A REMARK ON 1-QUASICONFORMAL MAPS

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1. Introduction. It is well known that if \( n \geq 3 \), every 1-quasiconformal map of a domain \( D \subset \mathbb{R}^n \) is the restriction of a Möbius transformation. For \( C^3 \)-maps this was already proved by Liouville in 1850. The general result is due to Gehring [Ge₁] and Rešetnjak [Re]. Their proofs are very deep; a more elementary proof has recently been given by Bojarski and Iwaniec [BI]. Mostow [Mo, (12.2)] pointed out that the case \( D = \mathbb{R}^n \) is much easier; another proof for this case has been given by Gehring [Ge₂]. The purpose of this note is to give a new and simple proof for this special case. It is based on the compactness properties of quasiconformal maps and on the fact that the 1-quasiconformal maps of \( \mathbb{R}^n \) form a group. It is also valid for \( n = 2 \).

2. Notation. For \( x \in \mathbb{R}^n \) and \( r > 0 \) we let \( S(x, r) \) denote the sphere \( \{ y \in \mathbb{R}^n : |y - x| = r \} \).

3. Lemma. Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a homeomorphism such that the image of each sphere \( S(x, r) \) is a sphere \( S(f(x), r) \). Then \( f \) is a similarity.

Proof. Let \( x, y \in \mathbb{R}^n \) with \( |x - y| = 2r > 0 \), and let \( z = (x + y)/2 \). Consider the sphere \( S_0 \) of radius \( r/2 \) which touches the spheres \( S_1 = S(x, r) \) and \( S_2 = S(x, 2r) \) at \( z \) and \( y \). Since \( fS_0 \) touches \( fS_1 \) and \( fS_2 \), \( f(z) \) lies on the line segment \( f(x)f(y) \). Since \( fS(z, r) \) is a sphere centered at \( f(z) \), \( f(z) = (f(x) + f(y))/2 \). Hence \( f \) preserves the midpoint of every line segment. By iteration and continuity, this implies that \( f \) is affine on every line. For each line \( L \), there is thus a number \( \lambda_L > 0 \) such that \( |f(a) - f(b)| = \lambda_L |a - b| \) for all \( a, b \in L \). Moreover, if the lines \( L \) and \( M \) intersect, \( \lambda_L = \lambda_M \). It follows that \( \lambda_L = \lambda \) is independent of \( L \).

4. Theorem. Let \( n \geq 2 \) and let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be 1-quasiconformal. Then \( f \) is a similarity.

Proof. By the preceding lemma, it suffices to show that \( f \) maps every sphere \( S(x, r) \) onto a sphere centered at \( f(x) \). With the aid of auxiliary similarity maps, we may assume that \( x = 0 = f(x) \), that \( r = 1 \), that \( f(e_1) = e_1 \), and that the open unit ball \( B^n \) is contained in \( fB^n \). Let \( W \) be the family of all 1-quasiconformal maps \( g: \mathbb{R}^n \to \mathbb{R}^n \) such that \( g(0) = 0, g(e_1) = e_1 \), and \( B^n \subset gB^n \). Since \( W \) is a closed nonempty normal family [Vä, 19.4, 21.3, 37.4], there is \( h \in W \) for which

\[
\text{m}(hB^n) = \max \{ \text{m}(gB^n) : g \in W \} = M < \infty.
\]

It suffices to show that $M = m(\overline{B}^n)$. If $M > m(\overline{B}^n)$, $\overline{B}^n$ is a proper subset of $h\overline{B}^n$, and hence $h\overline{B}^n$ is a proper subset of $hh\overline{B}^n$, which implies $m(hh\overline{B}^n) > M$. Since $hh \subseteq W$, this is a contradiction. □

5. Remark. The preceding theorem is also trivially true for $n=1$, if we, as usual, interpret the $K$-quasisymmetric functions $f: R^1 \rightarrow R^1$ as one-dimensional $K$-quasiconformal maps. On the other hand, the same proof gives the following more general result, which is nontrivial also for $n=1$. We allow the possibility that a quasiconformal map is sense-reversing.

6. Theorem. Let $n \geq 1$, let $K \geq 1$, and let $G$ be a group of $K$-quasiconformal maps of $R^n$ such that $G$ contains all similarity maps. Then $G$ is precisely the group of all similarity maps of $R^n$.

In the proof, we may assume that $G$ is closed, replacing it by $\overline{G}$. □

Actually, it is sufficient to assume that $G$ contains a group $S$ of similarities such that for each pair of distinct points $x, y \in R^n$ there is $g \in S$ such that $g(0) = x$, $g(e_i) = y$. The proof shows that every element of $G$ is then a similarity.

References


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