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GENERALIZED MEANS OF SUBHARMONIC FUNCTIONS

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1. Introduction

This paper is concerned with means of subharmonic functions over various bounded surfaces in Euclidean space $\mathbf{R}^n$ $(n \geqq 2)$. The simplest case is that of spherical means, which have played a fundamental rôle in the development of potential theory ever since the pioneering work of F. Riesz [26] in 1926. In particular, they have convexity properties, and their limiting behaviour for large radii may be used as a criterion for (e.g.) harmonic majorization in $\mathbf{R}^n$. A number of such properties are listed below in Theorem 11 (Section 12). However, if we wish to deal with a subharmonic function defined only in an unbounded proper subdomain of $\mathbf{R}^n$, then means over spheres with a common centre and arbitrarily large radii can no longer be considered.

In the half-space this problem was overcome by devising a "weighted" half-spherical mean, the development of which can be traced through papers by Ahlfors [1], Tsuji [27], Huber [19], Dinghas [11], Ahlfors [2], Kuran [22], [23] and Armitage [3], [4]. A corresponding cylindrical mean in the infinite strip, studied by Heins [17] and Brawn [8], [9], has only recently [5] been explored to an extent that approaches the half-spherical mean, and Fugard [13] has analogously investigated conical means in the infinite cone.

Each of these weighted means has been separately studied at some length, and shown to behave in a manner very similar to spherical means. In this paper we extend the work of [14] and present a unified theory of such means, which we define in terms of level surfaces of suitable functions. Some links may be seen here with work by Wu [29], who considers integral means of subharmonic functions over level curves of certain other harmonic functions in the plane. Also, in broad outline, there are similarities with recent work by Armitage [4] in the half-space. However, there is little in common with respect to the methods employed, as that paper relies heavily on a passage technique (due to Huber [19] and Kuran [22]), which is special to the half-space.

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As in [14], we shall first give the general theory, and then conclude with specific applications (Sections 12—14). However, in view of the more difficult nature of the work, we shall attempt to illuminate the general exposition by concurrent reference to the two-dimensional strip.

2. The framework

Points of $\mathbf{R}^n$ will be denoted by capital letters such as X, Y, Z, P, or Q; in particular, O will represent the origin of co-ordinates. When appropriate, X will be written in terms of its co-ordinates

$$X = (x_1, \ldots, x_n) = (X', x_n)$$

where $X' \in \mathbf{R}^{n-1}$. The closure and boundary of a subset A of $\mathbf{R}^n$ will be denoted by $\bar{A}$ and ∂A respectively, and, using $|X|$ to represent the Euclidean norm of X, we define

$$B(X, r) = \{Y \in \mathbf{R}^n\colon\ |Y - X| < r\}.$$

It will be convenient also to use $N(X)$ to denote the set of bounded open neighbourhoods of a point X in $\mathbf{R}^n$.

We recall that a bounded domain $\omega \subset \mathbf{R}^n$ is called a Lipschitz domain if $\partial\omega$ can be covered by right circular cylinders whose bases have positive distances from $\partial\omega$, and corresponding to each cylinder L, there is a co-ordinate system $(\tilde{X}', \tilde{x}_n)$ with $\tilde{x}_n$-axis parallel to the axis of L, a function $f\colon \mathbf{R}^{n-1} \to \mathbf{R}$ and a real number c such that

$$|f(\tilde{X}') - f(\tilde{Y}')| \leqq c\,|\tilde{X}' - \tilde{Y}'|$$

for all $\tilde{X}', \tilde{Y}' \in \mathbf{R}^{n-1}$,

$$L \cap \omega = \{X \in L\colon\ \tilde{x}_n > f(\tilde{X}')\}$$

and

$$L \cap \partial\omega = \{X \in L\colon\ \tilde{x}_n = f(\tilde{X}')\}.$$

(The extra generality of non-tangentially accessible domains (see [21]) is unnecessary for the type of applications we have in mind.)

An account of the Perron—Wiener—Brelot generalized solution of the Dirichlet problem is given in Helms' book [18, Chapter 8], and we shall adopt his notation. Thus, if f is resolutive on the boundary of an open set W, the Dirichlet solution is given by H_f^W.

Let Ω be an unbounded domain in $\mathbf{R}^n$ such that, for each $r > 0$, there is an open set $W_r' \supseteq B(O, r)$ for which $\Omega_r' = W_r' \cap \Omega$ is a Lipschitz domain. To avoid having to deal repeatedly with it as a special case, we shall exclude the possibility of $\Omega = \mathbf{R}^2$. We now state a number of lemmas, whose proofs will be given in Sections 6 and 7.

Lemma 1. *There exist*

(a) *a Green kernel G for Ω such that, if $X \in \Omega$, then $G(X, .)$ continuously vanishes on $\partial\Omega$, and*

(b) *at least one positive harmonic function h in Ω which continuously vanishes on $\partial\Omega$.*

In view of (b) above, we define h_* to be a (fixed) positive harmonic function in Ω which vanishes on $\partial\Omega$. We also let ν be a fixed (non-zero) Borel measure with compact support $E \subset \bar{\Omega}$.

Lemma 2. *The function $G(X, Y)/\{h_*(X)h_*(Y)\}$ has a positive, symmetric, jointly continuous extension to $(\bar{\Omega}\times\bar{\Omega})\setminus\{(X, Y)\colon X=Y\in\partial\Omega\}$ (continuous in the extended sense at points of the diagonal of $\Omega\times\Omega$), which we denote by $G^*(X, Y)$. Further, $h_*(.)G^*(., Y)$ is harmonic in $\Omega\setminus\{Y\}$.*

We define

$$\Phi(X) = \int_E G^*(X, Z)\, d\nu(Z) \quad (X\in\bar{\Omega}\setminus(E\cap\partial\Omega)),$$

and extend Φ to be defined on $\bar{\Omega}$ by writing

$$\Phi(X) = \liminf_{Y\to X} \Phi(Y) \quad (X\in E\cap\partial\Omega).$$

Clearly Φ is lower semicontinuous (l.s.c.) on $\bar{\Omega}$. We also have:

Lemma 3. *The function Φ is positive on $\bar{\Omega}$, and $h_*\Phi$ is superharmonic in Ω, harmonic in $\Omega\setminus E$ and continuously vanishes on $\partial\Omega\setminus E$.*

Definition 1. Let $\varkappa$ denote the (positive, possibly infinite) infimum of Φ on E, and let φ denote a (fixed) strictly decreasing mapping from $(0, +\infty)$ onto $(0, \varkappa)$ (which implies that φ is continuous and invertible). Since Φ is l.s.c. on $\bar{\Omega}$, there exists (for each $x>0$) an open set W_x such that

$$W_x\cap\bar{\Omega} = \{X\in\bar{\Omega}\colon\ \Phi(X) > \varphi(x)\}.$$

We shall suppose that each $\Omega_x = W_x\cap\Omega$ is a Lipschitz domain, and that, if $x<w$, then $\bar{\Omega}_w\setminus W_x$ is the disjoint union of the closures of finitely many Lipschitz domains. This will certainly be the case in our applications. We abbreviate the sets $\partial\Omega_x\cap\Omega$ and $W_x\cap\partial\Omega$ to σ_x and τ_x respectively, and denote harmonic measure with respect to Ω_x and $X\in\Omega_x$ by $\mu_{x,X}$. In view of [24, Théorème 25] and the fact that a cone internal to Ω_w with vertex at $Z\in\partial\Omega_w$ is non-thin at Z (see, for example, [20, Lemma (3.6)]), it follows that $\partial\Omega_x\cap\partial\Omega\setminus\tau_x$ has $\mu_{x,X}$-measure zero for any $X\in\Omega_x$. The Green kernel for Ω_x will be denoted by G_x.

Lemma 4.

(a) $E \subseteqq \bigcap_{x>0} \bar{\Omega}_x$;

(b) $\bar{\Omega} = \bigcup_{x>0} \bar{\Omega}_x$;

(c) $x < w \Rightarrow \bar{\Omega}_x \subset W_w\cap\bar{\Omega}$;

(d) $\Phi(X) = \varphi(x)$ *for* $X\in\sigma_x$.

For suitable functions f we define

$$I_{f,x}(X) = \int_{\sigma_x} f(Z)\, d\mu_{x,X}(Z),$$

and

$$H_{f,x}(X) = H_f^{\Omega_x}(X) - \int_{\tau_1} f(Z)\, d\mu_{1,X}(Z),$$

which are clearly harmonic in $\Omega_{\min\{x,1\}}$, provided that the integrals are finite. It was shown in [14, Lemma 1] that the quotients $H_{f,x}/h_*$ and $I_{f,x}/h_*$ can be continuously defined on $W_y \cap \bar{\Omega}$, where $y = \min\{x, 1\}$. Denoting these extended functions respectively by $\mathscr{H}_{f,x}$ and $\mathscr{I}_{f,x}$, we define

$$\mathscr{M}(f, x) = \int_E \mathscr{H}_{f,x}(X)\, dv(X)$$

and

$$\mathscr{N}(f, x) = \int_E \mathscr{I}_{f,x}(X)\, dv(X).$$

Let s be subharmonic in Ω and extend it to $\bar{\Omega}$ by

$$s(Z) = \limsup_{X \to Z} s(X) \quad (Z \in \partial\Omega).$$

If, for each $Z \in \partial\Omega$, there is a bounded neighbourhood of Z, whose intersection ω with Ω satisfies

(i) the restriction of s to $\partial\omega$ is resolutive for ω, and

(ii) $s \leqq H_s^\omega$ in ω,

then we say that $s \in \mathscr{L}\mathscr{D}$.

If $s \in \mathscr{L}\mathscr{D}$, then it follows from [15, Theorem 2 (i)] that s is resolutive for every Ω_x, and $s \leqq H_s^{\Omega_x}$ in Ω_x. Hence $H_{s,x}$, $\mathscr{H}_{s,x}$, $\mathscr{M}(s, x)$, $I_{s,x}$, $\mathscr{I}_{s,x}$, and $\mathscr{N}(s, x)$ all exist, and it is easy to see (cf. [14, Theorem 1]) that

(i) $\mathscr{M}(s, x)$ is an increasing[1]), real-valued function of x.

(ii) If also $s \leqq 0$ on $\partial\Omega$, then the same is true of $\mathscr{N}(s, x)$.

(iii) If h is harmonic in Ω and continuous on $\bar{\Omega}$, then $\mathscr{M}(h, x)$ is a constant function of x.

Lemma 5. *The function F_x, defined on $\bar{\Omega} \times \Omega_x$ by*

$$F_x(X, Y) = I_{h_*(\cdot) G^*(X, \cdot), x}(Y)/h_*(Y),$$

has a jointly continuous extension to $\bar{\Omega} \times (W_x \cap \bar{\Omega})$ such that $h_*(.)F_x(., Y)$ is harmonic in Ω_x for any Y. Further,

$$G^*(X, Y) = F_x(X, Y) \quad (X \notin W_x \cap \bar{\Omega}),$$

and

$$G^*(X, Y) = F_x(X, Y) + \lim_{(P,Q) \to (X,Y)} G_x(P, Q)/\{h_*(P) h_*(Q)\} \tag{1}$$

if $X, Y \in W_x \cap \bar{\Omega}$ and $X \neq Y$, the limit being unnecessary if both X and Y are in Ω_x.

The extended function of the above lemma will also be denoted by F_x.

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[1]) We use increasing in the wide sense.

We conclude this section by illustrating some of our definitions.

Example 1. Consider the two-dimensional case of the strip, so that

$$\Omega = (-1, 1)\times\mathbf{R}, \quad E = [-1, 1]\times\{0\}$$

$$dv(x_1, x_2) = 8\pi^{-2}\cos^2\left(\tfrac{1}{2}\pi x_1\right) dx_1\, d\delta_0(x_2)$$

and

$$h_*(x_1, x_2) = 2\pi^{-1}\cos\left(\tfrac{1}{2}\pi x_1\right)\cosh\left(\tfrac{1}{2}\pi x_2\right),$$

where δ_0 is the Dirac measure at the origin of $\mathbf{R}$. The Green kernel for Ω is well-known (see, for example, [7, Lemmas 3, 4] and use a simple conformal mapping); in particular,

$$G((x_1, x_2), (y_1, 0))$$
$$= 2\sum_{m=1}^{\infty} m^{-1}\sin\left[\tfrac{1}{2}m\pi(x_1+1)\right]\sin\left[\tfrac{1}{2}m\pi(y_1+1)\right]\exp\left(-\tfrac{1}{2}m\pi\,|x_2|\right).$$

If $x_2\neq 0$, then clearly the series converges uniformly in y_1, and so we can integrate term-by-term to obtain

$$\begin{aligned}\Phi(x_1, x_2) &= 2\int_{-1}^{1} G((x_1, x_2), (y_1,0))\cos\left(\tfrac{1}{2}\pi y_1\right)/\left\{\cos\left(\tfrac{1}{2}\pi x_1\right)\cosh\left(\tfrac{1}{2}\pi x_2\right)\right\} dy_1\\ &= 4\,\mathrm{sech}\left(\tfrac{1}{2}\pi x_2\right)\exp\left(-\tfrac{1}{2}\pi|x_2|\right)\\ &= 8\{1+\exp(\pi\,|x_2|)\}^{-1}.\end{aligned}$$

This remains valid for $x_2=0$ by the l.s. continuity of Φ. Thus $\varkappa=4$ and, defining $\varphi\colon (0, +\infty)\to(0, 4)$ by

$$\varphi(x) = 8\{1+\exp(\pi x)\}^{-1},$$

it follows that $\Omega_x=(-1, 1)\times(-x, x)$. The assumptions of Definition 1 are now easily seen to hold.

3. The generalized mean

If s is subharmonic in Ω, then the measure associated with s in Ω is given by $\mu_s=\gamma_n\Delta s$, where

$$\gamma_2 = (2\pi)^{-1}, \quad \gamma_n = \{(n-2)c_n\}^{-1} \quad (n\geqq 3),$$

c_n denoting the surface area of $\partial B(O, 1)$, and Δs is the distributional Laplacian of s in Ω. The following result associates a second measure, defined on $\partial\Omega$, with s.

Theorem 1. *If $s\in\mathcal{L}\mathcal{D}$, then there exists a unique measure λ_s on $\partial\Omega$ such that the least harmonic majorant of s in Ω_x is given by*

$$H_s^{\Omega_x}(Y)-h_*(Y)\int_{\tau_x}\{G^*(X, Y)-F_x(X, Y)\}\, d\lambda_s(X). \tag{2}$$

(We remark that, if s is subharmonic in an open set containing $\bar{\Omega}$, then the least harmonic majorant of s in Ω_x is given by $H_s^{\Omega_x}$, and so λ_s is the zero measure on $\partial\Omega$.)

Definition 2. We introduce a modified mean $\mathcal{M}^*(s, x)$ for $s \in \mathcal{LD}$, given by

$$\mathcal{M}^*(s, x) = \mathcal{M}(s, x) + \int_1^x \lambda_s(\tau_t)\, d\varphi(t),$$

where the latter term is a Riemann—Stieltjes integral.

Example 2. Following on from Example 1, we deduce from [14, Section 9] that

$$\mathcal{M}(s, x) = 2\pi^{-1} \operatorname{sech}\left(\tfrac{1}{2}\pi x\right) \int_{-1}^{1} \cos\left(\tfrac{1}{2}\pi x_1\right) \{s(x_1, x) + s(x_1, -x)\}\, dx_1$$

$$+ \int_1^x \operatorname{sech}^2\left(\tfrac{1}{2}\pi t\right) \int_{-t}^{t} \cosh\left(\tfrac{1}{2}\pi x_2\right) \{s(-1, x_2) + s(1, x_2)\}\, dx_2\, dt.$$

Since the derivative of $\varphi(x)$ is $-2\pi \operatorname{sech}^2\left(\tfrac{1}{2}\pi x\right)$, we have

$$\mathcal{M}^*(s, x) = 2\pi^{-1} \operatorname{sech}\left(\tfrac{1}{2}\pi x\right) \int_{-1}^{1} \cos\left(\tfrac{1}{2}\pi x_1\right) \{s(x_1, x) + s(x_1, -x)\}\, dx_1$$

$$+ \int_1^x \operatorname{sech}^2\left(\tfrac{1}{2}\pi t\right) \Big[\int_{-t}^{t} \cosh\left(\tfrac{1}{2}\pi x_2\right) \{s(-1, x_2) + s(1, x_2)\}\, dx_2$$

$$- 2\pi\lambda_s(\{-1, 1\} \times (-t, t))\Big]\, dt.$$

The following is a generalization of Nevanlinna's first fundamental theorem for subharmonic functions in $\mathbf{R}^n$ (see [16, p. 127]).

Theorem 2. *If* $s \in \mathcal{LD}$, *then*

$$\mathcal{M}^*(s, x) = \mathcal{N}(s, 1) - \int_1^x \int_{\Omega_t} h_*(Z)\, d\mu_s(Z)\, d\varphi(t).$$

Proofs of Theorems 1 and 2 may be found in Sections 8 and 9, respectively.

4. General results

Theorem 2 is used to deduce the main results of this paper.

Theorem 3. *Let* $s \in \mathcal{LD}$. *Then*

(i) $\mathcal{M}^*(s, x)$ *is increasing as a function of* x *and convex as a function of* $\varphi(x)$ *on* $(0, +\infty)$;

(ii) *if* $w > y > 0$ *and* s *is harmonic in* $\Omega_w \setminus \bar{\Omega}_y$, *then* $\mathcal{M}^*(s, x)$ *is a linear function of* $\varphi(x)$ *on* $[y, w]$;

(iii) $\mathcal{M}^*(s, x)$ *is constant on* $(0, +\infty)$ *if and only if* s *is harmonic in* Ω.

Theorem 4. *If* $s \in \mathscr{L}\mathscr{D}$, *then the following are equivalent:*
(i) *s has a harmonic majorant in* Ω;
(ii) $\mathscr{M}^*(s, x)$ *is bounded above on* $(0, +\infty)$;
(iii) $\int_{\Omega \setminus \Omega_1} h_*(X)\, \Phi(X)\, d\mu_s(X) < +\infty.$

Theorems 3 and 4 show that $\mathscr{M}^*(s, x)$ has "ideal" properties; that is, it behaves exactly like the ordinary spherical mean of subharmonic functions in $\mathbf{R}^n$ (of which it is a generalization). The major disadvantage of this mean is that λ_s has to be defined in a rather indirect fashion. Thus there is a case for discussing also the (slightly less satisfactory) properties of $\mathscr{M}(s, x)$, some of which have already been given in [14].

Theorem 5. (i) *If* $s \in \mathscr{L}\mathscr{D}$ *and* $\mathscr{M}(s, x)$ *is bounded above on* $(0, +\infty)$, *then s has a harmonic majorant in* Ω.
(ii) *Let s be subharmonic in an open set W containing* $\bar{\Omega}$. *Then s has a harmonic majorant in Ω if and only if* $\mathscr{M}(s, x)$ *is bounded above on* $(0, +\infty)$.

Part (i) holds since $\mathscr{M}^*(s, x) \leqq \mathscr{M}(s, x)$ (see Definition 2; φ is decreasing), and generalizes [14, Theorem 2]. Part (ii) is identical to [14, Theorem 3] and is immediate since, in this case $\lambda_s \equiv 0$.

Convexity results for $\mathscr{M}(s, x)$ were not considered in [14], but are now also easily derived.

Theorem 6. (i) *If* $s \in \mathscr{L}\mathscr{D}$, *then* $\mathscr{M}(s, x)$ *is increasing as a function of x, and convex as a function of* $\varphi(x)$ *on* $(0, +\infty)$.
(ii) *If also* $s \leqq 0$ *on* $\partial\Omega$, *then* $\mathscr{N}(s, x)$ *is increasing as a function of x, and convex as a function of* $\varphi(x)$ *on* $(0, +\infty)$.

Theorems 3, 4 and 6 are proved in Section 10.

5. Variant means

Analogous results for variants of the mean $\mathscr{N}(s, x)$ are now given.

Theorem 7. *If s is a non-negative subharmonic function in Ω which continuously vanishes on* $\partial\Omega$, *and* $1 \leqq p < +\infty$, *then the mean*

$$\mathscr{N}_p(s, x) = \{\mathscr{N}(h_*^{1-p} s^p, x)\}^{1/p}$$

is real-valued, convex as a function of $\varphi(x)$, *and increasing as a function of* $x > 0$.

Theorem 8. *If u is a positive superharmonic function in* Ω, *and* $p \in (-\infty, 0) \cup (0, 1)$, *then* $\mathscr{N}_p(u, x)$ *is real-valued, concave as a function of* $\varphi(x)$, *and decreasing as a function of* $x > 0$.

Theorem 9. *If* $s\in\mathscr{L}\mathscr{D}$ *and* $s\leqq 0$ *on* $\partial\Omega$, *then the "mean"*

$$\mathscr{N}_\infty(s,x)=\sup\{s(X)/h_*(X)\colon X\in\sigma_x\}$$

is real-valued, convex as a function of $\varphi(x)$, *and increasing as a function of* $x>0$.

Theorem 10. *If* $s\in\mathscr{L}\mathscr{D}$ *and* $s\leqq 0$ *on* $\partial\Omega$, *then the mean*

$$\mathscr{N}_E(s,x)=\log\mathscr{N}(h_*\exp(s/h_*),x)$$

is real-valued, convex as a function of $\varphi(x)$, *and increasing as a function of* $x>0$.

The proofs of these theorems are closely related, and are based on a technique of Fugard [12, Chapter 2] (or see [5, Theorems 7 and 8]). We shall illustrate this by giving the proof of Theorem 10 in Section 11. Theorem 9 is a generalization of Hadamard's Three Circles Theorem, and can equivalently be stated in terms of the infimum of u/h_* over σ_x for suitable superharmonic functions u. It is a little easier to prove, and the maximum principle can be used to establish the monotonicity part of the result.

6. Proofs of Lemmas 1—4

6.1. We shall make use of the following results.

Theorem A. (Boundary Harnack principle.) *Let* Ω' *be a bounded Lipschitz domain of which* P *is a fixed point,* A *be a relatively open subset of* $\partial\Omega'$, *and* W' *be a subdomain of* Ω' *satisfying* $\partial\Omega'\cap\partial W'\subseteqq A$. *Then there is a constant* c *such that, if* h_1 *and* h_2 *are two positive harmonic functions in* Ω' *vanishing on* A *and* $h_1(P)=h_2(P)$, *then* $h_1(X)\leqq ch_2(X)$ *for all* $X\in W'$.

Theorem B. *If* h_1 *and* h_2 *are positive harmonic functions on a bounded Lipschitz domain* Ω' *vanishing on a relatively open subset* A *of* $\partial\Omega'$, *then* h_1/h_2 *can be continuously extended to a strictly positive function defined on* $\Omega'\cup A$.

For Theorem A we refer to either Dahlberg [10, Theorem 4] or Wu [30, Theorem 1]. If the set A is empty, then the result reduces to the usual Harnack inequality [18, Theorem 2.16]. Alternative proofs for Theorem B can be found in [21, (7.9)] and [6, Theorem 2].

6.2. To prove Lemma 1, first note that Ω has a Green kernel. If $n\geqq 3$, this is immediate; if $n=2$, choose r such that $W_r'\cap\partial\Omega$ is non-empty. Since Ω_r' is Lipschitz, there exist Y and $\varepsilon>0$ such that $\bar B(Y,\varepsilon)\subseteqq W_r'\setminus\bar\Omega$, whence $\Omega\subseteqq\mathbf{R}^2\setminus\bar B(Y,\varepsilon)$ and so Ω has a Green kernel.

Denoting this kernel by G and letting $X\in\Omega$, we show that $G(X,.)$ vanishes on $\partial\Omega$. Fix r such that $X\in\Omega_r'$, let G_r' be the Green kernel for Ω_r', and define f_X on

$\partial\Omega_r'$ by setting it equal to $G(X, .)$ on $\partial\Omega_r'\cap\Omega$ and 0 elsewhere. Then the function

$$s(Y) = \begin{cases} G(X, Y) - G_r'(X, Y) - H_{f_X}^{\Omega_r'}(Y) & (Y\in\Omega_r') \\ 0 & (Y\in\Omega\setminus\Omega_r') \end{cases}$$

is easily seen to be a non-negative subharmonic minorant of $G(X, Y)$ in Ω (since Ω_r' is regular) and so is identically zero. Since $G_r'(X, .)$ and $H_{f_X}^{\Omega_r'}$ vanish on $W_r'\cap\partial\Omega$, so does $G(X, .)$, and since r may be arbitrarily large, part (a) is proved.

To prove (b), let Q be a fixed point in Ω and (Y_m) be an unbounded sequence of points in Ω. By choosing a suitable subsequence if necessary (compactness argument), we may assume that (Y_m) converges to a Martin boundary point of Ω. If $\partial\Omega$ is empty, the lemma is trivially true. Otherwise, let r and R be such that $\bar{\Omega}_r'\subset W_R'$ and $Q\in\Omega_R'$, and such that $W_r'\cap\partial\Omega$ is non-empty. Fix $P\in\Omega\setminus\bar{\Omega}_R'$. From Theorem A there is a constant c such that

$$G(Y_m, X)/G(Y_m, Q) \leqq cG(P, X)/G(P, Q) \quad (X\in\Omega_r')$$

whence

$$h(X) = \lim_{m\to\infty} G(Y_m, X)/G(Y_m, Q) \leqq c'G(P, X) \quad (X\in\Omega_r').$$

Thus h is a positive harmonic function in Ω_r' which vanishes on $W_r'\cap\partial\Omega$. Since r may be arbitrarily large, (b) is proved.

6.3. We now prove Lemma 2. Let $X_0, Y_0\in\bar{\Omega}$. Joint continuity clearly holds at (X_0, Y_0) unless at least one of X_0, Y_0 is in $\partial\Omega$. We shall consider the case where both X_0 and Y_0 are in $\partial\Omega$ and $X_0\neq Y_0$, the case where only one of X_0, Y_0 is in $\partial\Omega$ being similar and easier. It is clearly sufficient to show that $G^*(X, Y)$ has a limit as (X, Y) tends to (X_0, Y_0) from within $\Omega\times\Omega$.

Let $U_i\in N(X_0)$ and $V_i\in N(Y_0)$ $(i=1, 2, 3)$ be such that

(i) $\bar{U}_3\subset U_2\subset\bar{U}_2\subset U_1$ and similarly for V_i;

(ii) $\bar{U}_1\cap\bar{V}_1=\emptyset$;

(iii) the sets $U_i'=U_i\cap\Omega$ and $V_i'=V_i\cap\Omega$ are Lipschitz domains (this is possible because $X_0, Y_0\in W_r'$ for sufficiently large r). We denote harmonic measure for U_i' and $X\in U_i'$ by $\lambda_{i,X}$, and for V_i' and $Y\in V_i'$ by $\nu_{i,Y}$.

In view of (ii), $G(X, Y)$ is bounded above, by c say, for (X, Y) in $U_1'\times V_1'$. From Theorem A

$$\begin{aligned} G(X, Y) &= \int_{\partial U_1\cap\Omega} G(Z, Y)\, d\lambda_{1,X}(Z) \\ &\leqq c\lambda_{1,X}(\partial U_1\cap\Omega) \leqq c^i h_*(X) \end{aligned}$$

for $X\in U_2'$ and $Y\in V_1'$. Repeating this argument, we obtain

$$G(X, Y)/h_*(X) = \int_{\partial V_1\cap\Omega} G(X, Z)/h_*(X)\, d\nu_{1,Y}(Z) \leqq c^{ii} h_*(Y) \tag{3}$$

for $X\in U_2'$ and $Y\in V_2'$.

Let $\varepsilon'>0$. It follows from (3) and the joint continuity of G in $\Omega\times\Omega$ that there exists $\delta>0$ such that

$$|G(X_1,Y)-G(X_2,Y)|<\varepsilon' \quad (Y\in\partial V_2\cap\Omega)$$

for $X_1, X_2\in\partial U_2\cap\Omega$ satisfying $|X_1-X_2|<\delta$. Again using Theorem A

$$|G(X_1,Y)-G(X_2,Y)|\leqq\varepsilon' v_{2,Y}(\partial V_2\cap\Omega)<c^{iii}\varepsilon' h_*(Y) \quad (Y\in V_3'). \tag{4}$$

From Theorem B, the functions

$$\{G(X,.)/h_*(.)\colon X\in\partial U_2\cap\Omega\}$$

have continuous extensions to V_3'. Hence, from (3), (4) and the fact that $\partial U_2\cap\Omega$ is relatively compact, we can apply the Arzelà—Ascoli theorem to see that they are equicontinuous on $\overline{V}_3'$.

Let $\varepsilon>0$. Then there exists $V_\varepsilon\in N(Y_0)$ such that $\overline{V}_\varepsilon\subset V_3$ and

$$|G(X,Y_1)/h_*(Y_1)-G(X,Y_2)/h_*(Y_2)|<\varepsilon \quad (X\in\partial U_2\cap\Omega)$$

for $Y_1, Y_2\in V_\varepsilon'=V_\varepsilon\cap\Omega$, and so

$$|G(X,Y_1)/h_*(Y_1)-G(X,Y_2)/h_*(Y_2)|\leqq\varepsilon\lambda_{2,X}(\partial U_2\cap\Omega)<c^{iv}\varepsilon h_*(X)$$

for $X\in U_3'$. Thus we have

$$|G^*(X,Y_1)-G^*(X,Y_2)|<c^{iv}\varepsilon \quad (X\in U_3';\ Y_1,Y_2\in V_\varepsilon').$$

Correspondingly, we obtain U_ε' such that

$$|G^*(X_1,Y)-G^*(X_2,Y)|<c^{v}\varepsilon \quad (X_1,X_2\in U_\varepsilon';\ Y\in V_3'),$$

and so

$$|G^*(X_1,Y_1)-G^*(X_2,Y_2)|<(c^{iv}+c^{v})\varepsilon \quad (X_1,X_2\in U_\varepsilon';\ Y_1,Y_2\in V_\varepsilon'),$$

where c^{iv} and c^{v} are independent of ε. A completeness argument now shows that $G^*(X,Y)$ has a limit as $(X,Y)\to(X_0,Y_0)$.

The harmonicity of $h_*(.)G^*(.,Y)$ in $\Omega\setminus\{Y\}$ is clear if $Y\in\Omega$. If $Y\in\partial\Omega$, let (Y_m) be a sequence of points in Ω, converging to Y. In view of the joint continuity of G^*, the functions $h_*(.)G^*(.,Y_m)$ are locally uniformly bounded in Ω and so their limit is harmonic in Ω (see [18; Theorem 2.18]).

The positivity of G^* is a consequence of Theorem B, and the symmetry is obvious from the symmetry of G.

6.4. In Lemma 3, since G^* is positive and ν is non-zero, the positivity of Φ need be checked only at points Z of $E\cap\partial\Omega$. To do this, we choose r such that $E\subset W_r'$ and apply [24, Théorème 7'—16] and Theorem B to see that

$$\Phi(Z)=\{\liminf_{X\to Z} h_*(X)\Phi(X)/G_r'(X,Q)\}\{\lim_{X\to Z} G_r'(X,Q)/h_*(X)\}>0,$$

where Q is an arbitrary point of Ω_r'.

Let ν_1 and ν_2 be the restrictions of ν to $\partial\Omega\cap E$ and $\Omega\cap E$ respectively, and define

$$d\nu_3(Y) = \{h_*(Y)\}^{-1}\, d\nu_2(Y).$$

In view of Lemma 2, $h_*\Phi$ is clearly finite on $\Omega\setminus E$, and so the function $G\nu_3$ is a potential in Ω. It is immediate also from Lemma 2 that $h_*\Phi$ vanishes on $\partial\Omega\setminus E$, and so it remains only to show that

$$\int_{\partial\Omega\cap E} h_*(.)G^*(., Y)\, d\nu_1(Y)$$

is harmonic in Ω. In fact, Lemma 2 ensures that this function is continuous in Ω and Fubini's theorem shows that the mean-value equality holds for all sufficiently small spheres centred at any $X\in\Omega$.

6.5. Lemma 4 is straightforward to establish. Since, for each x, $\varphi(x)<\varkappa$, it follows that

$$E \subseteqq W_x\cap\bar{\Omega} \subseteqq \bar{\Omega}_x,$$

and so (a) holds. Part (b) is true because Φ is positive in $\bar{\Omega}$. Since Φ is continuous in $\bar{\Omega}\setminus E$, we have $\Phi(X)\geqq\varphi(x)$ for $X\in\bar{\Omega}_x$, and so

$$x < w \Rightarrow \varphi(x) > \varphi(w) \Rightarrow \bar{\Omega}_x \subset W_w\cap\bar{\Omega},$$

proving (c). Part (d) follows from the continuity of Φ at points of σ_x.

7. Proof of Lemma 5

Once the joint continuity of F_x is established, the harmonicity of $h_*(.)F_x(., Y)$ follows easily as in Lemma 2. Let

$$S_1 = (\Omega\setminus\bar{\Omega}_x)\times\Omega_x, \quad S_2 = \Omega_x\times\Omega_x$$

and

$$(X_0, Y_0)\in\bar{\Omega}\times(W_x\cap\bar{\Omega}).$$

We shall show that, as (X, Y) tends to (X_0, Y_0) from within $S_1\cup S_2$, the function $F_x(X, Y)$ tends to a limit, which equals $F_x(X_0, Y_0)$ if $Y_0\in\Omega_x$. Our proof falls naturally into three parts.

Case I: $X_0\in\bar{\Omega}\setminus\bar{\Omega}_x$. For $X\in\Omega\setminus\bar{\Omega}_x$, the function $h_*(.)G^*(X, .)$ is harmonic in Ω_x, continuous in $\bar{\Omega}_x$ and valued zero on τ_x (see Lemma 2). Thus, if $(X, Y)\in S_1$, then

$$F_x(X, Y) = h_*(Y)\,G^*(X, Y)/h_*(Y) = G^*(X, Y).$$

As $(X, Y)\to(X_0, Y_0)$, we have $F_x(X, Y)\to G^*(X_0, Y_0)$, and if $Y_0\in\Omega_x$, then $G^*(X_0, Y_0)=F_x(X_0, Y_0)$ as required.

Case II: $X_0 \in \Omega_x \cup \tau_x = W_x \cap \bar{\Omega}$. Let $\varepsilon > 0$. From the joint continuity of G^* there exists $U_\varepsilon \in N(X_0)$ such that $\bar{U}_\varepsilon \subset W_x$ and

$$|G^*(X_1, Z) - G^*(X_2, Z)| < \varepsilon$$

for X_1 and X_2 in $U_\varepsilon \cap \bar{\Omega}$ and Z in σ_x. Hence

$$|F_x(X_1, Y) - F_x(X_2, Y)| < \varepsilon$$

for X_1 and X_2 in $U_\varepsilon \cap \bar{\Omega}$ and $Y \in \Omega_x$. Since

$$G(X, Y) = I_{G(X,\cdot),x}(Y) + G_x(X, Y) \quad (X, Y \in \Omega_x), \tag{5}$$

it is clear that F_x is symmetric in $\Omega_x \times \Omega_x$ and so there also exists $V_\varepsilon \in N(Y_0)$ such that $\bar{V}_\varepsilon \subset W_x$ and

$$|F_x(X, Y_1) - F_x(X, Y_2)| < \varepsilon$$

for X in Ω_x and Y_1 and Y_2 in $V_\varepsilon \cap \Omega$. If we let $X_3 \in U_\varepsilon \cap \Omega$, then

$$\begin{aligned} |F_x(X_1, Y_1) - F_x(X_2, Y_2)| &\leqq |F_x(X_1, Y_1) - F_x(X_3, Y_1)| \\ &\quad + |F_x(X_3, Y_1) - F_x(X_3, Y_2)| + |F_x(X_3, Y_2) - F_x(X_2, Y_2)| \\ &< 3\varepsilon \end{aligned}$$

for X_1 and X_2 in $U_\varepsilon \cap \bar{\Omega}$ and $Y_1, Y_2 \in V_\varepsilon \cap \Omega$. A completeness argument shows that F_x has a limit as $(X, Y) \to (X_0, Y_0)$. If $Y_0 \in \Omega_x$, then choose $(X_2, Y_2) = (X_0, Y_0)$ to see that the limit is, in fact, $F_x(X_0, Y_0)$. Also, (1) now follows from (5) and Lemma 2.

Case III: $X_0 \in \bar{\Omega}_x \setminus W_x$. This is the most difficult case to prove. Let $Q, Y \in \Omega_x$. If $w > x$, then, as has already been observed in Section 2, $\Omega_w \setminus \bar{\Omega}_x$ is non-thin at X_0 in the minimal fine topology for Ω_w, whence

$$\int_{\sigma_x} \lim_{X \to X_0} G_w(X, Z)/G_w(X, Q)\, d\mu_{x,Y}(Z) = \lim_{X \to X_0} G_w(X, Y)/G_w(X, Q),$$

or, in view of Theorem B,

$$\int_{\sigma_x} \lim_{X \to X_0} G_w(X, Z)/h_*(X)\, d\mu_{x,Y}(Z) = \lim_{X \to X_0} G_w(X, Y)/h_*(X). \tag{6}$$

Also, from II, F_w is jointly continuous in $(W_w \cap \bar{\Omega}) \times (W_w \cap \bar{\Omega})$ and $h_*(.) F_w(X_0, .)$ is harmonic in Ω_w, and so

$$\int_{\sigma_x} h_*(Z) F_w(X_0, Z)\, d\mu_{x,Y}(Z) = h_*(Y)\, F_w(X_0, Y). \tag{7}$$

Thus, from (1), (6) and (7),

$$F_x(X_0, Y) = \int_{\sigma_x} h_*(Z)\, G^*(X_0, Z)\, d\mu_{x,Y}(Z)/h_*(Y) = G^*(X_0, Y) \quad (Y \in \Omega_x).$$

The lemma will follow if we show that

$$F_x(X, Y) \to G^*(X_0, Y_0) \quad ((X, Y) \to (X_0, Y_0);\ (X, Y) \in S_1 \cup S_2).$$

This is clearly true as (X, Y) tends to (X_0, Y_0) from within S_1, since $F_x = G^*$ there. If $(X, Y) \in S_2$, then it follows from (1) that we need only prove

$$G_x(X, Y)/\{h_*(X)h_*(Y)\} \to 0 \quad ((X, Y) \to (X_0, Y_0)). \tag{8}$$

We use the non-thinness of $\Omega_w \setminus \bar{\Omega}_x$ at X_0 and [24, Théorème 11] to observe that

$$G_x(X, Q)/G_w(X, Q) \to 0 \quad (X \to X_0;\ X \in \Omega_x)$$

and so, since $G_w(X, Q)/h_*(X)$ has a positive finite limit at X_0 (see Theorem B when $X_0 \in \partial\Omega$),

$$G_x(X, Q)/h_*(X) \to 0 \quad (X \to X_0;\ X \in \Omega_x). \tag{9}$$

Now, since $Y_0 \in W_x \cap \bar{\Omega}$ and $Q \in \Omega_x$, we can choose $z < y < x$ such that Y_0 is in $W_z \cap \bar{\Omega}$ and $Q \in \Omega_y$. Thus we may apply Theorem A with $\Omega = \Omega_y$ and $\Omega_0 = \Omega_z$ to obtain the existence of a positive constant c such that

$$G_x(X, Y)/G_x(X, Q) \leqq ch_*(Y)/h_*(Q)$$

and so

$$G_x(X, Y)/\{h_*(X)h_*(Y)\} \leqq c' G_x(X, Q)/h_*(X) \tag{10}$$

for $X \in \Omega_x \setminus \Omega_y$ and $Y \in \Omega_z$. Combining (9) and (10) yields (8) as required.

8. Proof of Theorem 1

8.1. We recall (see [20, Theorem (4.2)]) that, if Ω' is a Lipschitz domain, then every Martin boundary point of Ω' is minimal, and the set Δ_1 of Martin boundary points of Ω' can be put into one-to-one correspondence with $\partial\Omega'$ in such a way that the Martin topology on $\Omega' \cup \Delta_1$ is equivalent to the Euclidean topology on $\bar{\Omega}'$.

Lemma 6. *Let h be a non-negative harmonic function in a Lipschitz domain Ω', and let μ be the measure on $\partial\Omega'$ associated with it in the Martin representation. If A is a relatively open subset of $\partial\Omega'$, then h vanishes continuously on A if and only if $\mu(A) = 0$.*

The "only if" part follows from [30, Lemma 10]. The "if" part is trivial if A is empty. Otherwise, let $Z \in A$ and choose $W \in N(Z)$ such that $\Omega'' = W \cap \Omega'$ is a domain and $\partial\Omega'' \cap \partial\Omega' \subset A$. Let $Q \in \Omega' \setminus \bar{\Omega}''$ and $P \in \Omega''$. For a small positive value of ε, we can now apply Theorem A with Ω' replaced by $\Omega' \setminus \bar{B}(Q, \varepsilon)$ to deduce that

$$K(Y, X) \leqq cK(Y, P)\, G'(Q, X)/G'(Q, P) \tag{11}$$

for $Y \in \partial\Omega' \setminus A$ and $X \in \Omega''$, where G' is the Green kernel for Ω' and $K(.,.)$ denotes the Martin kernel on $\partial\Omega' \times \Omega'$ (by the Poincaré—Zaremba cone criterion [18, Theorem 8.27], $G(Q, .)$ vanishes on $\partial\Omega'$, and it is shown in [20] that $K(Y, .)$ vanishes on $\partial\Omega'\setminus\{Y\}$). Integrating both sides of (11) with respect to $d\mu(Y)$, it follows that

$$h(X) \leqq c' h(P) G'(Q, X) \quad (X \in \Omega'')$$

and so h vanishes at Z as required.

Lemma 7. *If λ and μ are measures on τ_x such that, for all $Y \in \Omega_x$, we have*

$$\int_{\tau_x} \{G^*(X, Y) - F_x(X, Y)\}\, d\lambda(X) = \int_{\tau_x} \{G^*(X, Y) - F_x(X, Y)\}\, d\mu(X),$$

then $\lambda = \mu$.

Let

$$h_\lambda(Y) = h_*(Y) \int_{\tau_x} \{G^*(X, Y) - F_x(X, Y)\}\, d\lambda(X),$$

and define λ' on τ_x by

$$d\lambda'(X) = \{\lim_{Z \to X} G_x(Z, Q)/h_*(Z)\}\, d\lambda(X),$$

where Q is a fixed point of Ω_x (see Theorem B). From Lemmas 2 and 5, h_λ is harmonic in Ω_x and, from (1),

(12)
$$\begin{aligned} h_\lambda(Y) &= \int_{\tau_x} \lim_{Z \to X} G_x(Z, Y)/h_*(Z)\, d\lambda(X) \\ &= \int_{\tau_x} \lim_{Z \to X} G_x(Z, Y)/G_x(Z, Q)\, d\lambda'(X). \end{aligned}$$

Defining h_μ and μ' in a similar manner, we obtain an equation analogous to (12). Since $h_\lambda = h_\mu$ by hypothesis, it follows from the uniqueness of the Martin representation for h_λ in Ω_x that $\lambda' = \mu'$, whence $\lambda = \mu$.

8.2. The proof of Theorem 1 will now be given. Let $s \in \mathcal{L}\mathcal{D}$ and $w > x > 0$. From [15, Theorem 2(i)], the restriction of s to $\partial\Omega_w$ is resolutive and $H_s^{\Omega_w} - s$ is non-negative and superharmonic in Ω_w. It follows from the Riesz—Martin decomposition and Theorem B that there is a measure λ_s on τ_w such that

(13)
$$H_s^{\Omega_w}(Y) - s(Y) = h(Y) + \int_{\Omega_w} G_w(X, Y)\, d\mu_s(X) + \int_{\tau_w} \lim_{Z \to X} G_w(Z, Y)/h_*(Z)\, d\lambda_s(X),$$

where h is non-negative and harmonic in Ω_w and continuously vanishes on τ_w (see Lemma 6). For any $y < w$, let $Q \in \Omega_w \setminus \bar{\Omega}_y$ and apply Theorem A to show that there is a positive constant c such that

$$+\infty > \int_{\Omega_y \cup \sigma_y} G_w(Q, X)\, d\mu_s(X) \geqq c \int_{\Omega_y \cup \sigma_y} h_*(X)\, d\mu_s(X).$$

It follows that we can define a measure ν_s on the Borel subsets A of $\Omega_w \cup \tau_w$ by

(14)
$$\nu_s(A) = \int_{A \cap \Omega_w} h_*(Z)\, d\mu_s(Z) + \lambda_s(A \cap \tau_w).$$

This and (1) enable us to rewrite (13) as

(15) $$H_s^{\Omega_w}(Y)-s(Y) = h(Y)+\int_{\Omega_w\cup\tau_w} h_*(Y)\{G^*(X,Y)-F_w(X,Y)\}\,dv_s(X).$$

Now observe that

$$\int_{\sigma_x} H_s^{\Omega_w}(Z)\,d\mu_{x,Y}(Z) = H_s^{\Omega_w}(Y)-\int_{\tau_x} s(Z)\,d\mu_{x,Y}(Z).$$

Also, since $h_*(\cdot)F_w(X,\cdot)$ is harmonic in Ω_w and continuously vanishes on τ_w (see Lemma 5),

$$\int_{\sigma_x} h_*(Z)F_w(X,Z)\,d\mu_{x,Y}(Z) = h_*(Y)F_w(X,Y).$$

It now follows that, if we integrate (15) with respect to harmonic measure on σ_x (relative to Ω_x) and use Lemma 5, we obtain

(16) $$H_s^{\Omega_w}(Y)-H_s^{\Omega_x}(Y) = h(Y)+\int_{(W_w\setminus W_x)\cap\bar\Omega} h_*(Y)\{G^*(X,Y)-F_w(X,Y)\}\,dv_s(X)$$
$$+\int_{W_x\cap\bar\Omega} h_*(Y)\{F_x(X,Y)-F_w(X,Y)\}\,dv_s(X).$$

Subtracting (16) from (15) yields

(17) $$H_s^{\Omega_x}(Y)-s(Y) = \int_{W_x\cap\bar\Omega} h_*(Y)\{G^*(X,Y)-F_x(X,Y)\}\,dv_s(X),$$

and so (2) holds for $x<w$ and λ_s is uniquely (by Lemma 7) defined on τ_w. Since w may be arbitrarily large, λ_s can be defined on all of $\partial\Omega$ (see Lemma 4 (b)).

9. Proof of Theorem 2

9.1. We require the following lemma.

Lemma 8. *The function*

$$\Phi_x(X) = \int_E F_x(X,Y)\,dv(Y)$$

has the constant value $\varphi(x)$ *on* $\bar\Omega_x$.

From Lemma 5, Φ_x is continuous on $\bar\Omega_x$ and, if $X\in\sigma_x$, then

$$\Phi_x(X) = \int_E G^*(X,Y)\,dv(Y) = \Phi(X) = \varphi(x)$$

(see Lemma 4 (d)). Further, by Fubini's theorem and Lemma 5, $h_*\Phi_x$ satisfies the mean-value equality for balls whose closures are contained in Ω_x. Thus

$$h^*(\cdot)\{\Phi_x(\cdot)-\varphi(x)\}$$

is harmonic in Ω_x, continuously vanishing on $\partial\Omega_x$, and so it is identically zero in $\bar\Omega_x$. Hence $\Phi_x(\cdot)=\varphi(x)$ in $\bar\Omega_x\cap\Omega$ and so also in $\bar\Omega_x$ by the continuity of Φ_x.

9.2. To prove Theorem 2, we suppose $x>1$, the case $x<1$ being similar and the case $x=1$ being trivial. Let ν_s be defined on $W_x\cap\bar{\Omega}$ by (14) (with $w=x$). Subtracting (17) when $x=1$ from (17) as it stands yields

$$H_s^{\Omega_x}(Y)-H_s^{\Omega_1}(Y) = h_*(Y)\int_{(W_x\setminus W_1)\cap\bar{\Omega}}\{G^*(X,Y)-F_x(X,Y)\}\,d\nu_s(X)$$
$$+h_*(Y)\int_{W_1\cap\bar{\Omega}}\{F_1(X,Y)-F_x(X,Y)\}\,d\nu_s(X),$$

whence, by the joint continuity of G^*, F_x and F_1 (see Lemmas 2 and 5),

$$\mathscr{H}_{s,x}(Y)-\mathscr{I}_{s,1}(Y) = \int_{(W_x\setminus W_1)\cap\bar{\Omega}}\{G^*(X,Y)-F_x(X,Y)\}\,d\nu_s(X)$$
$$+\int_{W_1\cap\bar{\Omega}}\{F_1(X,Y)-F_x(X,Y)\}\,d\nu_s(X)$$

for $Y\in E$. If we now integrate this equation with respect to ν and apply Fubini's theorem (recall that the integrands are jointly continuous and non-negative on the range of double integration), we obtain

$$\mathscr{M}(s,x)-\mathscr{N}(s,1) \tag{18}$$
$$=\int_{(W_x\setminus W_1)\cap\bar{\Omega}}\{\Phi(X)-\Phi_x(X)\}\,d\nu_x(X)+\int_{W_1\cap\bar{\Omega}}\{\Phi_1(X)-\Phi_x(X)\}\,d\nu_s(X)$$
$$=\int_{(W_x\setminus W_1)\cap\bar{\Omega}}\{\Phi(X)-\varphi(x)\}\,d\nu_s(X)+\{\varphi(1)-\varphi(x)\}\,\nu_s(W_1\cap\bar{\Omega}),$$

the second equality being a consequence of Lemma 8.

We now define
$$\alpha_s(t)=\nu_s(W_t\cap\bar{\Omega})\quad(t\in[1,x]),$$

which allows (18) to be rewritten as

$$\mathscr{M}(s,x)-\mathscr{N}(s,1)=\int_1^x\varphi(t)\,d\alpha_s(t)-\varphi(x)\,\alpha_s(x)+\varphi(1)\,\alpha_s(1),$$

since φ is continuous and decreasing, and α_s is of bounded variation on $[1,x]$. Integrating by parts, this yields

$$\mathscr{M}(s,x)-\mathscr{N}(s,1)=-\int_1^x\alpha_s(t)\,d\varphi(t)$$
$$=-\int_1^x\lambda_s(\tau_t)\,d\varphi(t)-\int_1^x\int_{\Omega_t}h_*(Z)\,d\mu_s(Z)\,d\varphi(t),$$

and the result follows from the definition of $\mathscr{M}^*(s,x)$.

10. Proofs of Theorems 3, 4 and 6

10.1. To prove Theorem 3, we observe from Theorem 2 that

(19) $$\mathcal{M}^*(s, x) = \mathcal{N}(s, 1) - \int_1^x \int_{\Omega_t} h_*(Z)\, d\mu_s(Z)\, d\varphi(t).$$

Since φ is decreasing, the double integral is decreasing, and so $\mathcal{M}^*(s, x)$ is increasing. Next note that the integrand

$$\int_{\Omega_t} h_*(Z)\, d\mu_s(Z)$$

is right continuous with respect to $\varphi(t)$, so that the double integral in (19) is right differentiable with respect to $\varphi(x)$, and

(20) $$\frac{d\mathcal{M}^*(s, x)}{d\varphi(x)} = -\int_{\Omega_x} h_*(Z)\, d\mu_s(Z)$$

holds on $(0, +\infty)$ if the derivative is understood as a right derivative. Since the right hand side of (20) increases as $\varphi(x)$ increases, it follows that $\mathcal{M}^*(s, x)$ is convex as a function of $\varphi(x)$ on $(0, +\infty)$, proving (i).

Further, if s is harmonic in $\Omega_w \setminus \bar{\Omega}_y$, then $\mu_s(\Omega_w \setminus \bar{\Omega}_y)$ is zero, and it follows from (19) that $\mathcal{M}^*(s, x)$ is a linear function of $\varphi(x)$ on $(y, w]$, and so on $[y, w]$ by the continuity of $\mathcal{M}^*(s, x)$ on $(0, +\infty)$.

Finally, $\mathcal{M}^*(s, x)$ is constant if and only if $\mu_s(\Omega_x)$ is zero for all x, which is equivalent to s being harmonic in Ω.

10.2. To prove Theorem 4, we begin by obtaining some inequalities. Let $y > 1$ and $Q \in \Omega_y \setminus \bar{\Omega}_1$. Since $s \in \mathcal{L}\mathcal{D}$, it follows that s has a harmonic majorant in Ω_y (for example, $H_s^{\Omega_y}$) and so, using Theorem A to compare $G(Q, \cdot)$ with $G_y(Q, \cdot)$ in Ω_1, we have

(21) $$\int_{\Omega_1} G(Q, X) d\mu_s(X) \leqq c' \int_{\Omega_1} G_y(Q, X)\, d\mu_s(X) < +\infty.$$

Let $P \in \Omega_1$. Using Theorem A again, there is a positive constant c such that

$$c^{-1} h_*(Y)/h_*(P) \leqq G(X, Y)/G(X, P) \leqq c h_*(Y)/h_*(P)$$

for $X \in \Omega \setminus \Omega_1$ and $Y \in \Omega_{1/2}$, and so, from Lemma 2,

$$c'' G(X, P) \leqq h_*(X)\, G^*(X, Y) \leqq c''' G(X, P),$$

for $X \in \Omega \setminus \Omega_1$ and $Y \in \bar{\Omega}_{1/2}$, whence

(22) $$c'' v(E)\, G(X, P) \leqq h_*(X)\, \Phi(X) \leqq c''' v(E)\, G(X, P)$$

for $X \in \Omega \setminus \Omega_1$.

We now show that (i) and (iii) are equivalent. The function s has a harmonic majorant in Ω if and only if $G\mu_s$ is a potential in Ω. From (21) this is equivalent to

$$\int_{\Omega\setminus\Omega_1} G(X, P)\, d\mu_s(X) < +\infty,$$

which, in turn, is equivalent to (iii) because of (22).

It remains to show that (ii) and (iii) are equivalent. Let $x > 1$. From (18) and the integration by parts employed at the end of the proof of Theorem 2, we can write

$$\begin{aligned}
(23)\qquad \mathcal{M}^*(s, x) &= \mathcal{N}(s, 1) + \{\varphi(1) - \varphi(x)\} \int_{\Omega_1} h_*(X)\, d\mu_s(X) \\
&\quad + \int_{\Omega_x\setminus\Omega_1} \{\Phi(X) - \varphi(x)\}\, h_*(X)\, d\mu_s(X) \\
&\leqq \mathcal{N}(s, 1) + \varphi(1) \int_{\Omega_1} h_*(X)\, d\mu_s(X) + \int_{\Omega_x\setminus\Omega_1} \Phi(X) h_*(X)\, d\mu_s(X).
\end{aligned}$$

Thus (iii) implies (ii). On the other hand, the function

$$\psi(x) = \varphi^{-1}(2\varphi(x))$$

is defined for all sufficiently large x, and

$$\int_{\Omega_x\setminus\Omega_1} \{\Phi(X) - \varphi(x)\} h_*(X)\, d\mu_s(X) \geqq \tfrac{1}{2} \int_{\Omega_{\psi(x)}\setminus\Omega_1} \Phi(X)\, h_*(X)\, d\mu_s(X),$$

since $\Phi(X) > \varphi(\psi(x))$ on $\Omega_{\psi(x)}$. Therefore, from (23),

$$\mathcal{M}^*(s, x) \geqq \mathcal{N}(s, 1) + \tfrac{1}{2} \int_{\Omega_{\psi(x)}\setminus\Omega_1} \Phi(X)\, h_*(X)\, d\mu_s(X),$$

and so (ii) implies (iii).

10.3. It is now straightforward to deduce Theorem 6. To show (i), we recall that

$$\mathcal{M}(s, x) - \mathcal{M}^*(s, x) = -\int_1^x \lambda_s(\tau_t)\, d\varphi(t).$$

Since φ is decreasing, the right hand side is increasing as a function of x. Further, its right derivative with respect to $\varphi(x)$ increases as $\varphi(x)$ increases, so that it is convex with respect to $\varphi(x)$. The result now follows from Theorem 3 (i).

In the case of (ii), $I_{s,x}$ is a harmonic majorant of s in Ω_x, and as in Theorem 1, there exists a measure λ_s' on $\partial\Omega$ such that the least harmonic majorant of s in Ω_x is given by

$$I_{s,x}(Y) - h_*(Y) \int_{\tau_x} \{G^*(X, Y) - F_x(X, Y)\}\, d\lambda_s'(X).$$

The argument of Theorem 2 now yields that

$$\mathcal{N}(s, x) = \mathcal{N}(s, 1) - \int_1^x \left\{\lambda_s'(\tau_t) + \int_{\Omega_t} h_*(Z)\, d\mu_s(Z)\right\} d\varphi(t),$$

and the result follows as in Theorem 3 (i).

11. Proof of Theorem 10

11.1. The following lemma is required.

Lemma 9. *Let* $0<a<z$ *and* S *be a function which is non-negative and subharmonic in* $\Omega_z\setminus\bar{\Omega}_a$, *and vanishes continuously on* $\tau_z\setminus\tau_a$. *Then the mean* $\mathcal{N}(S, x)$ *is real-valued and convex as a function of* $\varphi(x)$ *for* $x\in(a, z)$.

To see this, let $a<b<c<d<y<z$ and define

$$S_0(X)=\begin{cases}\mu_{y,X}(\tau_a) & \text{if } X\in\Omega_y\\ 1 & \text{if } X\in\bar{\tau}_a\\ 0 & \text{elsewhere in } \bar{\Omega}.\end{cases}$$

Clearly $S_0\in\mathcal{L}\mathcal{D}$, and from Theorem A there is a positive constant c' such that

$$S(X)\leqq H_S^{\Omega_y\setminus\bar{\Omega}_b}(X)\leqq c'S_0(X)\quad (X\in\Omega_d\setminus\bar{\Omega}_c),$$

the first inequality being a consequence of [15, Theorem 2 (i)] and the fact that $\bar{\Omega}_y\setminus W_b$ is the disjoint union of the closures of finitely many Lipschitz domains (see Definition 1). Hence the function

$$S'(X)=\begin{cases} c'S_0(X) & \text{if } X\in\bar{\Omega}_c\\ \max\{c'S_0(X), S(X)\} & \text{if } X\in(W_z\cap\Omega)\setminus\bar{\Omega}_c\end{cases}$$

is subharmonic in Ω_z, equal to S in $\Omega_z\setminus\bar{\Omega}_y$, and satisfies

$$\limsup_{X\to Z} S'(X)=S'(Z)\leqq c'\quad (Z\in\tau_z).$$

Now suppose that $z>1$. If $x\in(y, z)$, then

$$\mathcal{M}(S', x)=\mathcal{N}(S, x)+\mathcal{M}(S_0, x). \tag{24}$$

Since $S_0\equiv 0$ in $(W_z\cap\bar{\Omega})\setminus\bar{\Omega}_y$, it follows from Theorem 3 (ii) that $\mathcal{M}^*(S_0, x)$ is a linear function of $\varphi(x)$ on (y, z). Further, it is easily seen from Lemma 6 and the proof of Theorem 1 that

$$\lambda_{S_0}(\tau_z\setminus\tau_y)=0,$$

and so $\mathcal{M}(S_0, x)$ is also a linear function of $\varphi(x)$ on (y, z) (see Definition 2). In addition, Theorem 6 (i) shows that $\mathcal{M}(S', x)$ is a convex function of $\varphi(x)$ on $(0, z)$ (the fact that S' is not defined on all of $\bar{\Omega}$ is immaterial). Hence, from (24), $\mathcal{N}(S, x)$ is a convex function of $\varphi(x)$ on (y, z), and so on (a, z) since $y\in(a, z)$ is arbitrary.

Finally, we point out that, if $z\leqq 1$, then we could define

$$H_{s,x}(X)=H_s^{\Omega_x}(X)-\int_{\tau_{z/2}} s(Z)\, d\mu_{z/2,X}(Z)$$

and corresponding means $\mathcal{M}_z(s, x)$ and $\mathcal{M}_z^*(s, x)$ to avoid the problem of S' and $\lambda_{S'}$ not being defined on τ_1.

11.2. We now prove Theorem 10. Routine differentiation yields that

$$\Delta\{h_* \exp(s/h_*)\} \geqq 0$$

in Ω if $s \in C^2(\Omega)$. If $X \in \Omega$, take a decreasing sequence (s_m) of C^2 subharmonic functions, and it follows easily that $h_* \exp(s/h_*)$ is u.s.c. in Ω and satisfies the mean-value inequality for balls whose closures are contained in Ω.

Let $z > y > 0$. Using the fact that $s \leqq 0$ on $\partial\Omega$ and Theorem A, there is a positive constant c such that

$$s(X) \leqq I_{s^+, z}(X) \leqq c h_*(X) \quad (X \in \Omega_y),$$

whence $h_* \exp(s/h_*)$ vanishes continuously on τ_y, and so (y being arbitrary) on all of $\partial\Omega$. It follows from Theorem 6 (i) that

$$\mathcal{N}(h_* \exp(s/h_*), x)$$

is increasing as a function of x, and so the same is true of $\mathcal{N}_E(s, x)$.

Let $0 < a < y < w$, and note that (see Lemma 3) the function

$$S = h_* \exp\{k\Phi + s/h_*\},$$

where

$$k = \{\mathcal{N}_E(s, w) - \mathcal{N}_E(s, y)\}/\{\varphi(y) - \varphi(w)\}, \tag{25}$$

is subharmonic in $\Omega \setminus \bar{\Omega}_a$ and vanishes continuously on $\partial\Omega \setminus \tau_a$. From Lemma 9, $\mathcal{N}(S, x)$ is real-valued and convex as a function of $\varphi(x)$ on $(a, +\infty)$. Using Lemma 4 (d), if $x \in (y, w)$, then

$$\exp\{k\varphi(x)\} \exp\{\mathcal{N}_E(s, x)\} \leqq \left\{\frac{\varphi(x) - \varphi(w)}{\varphi(y) - \varphi(w)}\right\} \exp\{k\varphi(y)\} \exp\{\mathcal{N}_E(s, y)\}$$

$$+ \left\{\frac{\varphi(y) - \varphi(x)}{\varphi(y) - \varphi(w)}\right\} \exp\{k\varphi(w)\} \exp\{\mathcal{N}_E(s, w)\}$$

which, upon rearranging, using (25) and taking logs, yields

$$\mathcal{N}_E(s, x) \leqq \left\{\frac{\varphi(x) - \varphi(w)}{\varphi(y) - \varphi(w)}\right\} \mathcal{N}_E(s, y) + \left\{\frac{\varphi(y) - \varphi(x)}{\varphi(y) - \varphi(w)}\right\} \mathcal{N}_E(s, w)$$

as required.

12. Applications to the whole space

In this and subsequent sections, when $(n-1)$-dimensional surface area measure on the boundary of a domain exists, it will be denoted by σ. Thus, in particular, the spherical mean of a suitably defined function f is given by

$$\mathcal{L}(f\colon X, r) = c_n^{-1} r^{1-n} \int_{\partial B(X, r)} f(Z)\, d\sigma(Z),$$

where c_n denotes the surface area of $\partial B(O, 1)$.

Let $\Omega=\mathbf{R}^n$ $(n\geqq 3)$ and $h_*\equiv 1$. First consider $E=\{O\}$, and ν to be the Dirac measure at the origin. Clearly $\Phi(X)=|X|^{2-n}$ so that $\varkappa=+\infty$ and, if we take $\varphi(x)=x^{2-n}$, then $\Omega_x=B(O,x)$ for all x, and

$$\mathscr{M}^*(s,x)=\mathscr{M}(s,x)=\mathscr{N}(s,x)=H_s^{B(O,x)}(O)=\mathscr{L}(s\colon O,x).$$

The following well-known results are now seen to be special cases of the results in Sections 4 and 5.

Theorem 11. *Let s be subharmonic in $\mathbf{R}^n$ $(n\geqq 3)$ and u be positive and superharmonic. Then*

(i) *$\mathscr{L}(s\colon O,r)$ is convex as a function of r^{2-n} and increasing as a function of r;*

(ii) *if $R_2>R_1>0$ and s is harmonic in $B(O,R_2)\setminus\bar{B}(O,R_1)$, then $\mathscr{L}(s\colon O,r)$ is a linear function of r^{2-n} on $[R_1,R_2]$;*

(iii) *s has a harmonic majorant in $\mathbf{R}^n$ if and only if $\mathscr{L}(s\colon O,r)$ is bounded above for $r>0$, which in turn is equivalent to*

$$\int_{\mathbf{R}^n}(1+|X|)^{2-n}\,d\mu_s(X)<+\infty;$$

(iv) *the expressions*

$$\sup\{s(X)\colon |X|=r\}$$

and

$$\log\mathscr{L}(\exp s\colon O,r)$$

are convex as functions of r^{2-n} and increasing as functions of $r>0$;

(v) *if $s\geqq 0$ and $p\geqq 1$, then the same is true of*

$$\{\mathscr{L}(s^p\colon O,r)\}^{1/p};$$

(vi) *if $p\in(-\infty,0)\cup(0,1)$, then*

$$\{\mathscr{L}(u^p\colon O,r)\}^{1/p}$$

is concave as a function of r^{2-n} and decreasing as a function of $r>0$.

It is natural to ask what results could be obtained for different choices of E and ν. The simplest cases to consider would be when E is an m-dimensional ball, where $0<m\leqq n-1$, and ν is symmetrically distributed on E. In order to simplify the discussion, we shall restrict ourselves to the case $\Omega=\mathbf{R}^3$, and again let $h_*\equiv 1$.

Example 3. (i) Fix $c>0$ and let

$$E=\{X\in\mathbf{R}^3\colon x_1=x_2=0 \text{ and } |x_3|\leqq c\}.$$

It will be convenient to work in prolate spheroidal polar co-ordinates, so that

$$x_1=c\sinh\eta\sin\theta\cos\psi,$$

$$x_2=c\sinh\eta\sin\theta\sin\psi,$$

$$x_3=c\cosh\eta\cos\theta,$$

where

$$0 \leq \eta < +\infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \psi < 2\pi.$$

Choosing $\varphi(x) = \log \coth\left(\frac{1}{2}x\right)$, it is routine to deduce that Ω_x is the region bounded by the prolate spheroid

$$x_3^2/\cosh^2 x + (x_1^2 + x_2^2)/\sinh^2 x = c^2,$$

and that

$$\mathcal{M}(s, x) = \mathcal{N}(s, x) = (4\pi)^{-1} c \int_0^\pi \int_0^{2\pi} s(x, \theta, \psi) \sin\theta \, d\psi \, d\theta.$$

A theorem analogous to Theorem 11 can now be written down for the prolate spheroidal mean $\mathcal{M}(s, x)$; convexity is in terms of $\log \coth\left(\frac{1}{2}x\right)$.

(ii) If similar calculations are performed for

$$E = \{X \in \mathbf{R}^3 \colon x_1^2 + x_2^2 \leq c^2 \text{ and } x_3 = 0\},$$

analogous results for an oblate spheroidal mean are obtained. Details are left to the reader.

13. Applications to the half-space

Let $\Omega = \mathbf{R}^{n-1} \times (0, +\infty)$ $(n \geq 2)$, let $E = \{O\}$ and ν be the mass $c_n/(2n)$ at O, and $h_*(X) = x_n$ in Ω. From [25, Lemma 1],

$$2\gamma_n^{-1} c_n^{-1} x_n y_n |(X', -x_n) - Y|^{-n} \leq G(X, Y) \leq 2\gamma_n^{-1} c_n^{-1} x_n y_n |X - Y|^{-n},$$

where γ_n is as defined in Section 3. Hence

$$\Phi(X) = c_n (2n)^{-1} G^*(X, O) = \gamma_n^{-1} n^{-1} |X|^{-n}.$$

Thus $\varkappa = +\infty$ and, defining $\varphi(x) = \gamma_n^{-1} n^{-1} x^{-n}$ for $x \in (0, +\infty)$, it follows that $\Omega_x = B(O, x) \cap \Omega$. It now follows from [14, Section 8] that

$$\mathcal{N}(s, x) = x^{-n-1} \int_{\sigma_x} y_n s(Y) \, d\sigma(Y),$$

and

$$\mathcal{M}(s, x) = \mathcal{N}(s, x) + \int_1^x t^{-n-1} \int_{\tau_t} s(Y) \, d\sigma(Y) \, dt.$$

In this context, Theorems 2—6 are improvements of the main results of [4].

14. Applications to the infinite cylinder

Instead of deducing known results concerning the infinite strip, [5], and infinite cone, [13], we follow the pattern of [14] and derive previously unpublished results for the infinite cylinder.

Let $\Omega = \{(X', x_n) \colon |X'| < 1\}$, $(n \geq 2)$. We shall employ the Bessel function $J_{(n-3)/2}$ defined in Watson [28, pp. 40—42], the least positive zero of which will be denoted by a_n. We write

$$\psi(t) = t^{(3-n)/2} J_{(n-3)/2}(a_n t) \quad (t > 0)$$

and

$$b_n = a_n J_{(n-1)/2}(a_n) > 0,$$

(see [28, p. 45 (4) and p. 479 § 15.22]). Recalling (see [14, Lemma 3]) that the functions $\psi(|X'|)\exp(\pm a_n x_n)$ are positive and harmonic in Ω and vanish on $\partial\Omega$, we can define

$$E = (X', x_n)\colon\ |X'| < 1,\ x_n = 0\},$$

$$dv(X) = 2\{\psi(|X'|)\}^2\, dX'\, d\delta_0(x_n) \quad (X \in E),$$

and

$$h_*(X) = \psi(|X'|)\cosh(a_n x_n),$$

where δ_0 denotes the Dirac measure at the origin of $\mathbf{R}$.

Next we determine Φ, and hence Ω_x. Clearly the function

$$v(X) = a_n^{-1}\exp(-a_n|x_n|)\psi(|X'|) \quad (X \in \Omega)$$

is positive and superharmonic in Ω, harmonic in $\Omega \setminus E$, bounded above on $\bar{\Omega}$, and continuously vanishing on $\partial\Omega$. From a result of Bouligand [18, Corollary 9.20], the greatest harmonic minorant of v in Ω is zero, and so v is the potential whose measure is given by $\mu = -\gamma_n \Delta v$. If we now let Ψ be a C^∞ function with compact support in Ω, it follows from Green's theorem (as in [14, Section 9]) that

$$(\Delta v)(\psi) = -2\int_{\{|X'|<1\}} \Psi(X', 0)\, |X'|^{(3-n)/2} J_{(n-3)/2}(a_n|X'|)\, dX',$$

whence

$$d\mu(X) = 2\gamma_n |X'|^{(3-n)/2} J_{(n-3)/2}(a_n|X'|)\, dX'\, d\delta_0(x_n),$$

and so

$$\gamma_n^{-1} v(X) = \int_E G(X, Y)/h_*(Y)\, dv(Y) = h_*(X)\Phi(X).$$

Hence, dividing through by $h_*(X)$,

$$\Phi(X) = \gamma_n^{-1} a_n^{-1}\exp(-a_n|x_n|)\operatorname{sech}(a_n x_n),$$

$$= \gamma_n^{-1} a_n^{-1}\{1 - \tanh(a_n|x_n|)\},$$

and so $\varkappa = \gamma_n^{-1} a_n^{-1}$. If we define

$$\varphi(x) = \gamma_n^{-1} a_n^{-1}\{1 - \tanh(a_n x)\},$$

it follows that

$$\Omega_x = \{X \in \Omega\colon\ |x_n| < x\},$$

and so, from [14, Section 9],

$$\mathcal{N}(s, x) = \operatorname{sech}(a_n x)\int_{\sigma_x} \psi(|X'|)\, s(X)\, d\sigma(X),$$

and

$$\mathscr{M}(s, x) = \mathscr{N}(s, x) + b_n \int_1^x \operatorname{sech}^2 (a_n t) \int_{\tau_t} s(X) \cosh (a_n x_n)\, d\sigma(X)\, dt.$$

The results of Sections 4 and 5 may now be applied to subharmonic functions in the infinite cylinder, and convexity as a function of $\varphi(x)$ can clearly be equivalently stated as convexity as a function of $\tanh (a_n x)$.

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