frac{(1 + o(1))N(r_k/2, 0, f) - (1 + o(1))T(r_k/2, f)}{k \to \infty}.
\end{equation}

From (3.4) we deduce that

\[ N(r, \infty, f) = (1 - d + o(1))N(r, 0, f) \quad (r \to \infty), \]

which together with (3.14) implies that \( \delta(\infty, f) = d \). This together with (3.13), (3.14) and (3.11) shows that

\begin{equation}
\log (r \mu(r, f)) \equiv (\delta(\infty, f) + o(1)) \frac{T(r, f)}{n(r, 0, f)} \quad \text{as } r \to \infty.
\end{equation}

From (3.11), (3.12), (3.14) and (3.3) we deduce that

\[ \log (r \mu(r, f)) = O \left( \frac{T(r, f)}{(\log r)^{\delta-1}} \right) \quad (r \to \infty), \]

which proves (1.10), and from (3.11), (3.12), (3.2) and (3.3) we get

\[ \log (r \mu(r, f)) = O(\varphi(r)(\log r)^{2-\delta}) \quad (r \to \infty), \]

which proves (1.8).

From (3.3) we deduce that

\[ n(r, 0, f) = O((\log r)^{t-1}) \quad (r \to \infty), \]

which together with (3.7) and (3.14) implies that

\[ T(r, f) = (1 + o(1))N(r, 0, f) \]

\[ = O \left( \int_1^r \frac{1}{x^{\delta-1} x^{-1} dx} \right) \]

\[ = O((\log r)^{\delta}) \quad (r \to \infty). \]

Combining (3.15) and (1.6) we get (1.9). Theorem 2 is proved.

**4. Some lemmas**

**Lemma 1.** Let \( \varphi(r) \) be an increasing function of \( r \) such that \( \varphi(r) \to \infty \) as \( r \to \infty \). We choose \( r_0 = s_0 = 100 \), and for \( n \geq 1 \), \( r_n \) and \( s_n \) are chosen such that

\begin{equation}
r_n > \exp \exp \exp (r_{n-1}),
\end{equation}

\begin{equation}
\varphi(\sqrt[3]{r_n}) > s_{n-1}
\end{equation}

and

\begin{equation}
s_n = \lfloor s_{n-1} \log r_n \rfloor.
\end{equation}
We set 
\[ g(z) = \prod_{n=1}^{\infty} (1 - z/r_n)^{s_n}. \]

Then
\[ T(r, g) = O(\varphi(r)(log r)^g) \quad (r \to \infty), \]
\[ \limsup_{r \to \infty} \frac{r \mu(r, g)}{N(r/9, 0, g)} \leq 5, \]
\[ N(r/9, 0, g) = (1 + o(1))s_p \quad (p \to \infty), \]
and
\[ z^2g(z) \to 0 \]
as \( z \to \infty \) through the union of the discs \( |z-r_p| \leq r_p/3. \)

**Proof.** Let \( r_p/4 \leq |z| \leq 4r_p \). We have
\[ \log |g(z)| = (1 + o(1))s_{p-1} \log r_p + s_p \log |(z-r_p)/r_p| \quad (p \to \infty). \]

If \( |z-r_p| \leq 2r_p/5 \), we get from (4.3) and (4.8)
\[ \log |z^2g(z)| \equiv (1 + o(1))s_{p-1} \log r_p - s_p \log (5/2) \]
\[ = (1 - \log (5/2) + o(1)) s_{p-1} \log r_p \equiv 1 + o(1) \quad (p \to \infty). \]

If \( |z-r_p| \leq r_p/3 \), we deduce from (4.3) and (4.8) that
\[ \log |z^2g(z)| \equiv (1 + o(1))s_{p-1} \log r_p - s_p \log 3 \]
\[ = -(\log 3 - 1 + o(1)) s_{p-1} \log r_p \equiv -1 + o(1) \quad (p \to \infty), \]
which proves (4.7).

Using the minimum principle, we deduce from (4.9) that
\[ |z^{-2}g(z)| \to \infty \]
as \( z \to \infty \) outside the union of the discs \( |z-r_p| < 2r_p/5. \)

From (4.11) it follows that
\[ |z|e(g(z)) \equiv |zg'(z)||g(z)|^{-2} \]
\[ = \left| (2\pi i)^{-1}z \int_{|w-z|=1} \frac{1/g(z)}{(w-z)^2} dw \right| = o(1) \]
as \( z \to \infty \) outside the union of the discs \( |z-r_p| < r_p/2. \) and from (4.7) we get
\[ |z|e(g(z)) \equiv |zg'(z)| \]
\[ = \left| (2\pi i)^{-1}z \int_{|w-z|=1} g(z)(w-z)^{-2} dw \right| = o(1) \]
as \( z \to \infty \) through the union of the discs \( |z-r_p| \leq r_p/4. \)
Let $r_p/4 < |z - r_p| < r_p/2$. We have

\[
|z|g'(z)g(z) = |z| s_k(z - r_k)^{-1} \]

\[
\leq s_p|z||z - r_p|^{-1} + 4s_p + o(1) \equiv (5 + o(1))s_p \quad (p \to \infty).
\]

Let $r_p/100 < r \leq r_p$. From (4.1) and (4.3) we get

\[
N(r, 0, f) = (1 + o(1))s_p \log r
\]

\[
= (1 + o(1))s_p \log r_p = (1 + o(1))s_p \quad (p \to \infty),
\]

which proves (4.6), and together with (4.14), (4.12) and (4.13) shows that

\[
r\mu(r, f) \equiv (5 + o(1))N(r/9, 0, f) \quad (r \to \infty),
\]

which proves (4.5).

From (4.1), (4.2) and (4.3) we get for $r_p^{1/8} \leq r \leq r_p^{1/8}$

\[
T(r, f) \equiv (1 + o(1)) \log M(r, f) \equiv (1 + o(1))s_p \log r
\]

\[
\leq (2 + o(1))s_p \log r \leq (2 + o(1)) \varphi(r) \log r^a,
\]

which proves (4.4). Lemma 1 is proved.

The following lemma is proved in [11].

Lemma 2. Let $k$ be a positive integer, $g(z) = (1 - z^{nk})^{-1}$, $g_p(z) = g(2^{-p/k}z)$ for $p = 1, \ldots, k$, and

\[
f_k(z) = \sum_{p=1}^{k} (-1)^p g_p(z).
\]

Then $n(r, \infty, f_k) = 8k^2$ for $r \geq 2$,

\[
q(f_k(z)) < 72k
\]

for all $z$ in the finite complex plane $C$, and if $|z| \leq 4$, then

\[
|f_k(z)| \equiv |z|^{6k}.
\]

5. Proof of Theorem 3

Let $\varphi(r)$ and $d$ be as in Theorem 3. If $d = 1$, we choose $f(z) = g(z)$, where $g$ is the function of Lemma 1, and deduce from Lemma 1 that $f$ satisfies the assertions of Theorem 3.

Let us suppose that $0 < d < 1$. Let $g$, $s_p$, and $r_p$ be as in Lemma 1. We set

\[
b = 1/d - 1,
\]

\[
q_p = 1 + \frac{[(bs_p/8)^{1/2}]}{
\]

\[
h_p(z) = f_{q_p}(8r_p^{-1}p^2(z - r_p)).
\]

\[
(5.1)
\]

\[
(5.2)
\]
where $f_{q_p}$ is as in Lemma 2, and

$$h(z) = \sum_{p=1}^{\infty} h_p(z).$$

We set $f = g + h$.

It follows from Lemma 2 that

$$q(h_p(z)) = 8r_p^{-1} p^2 q(f_{q_p}(8r_p^{-1} p^{2}(z - r_p))) \leq 576r_p^{-1} p^2 q_p$$

for all $z$, and if $|z - r_p| \equiv r_p/p$, then

$$|h_p(z)| \leq \frac{r_p}{4p^2(z - r_p)} \left| \frac{1}{q_p} \right| \equiv \min (p^{-2}, r_p|z - r_p|^{-1}).$$

Since the series $\sum p^{-2}$ is convergent and, for any fixed $p$, $r_p|z - r_p|^{-1} \to 0$ as $z \to \infty$, we deduce from (5.4) that

$$|h(z)| \equiv \sum_{p=1}^{\infty} |h_p(z)| \to 0$$

as $z \to \infty$ outside the discs $|z - r_p| \equiv r_p/p$, and that

$$|h(z) - h_p(z)| \equiv o(1) \quad (p \to \infty)$$

in $|z - r_p| \equiv r_p/2$.

Let $|z - r_p| \equiv r_p/3$. We write $f(z) = h_p(z) + H_p(z)$. Since $H_p = g + h - h_p$, we deduce from (5.6) and Lemma 1 that

$$|H_p(z)| \equiv o(1) \quad (p \to \infty)$$

and, integrating along the circle $|w - r_p| = r_p/3$, that

$$|H'_p(z)| = |(2\pi i)^{-1} \int H_p(w)(w - z)^{-2} dw| \equiv o(r_p^{-1}) \quad (p \to \infty)$$

in $|z - r_p| \equiv r_p/6$. Since

$$q(f(z)) \equiv \frac{|h'_p(z)|}{1 + |h_p(z) + H_p(z)|} + |H'_p(z)|,$$

we get from (5.7) and (5.8)

$$q(f(z)) \equiv (1 + o(1))q(h_p(z)) + o(r_p^{-1}) \quad (p \to \infty),$$

which together with (5.3), (5.2) and Lemma 1 implies that

$$|z|q(f(z)) \equiv O(p^2 s_p) = O(p^2 s_p^{1/2}) = o(s_p) = o(N(|z|, 0, g)) \quad (p \to \infty)$$

in $|z - r_p| \equiv r_p/6$.

Integrating along the circle $|w - z| = |z|/24$, we deduce from (5.5) that

$$|h'(z)| = \left| (2\pi i)^{-1} \int h(z)(w - z)^{-2} dw \right| = o(|z|^{-1})$$

as $z \to \infty$ outside the discs $|z - r_p| \equiv r_p/6$. Since

$$q(f(z)) \equiv \frac{|g'(z)|}{1 + |g(z) + h(z)|} + |h'(z)|,$$
we get from (5.5), (5.10) and Lemma 1

\begin{equation}
|z|g(f(z)) \equiv (1 + o(1))|z|g(g(z)) + o(1)
\end{equation}

\begin{equation}
\leq (5 + o(1))N(|z|, 0, g)
\end{equation}

as $z \rightarrow \infty$ outside the discs $|z - r_p| < r_p/6$. Combining (5.9) and (5.11) we get

\begin{equation}
r \mu(r, f) \equiv (5 + o(1))N(r, 0, g) \quad (r \rightarrow \infty).
\end{equation}

Let $r_p(1 - 1/p) \equiv r \leq r_{p+1}(1 - (p + 1)^{-1})$. It follows from (5.2) and Lemma 2 that

\begin{equation}
N(r, \infty, h) \equiv (8 + o(1))q_{p-1}^2 \log r + 8q_{p}^2 \log \left( r/(r_p - r_p/p) \right)
\end{equation}

\begin{equation}
= (b + o(1))s_{p-1} \log r + (b + o(1))s_{p} \log^+ (r/r_p) \quad (p \rightarrow \infty)
\end{equation}

and that

\begin{equation}
N(r, \infty, h) \equiv (8 + o(1))q_{p-1}^2 \log r + 8q_{p}^2 \log^+ \left( r/(r_p + r_p/p) \right)
\end{equation}

\begin{equation}
= (b + o(1))s_{p-1} \log r + (b + o(1))s_{p} \log^+ (r/r_p) \quad (p \rightarrow \infty).
\end{equation}

Since

\begin{equation}
N(r, 0, g) = (1 + o(1))s_{p-1} \log r + s_{p} \log^+ (r/r_p),
\end{equation}

for these values of $r$ we deduce that

\begin{equation}
N(r, \infty, h) = (b + o(1))N(r, 0, g)
\end{equation}

\begin{equation}
= (b + o(1))s_{p-1} \log r + (b + o(1))s_{p} \log (r/r_p) \quad (p \rightarrow \infty)
\end{equation}

for $r_p \equiv r \leq r_{p+1}$. Using the first main theorem of the Nevanlinna theory, we deduce from (5.5), (5.13) and (5.14) that if $r_p(1 - 1/p) \equiv r \leq r_{p}(1 + 1/p)$ then

\begin{equation}
m(r, \infty, h) = T(r, h) - N(r, \infty, h)
\end{equation}

\begin{equation}
\leq T(r_p(1 + 1/p), h) - N(r_p(1 - 1/p), \infty, h)
\end{equation}

\begin{equation}
= N(r_p(1 + 1/p), h) - N(r_p(1 - 1/p), h) + o(1)
\end{equation}

\begin{equation}
= o(s_{p-1} \log r) + o(s_{p})
\end{equation}

\begin{equation}
= o(N(r, 0, g)) \quad (p \rightarrow \infty),
\end{equation}

which together with (5.5) implies that

\begin{equation}
m(r, h) = o(N(r, 0, g)) = o(T(r, f)) \quad (r \rightarrow \infty).
\end{equation}

Since $g$ is an entire function and

\begin{equation}
|m(r, f) - m(r, g)| \equiv m(r, h) + \log 2,
\end{equation}

we get from (5.16)

\begin{equation}
m(r, f) = m(r, g) + o(N(r, 0, g))
\end{equation}

\begin{equation}
= (1 + o(1))T(r, g) \quad (r \rightarrow \infty).
\end{equation}
Since \( N(r, f) = N(r, h) \) for all \( r > 0 \), we get from (5.15) and (5.17)

\[
\frac{m(r, f)}{T(r, f)} = \frac{T(r, g)}{T(r, g) + b N(r, 0, g)} + o(1)
\equiv (1 + b)^{-1} + o(1) = d + o(1) \quad (r \to \infty),
\]

which implies that \( \delta(\infty, f) \equiv d \), and since

\[
T(r^2_p, g) = (1 + o(1)) N(r^2_p, 0, g) \quad (p \to \infty),
\]

we get

\[
\frac{m(r^2_p, f)}{T(r^2_p, f)} \to d \quad \text{as} \quad p \to \infty,
\]

which implies that \( \delta(\infty, f) \equiv d \). These estimates imply that \( \delta(\infty, f) = d \).

Since \( \delta(\infty, f) > 0 \), it follows from (5.17) and Lemma 1 that

\[
T(r, f) = O(m(r, f)) = O(T(r, g)) = O(\varphi(r)(\log r)^2) \quad (r \to \infty),
\]

which proves (1.11).

From (5.12), (5.15) and (5.17) we get

\[
\frac{r \mu(r, f)}{T(r, f)} \equiv \frac{5 N(r, 0, g)}{b N(r, 0, g) + T(r, g)} + o(1)
\equiv 5(1 + b)^{-1} + o(1) = 5d + o(1) \quad (r \to \infty),
\]

which proves (1.12). Theorem 3 is proved.

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References

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