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DISCRETE QUASICONFORMAL GROUPS THAT ARE NOT THE QUASICONFORMAL CONJUGATES OF MÖBIUS GROUPS*

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1. Introduction

In this paper we provide examples of discrete quasiconformal groups that are not the quasiconformal conjugates of Möbius groups. Gehring and Palka [G. P.] first asked the question whether every such group was in fact the quasiconformal conjugate of a conformal, or Möbius, group.

In the case of quasiconformal groups acting on subsets of the Riemann sphere the question was answered in the affirmative by Sullivan [S] and Tukia [T. 2]. The idea of the proof was to construct, for a given quasiconformal group G, a G-invariant measurable Riemannian structure in which G acts conformally, that is a measurable map $\mu : R \to S$, where S is the space of positive definite symmetric $n\times n$ matrices with determinant 1, such that for each $g\in G$

$$\mu(x) = |\det g'(x)|^{-2/n} \cdot g'(x)^t \mu(g(x))\, g'(x).$$

When $n=2$ the measurable Riemann mapping theorem implies that this structure is in fact the pull-back, under a quasiconformal mapping, of the standard conformal structure, and so the group G can be conjugated by a quasiconformal mapping, so as to be conformal.

No such measurable Riemann mapping theorem is true in higher dimensions, however Tukia [T. 4] has shown that if the measurable G-invariant Riemannian structure is continuous at a limit point of the group (or approximately continuous at a conical limit point), then the group is in fact the quasiconformal conjugate of a Möbius group. He does this by blowing up neighbourhoods of a limit point, where the action of G is like a Möbius group. In particular this implies that every cocompact Fuchsian quasiconformal group is quasiconformally conjugate to a quasiconformal group whose action is conformal on S^{n-1}. We show that the cocompactness hypothesis cannot be removed.

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Previously Tukia [T. 3] had given examples of quasiconformal groups of $\bar{R}^n$, $n \geqq 3$, which were not the quasiconformal conjugates of any Möbius group. His examples were not discrete and depended heavily on that fact to require that the orbit of a point be a particularly nasty $(n-1)$-cell in R^n. The orbit of a point under the quasiconformal conjugate of a Möbius group was then shown not to have such bad behaviour.

Gehring and Martin [G. M.] began to study discrete quasiconformal groups. We found many similarities between conformal and quasiconformal groups, particularly in the structure, classification of the elements, the associated Poincaré series and behaviour of the limit sets. In particular any loxodromic element of a quasiconformal group was quasiconformally conjugate to a loxodromic Möbius transformation. The natural question was then whether every discrete quasiconformal group is quasiconformally conjugate to a Möbius group.

We pursue Tukia's ideas in the discrete case and show in fact, that every discrete subgroup of maximal rank in Tukia's group is not the quasiconformal conjugate of any Möbius group. We modify Tukia's construction a little so as to be able to provide non-elementary (uncountably many limit points) examples as well. In the process we obtain some results on the conjugacy of Möbius groups to translation groups and find an example of a locally quasiconformally flat $(n-1)$-cell in R^n, which is topologically flat but not quasiconformally flat. We finally appeal to the L. Q. C. Hauptvermutung in the cocompact case, together with the results of Farrell and Hsiang [F. H.] to show that in some sense our examples are best possible.

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1.1. Definitions and notation

We denote by Möb (n) the group of all Möbius transformations of $\bar{R}^n$, and a Möbius group is a subgroup of Möb (n). $E(n)$, $O(n)$ and $A(n)$ denote the groups of euclidean isometries, orthogonal transformations and affine mappings respectively. Our standard reference for definitions and results concerning quasiconformal mappings is [V]. We remark here, however, that conformal (or 1-quasiconformal) homeomorphisms of $\bar{R}^n$ are precisely the Möbius transformations. We also note that the subgroup of Möb (n) fixing B^n, the unit ball, is precisely the group of hyperbolic isometries of B^n.

A group G of self homeomorphisms of a domain $U \subset \bar{R}^n$ is called a quasiconformal group if there is some finite K such that each $g \in G$ is K-quasiconformal. We then observe that if H is a Möbius group acting on $U \subset \bar{R}^n$, then for any K-quasiconformal embedding $f\colon U\to\bar{R}^n$, the group

$$fHf^{-1}$$

is a K^2-quasiconformal group of $f(U)$.

If U is the unit ball or the upper half-space of R^n, then a quasiconformal group acting on U will be called a Fuchsian quasiconformal group. By a K-quasiconformal, or just quasiconformal, hyperplane we mean the image of R^{n-1} under a K-quasiconformal homeomorphism of R^n. For two groups A and B we denote by $A\rtimes B$ the semidirect product of the groups A and B. If A and B act as a group of self homeomorphisms of $U\subset\bar{R}^n$, then we can define the action of $A\rtimes B$ on U as

$$(a, b)(x) = a\big(b(x)\big) \quad \text{for} \quad (a, b)\in A\rtimes B.$$

Since each affine transformation $C\colon\bar{R}^n\to\bar{R}^n$ (we set $C(\infty)=\infty$) is quasiconformal, the groups

$$A(n) \quad \text{and} \quad A(n)\rtimes \text{Möb}\,(n)$$

are groups of self homeomorphisms of $\bar{R}^n$ for which every element is quasiconformal but neither is a quasiconformal group. Notice that the group structure of $A(n)\rtimes\text{Möb}\,(n)$ is not the usual composition of homeomorphisms.

For a proper subdomain D of $\bar{R}^n$ we define the quasihyperbolic distance in D as the distance function, $k_D(x, y)$, associated with the generalized Riemannian metric $d(x, \partial D)^{-2}|dx|^2$, where $d(x, \partial D)$ is the euclidean distance from $x\in D$ to the boundary of D. Thus for two points $x_1, x_2\in D$

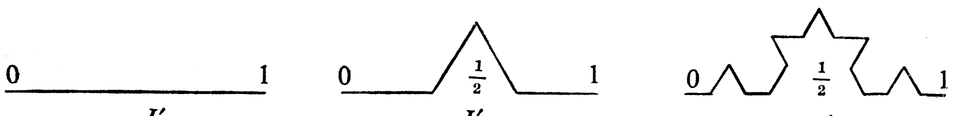
$$k_D(x_1, x_2) = \inf_{\alpha}\int_{\alpha} d(x, \partial D)^{-1}\,|dx|,$$

where the infinum is taken over all locally rectifiable arcs α joining x_1 to x_2 in D. The quasihyperbolic metric is the natural generalization of the hyperbolic metric to domains in space. The basic properties of this metric can be found in [G. O.] and [M].

2. Tukia's construction and a modification

We begin with an outline of Tukia's construction (see [T. 3, Section 3]) and pay particular attention to the details that concern us. In order to construct the discrete non-elementary examples in the latter section we need to modify the construction so that the map F' that Tukia obtains is in fact conformal in a uniformly large neighbourhood of each point of $\{n+i\colon n\in Z\}$, while the map remains unchanged outside some slightly larger set. To do this we need a quantitative version of the annulus theorem for quasiconformal mappings, for we must alter F' infinitely often and yet still ask that it be K-quasiconformal for some finite K.

We first construct the well-known, non-rectifiable quasiconformal arc J as follows. J' is the limit of the arcs $J'_0, J'_1, J'_2 \dots$ illustrated below.



J'_0 J'_1 J'_2 ...

Since $J'/3$ is a subarc of J' and likewise J' is a subarc of $3J'$, if we set

$$J = \bigcup_{i \geqq 0} 3^i (J' \cup (-J'))$$

we obtain an open arc J. There is a natural map

$$f\colon [0, 1] \to J',$$

such that

$$f(4^i x) = 3^i f(x),$$

if $i \geqq 0$ and $0 \leqq x \leqq 4^i x \leqq 1$. We can then define an extension $f_1\colon \boldsymbol{R} \to J$ by

$$f_1(\pm 4^i x) = \pm 3^i f(x), \tag{2.1}$$

if $i \geqq 0$ and $x \in [0, 1]$.

This map f_1 is then a quasisymmetric embedding of $\boldsymbol{R}$ into $\boldsymbol{R}^2$. That is there is a constant $H \geqq 1$ such that

$$|f_1(a) - f_1(x)| \leqq H |f_1(b) - f_1(x)|$$

for all $a, b, x \in \boldsymbol{R}$ satisfying $|a-x| \leqq |b-x|$.

Such quasisymmetric embeddings can be extended (see (11) of [T.3]) to a quasiconformal homeomorphism $F_1\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $F_1|\boldsymbol{R} = f_1$, in such a way that F_1 is bilipschitz in the quasihyperbolic metric, i.e. there is an $L \geqq 1$ such that if U is a component of $\boldsymbol{R}^2 \setminus \boldsymbol{R}$ and $U' = F_1(U)$, then for all $x, y \in U$

$$\frac{1}{L} k_U(x, y) \leqq k_{U'}(F_1(x), F_1(y)) \leqq L k_U(x, y). \tag{2.2}$$

It follows from this and the above (see (5) of [T.3]) that there is a constant $M \geqq 1$ such that for all $u, v \in \boldsymbol{R}$

$$\frac{1}{M} |v|^\alpha \leqq d(F_1(u, v), J) \leqq M |v|^\alpha \tag{2.3}$$

where $\alpha = \log 3/\log 4$ ($1/\alpha$ = Hausdorff dimension of J).

For details of the preceeding discussion see [T.3, Section 3]. Our first observation is that

2.4. Lemma. *For all $i \geqq 0$ and $x \in \boldsymbol{R}$*

$$f_1(\pm 4^i x) = \pm 3^i f_1(x).$$

Proof. Let $x\in\boldsymbol{R}$, for simplicity we may assume $x>0$. Then there is an $m\geqq 0$ such that $y=4^{-m}x\in[0, 1]$. Since $4^m y=x$ we see from (2.1) that

$$f_1(x)=f_1(4^m y)=3^m f(y)$$

and hence

$$f_1(4^i x)=f_1(4^{i+m} y)=3^{i+m} f(y)=3^i f_1(x).$$

It follows easily from (2.3) that for all $n\in\boldsymbol{Z}$

$$1/M \leqq d\big(F_1(n, 1), J\big) \leqq M. \tag{2.5}$$

We now need the following version of the annulus theorem. It is a consequence of the quantitative version of the annulus theorem (see [T. V., Theorem 5.8]).

2.6. Theorem. *Let $L\geqq 1$ and $d>0$. Then there are L_1 and $\delta>0$, depending on n, such that for each L-bilipschitz embedding $f\colon B^n(d)\to\boldsymbol{R}^n$ with $f(0)=0$, there is an L_1-bilipschitz embedding $g\colon B^n(d)\to\boldsymbol{R}^n$ with the following properties:*

(1) $g=f$ *near* $S^{n-1}(d)$.

(2) $g|B^n(\delta)=$ *identity.*

Proof. Since f is L-bilipschitz $d\big(f(0), f(S^{n-1}(d))\big)\geqq d/L$. We denote by $A(a, b)$ the annulus $\{x\in\boldsymbol{R}^n\colon a<|x|<b\}$. Let $a=d/(10L)$, $b=d/(8L)$ and $c=8d/10$. Define

$$f_1(x)=\begin{cases} f(x), & x\in A(c, d)\\ x, & x\in A(a, b).\end{cases}$$

We claim f_1 is a $4L$-bilipschitz embedding, thus let $x, y\in A(a, b)\cup A(c, d)$. The result is trivial if both x and y lie in one of $A(c, d)$ or $A(a, b)$, and so we suppose $x\in A(a, b)$, $y\in A(c, d)$. Then

$$\tfrac{1}{2}d \leqq c-b \leqq |x-y| \leqq d+b \leqq 2d$$

and since $f_1(x)=x$,

$$\frac{1}{2L}d \leqq \frac{c}{L}-b \leqq |f_1(x)-f_1(y)| \leqq Ld+b \leqq 2Ld.$$

Thus

$$\frac{1}{4L}|x-y| \leqq |f_1(x)-f_1(y)| \leqq 4L|x-y|.$$

The diameter of $f_1\big(S^{n-1}(d)\big)$ is the same as that of $f\big(S^{n-1}(d)\big)$ and so is no more than Ld, while the diameter of $f_1\big(S^{n-1}(a)\big)$ is a, the ratio of these diameters is then at most $10L^2$. We now appeal to the quantitative version of the bilipschitz annulus theorem alluded to in Remark 5.9 in [T. V.] following the quasiconformal version Theorem 5.8.

We thus obtain an embedding $g: B^n(d) \to \boldsymbol{R}^n$ with the following properties:

(1) g is L_1-bilipschitz.

(2) $g = f_1 = f$ near $S^{n-1}(d)$.

(3) $g|S^{n-1}(a) = f_1|S^{n-1}(a) =$ identity.

The constant L_1 depends only on the constants L, a, b, c, d, n and the ratio we have bounded by $10L^2$. Thus L_1 depends only on L, n and d. We are done once we set $\delta = a$ and extend g by the identity on $B^n(\delta)$.

2.7. Corollary. *Let D be a proper subdomain of $\boldsymbol{R}^n$, $L \geqq 1$, $d > 0$ and $0 < \varepsilon \leqq R < \infty$. Then there are L_1 and $\delta > 0$, depending on n, with the following properties. If $f: D \to D'$ is a homeomorphism which is L-bilipschitz in the quasihyperbolic metrics of D and D' and if $x \in D$ with $d = d(x, \partial D)$ and $\varepsilon \leqq d(f(x), \partial D) \leqq R$, then there is a homeomorphism $g: D \to D'$ with the following properties:*

(1) *g is L_1-bilipschitz in the quasihyperbolic metrics of D and D'.*
(2) *$g|D \setminus B^n(x, a) = f|D \setminus B^n(x, a)$, where $a = (d/2L) \log (1 + \varepsilon/5R) \leqq d$.*
(3) *$g|B^n(x, \delta)$ is a translation, i.e. $g(y) = y + g(x)$, $y \in B^n(x, \delta)$.*

Proof. We first show that the hypotheses imply that f is L_2-bilipschitz in the euclidean metric in a neighbourhood of x and that both L_2 and the size of this neighbourhood depend only on the constants in the hypotheses. We may pre and post compose f with translations so that $x = f(x) = 0$. We must then show

(2)′ $g|D \setminus B^n(a) = f|D \setminus B^n(a)$.

(3)′ $g|B^n(\delta) =$ identity.

Let m be so large that $m \geqq 5$ and

$$R(\exp (2L/m) - 1) \leqq \varepsilon/5.$$

Notice that $d/m \leqq a \leqq d$. Let $A = B^n(d/m)$. We easily obtain from Lemma 6.5 of [T. V.], where we have $c_1 \leqq 2$ since $d(A)/d(A, \partial B^n(d)) \leqq 1/2$,

$$\frac{1}{2d} |u - v| \leqq k_D(u, v) \leqq \frac{2}{d} |u - v| \quad \text{for all} \quad u, v \in A. \tag{2.8}$$

Similarly if $B = B^n(\varepsilon/5)$, then

$$\frac{1}{3R} |u - v| \leqq k_{D'}(u, v) \leqq \frac{3}{\varepsilon} |u - v| \quad \text{for all} \quad u, v \in B. \tag{2.9}$$

Thus if we set $L_2 = \max \{6Ld/\varepsilon, 6LR/d\}$, then the inequalities (2.8) and (2.9) together with the fact that f is L-bilipschitz in the quasihyperbolic metric will imply that $f|A$ is L_2-bilipschitz in the euclidean metric, provided we show $f(A) \subset B$. To see this suppose $z\in S^{n-1}(d/m)$. Then from (2.8), $k_D(0,z)\leqq 2/m$, and so $k_{D'}(0,f(z))\leqq 2L/m$. Since $d(0,\partial D)\leqq R$, we see from [G. O., (1.2)] that

$$k_{D'}(0,f(z))\geqq \log\left(1+|f(z)|/R\right).$$

Thus

$$|f(z)|\leqq R\left(\exp(2L/m)-1\right)\leqq \varepsilon/5,$$

the latter inequality following from the choice of m.

We can now appeal to Theorem 2.6 with d replaced by d/m, and find an L_3 and a $\delta>0$, depending only on L_1, d/m and n, and an L_3-bilipschitz embedding $g\colon B^n(d/m)\to B^n(d/m)$ agreeing with f near $S^{n-1}(d,m)$ and the identity on $B^n(\delta)$. The map g has all the desired properties provided we show that g is L_1-bilipschitz in the quasihyperbolic metric. This is clear for points in or sufficiently near $D\setminus B^n(d/m)$, since there $f=g$, while in $B^n(d/m)$ we can again put together the estimates (2.8) and (2.9) with the fact that g is L_3-bilipschitz in the euclidean metric of $B^n(d/m)$, to see that $g|B^n(d/m)$ is $\max\{6L_3d/\varepsilon, 6L_3R/d\}$-bilipschitz in the quasihyperbolic metrics of D and D'. The result will now follow immediately from Lemma 6.21 of [T. V.] which says that a homeomorphism of D onto D' which is locally L-bilipschitz in the quasihyperbolic metrics of D and D' is actually L-bilipschitz in these metrics. The proof is complete.

We now turn back to Tukia's construction. Let F_1 be the map of (2.2). We observe that if in Corollary 2.7, $d=1$, $\varepsilon=1/M$ and $R=M$, then the number a obtained is less than 1/4. Let U denote the upper half-plane and consider

$$F_1\colon U\to F_1(U). \tag{2.10}$$

Corollary 2.7 enables us to alter F_1 to obtain a map $F_2\colon U\to F_1(U)$ and a $\delta>0$ such that

$$\begin{cases} (1) & F_2|U\setminus\bigcup_{m\in Z}B^2((m,1),a)=F_1|U\setminus\bigcup_{m\in Z}B^2((m,1),a)\\ (2) & F_2|B^2((m,1),\delta)\ \text{ is a translation, for all }\ m\in\boldsymbol{Z}.\end{cases} \tag{2.11}$$

We may do this since we change nothing outside of $\bigcup_{m\in Z}B^2((m,1),a)$ and we further note that by Corollary 2.7, F_2 is locally L_1-bilipschitz in the quasihyperbolic metric, where L_1 depends only on L, $d((m,1),\partial D)=1$, and the constant M of (2.5), δ too depends only on these quantities. F_2 is then L_1-bilipschitz in the quasihyperbolic metrics of U and $F_1(U)=F_2(U)$ as noted above by Lemma 6.21 of [T. V.].

We now extend the map F_2 to the lower half-plane via F_1. We observe that we only used the quantitative version of the annulus theorem when $n=2$ to obtain our map F_2. In this dimension the theorem is somewhat easier to obtain and does not depend on the deep results of Sullivan.

For $x\in\boldsymbol{R}^n$ (henceforth $n\geqq 3$) we set $x=(z,y)\in\boldsymbol{R}^2\times\boldsymbol{R}^{n-2}$. We then define

$$\begin{cases} F(x)=(F_1(z),y)\\ F_*(x)=(F_2(z),y).\end{cases} \tag{2.12}$$

Then F and F_* are topological homeomorphisms of $\boldsymbol{R}^n$, neither of these maps is quasiconformal as we shall see later.

Let T be the following group of translations.

$$T = \{x \to x+a \colon a = (a_1, 0, a_3, \ldots, a_n),\ a_i \in \boldsymbol{R}\}.$$

Tukia's main results are then

2.13. Theorem. *The group $G = F \circ T \circ F^{-1}$ is a Lipschitz (and hence quasiconformal) group of $\boldsymbol{R}^n$ acting transitively on the $(n-1)$-cell $S = J \times \boldsymbol{R}^{n-2}$.*

Rickman had earlier observed

2.14. Theorem. *The $(n-1)$-cell $S = J \times \boldsymbol{R}^{n-2}$ is not locally quasiconformally flat. In particular there is no quasiconformal self homeomorphism g of $\boldsymbol{R}^n$ such that $g(\boldsymbol{R}^{n-1}) = S$.*

The Bieberbach theorems now imply that for such groups as G the orbit of a point should be a quasiconformal hyperplane if in fact the group is the quasiconformal conjugate of a Möbius group. This cannot be the case for S is the orbit of the origin under G. Hence

2.15. Theorem. *The group G is not the conjugate of any Möbius group by any quasiconformal self homeomorphism of $\boldsymbol{R}^n$.*

Notice that the group G easily extends to a quasiconformal group of $\bar{\boldsymbol{R}}^n$. One may prove exactly as in the proof of Theorem 2.13 (see [T. 3, Theorem 2]) that the group $H = F_* T F_*^{-1}$ is also Lipschitz group of $\boldsymbol{R}^n$ acting transitively on the $(n-1)$-cell S. In fact the group $H|S = G|S$.

We then easily obtain the following

2.16. Theorem. *The group H is a Lipschitz group of $\boldsymbol{R}^n$ which is not the quasiconformal conjugate of any Möbius group.*

We went to the trouble to construct H, since a certain discrete subgroup of H will act conformally in a uniformly large neighbourhood of some point. This will enable us to construct the discrete non-elementary examples we seek.

3. The discrete examples

3.1. Lemma. *Let G be a discrete group of Möbius transformations acting on $\bar{\boldsymbol{R}}^n$ which is the topological conjugate of a group of translations T of rank n. Then there is*

$$g \in A(n) \rtimes \text{Möb}\,(n)$$

such that

$$gGg^{-1} = T.$$

In particular the map g is quasiconformal.

Proof. Let f be a topological homeomorphism such that $G=fTf^{-1}$. If $h\in G$, then there is some $t\in T$ so that $h=ftf^{-1}$. Hence

$$fix\,(h) = fix\,(ftf^{-1}) = f(fix(t)) = f(\infty).$$

Thus every $h\in G$ has the unique fixed point $f(\infty)$. Let Ψ be a Möbius transformation for which $\Psi(f(\infty))=\infty$. Then $G'=\Psi^{-1}G\Psi$ is a discrete group of Möbius transformation which fix infinity and so each $h\in G'$ is a similarity and so has the form

$$h(x) = rAx+b,$$

for some $A\in O(n)$, $r\in \mathbf{R}$ and $b\in \mathbf{R}^n$. If $r\neq 1$, then $(rA-I)$ is invertible and so h has the fixed point $-(rA-I)^{-1}b$ which is impossible. Thus $r=1$ and so

$$G' \subset E(n).$$

As the topological conjugate of T, which is a discrete group of translations of rank n and so a crystallographic group G' is a uniform discrete subgroup of $E(n)$ isomorphic to the crystallographic group T. Thus by the Bieberbach theorems [W, 3.2.2] there is an affine map B such that

$$BG'B^{-1} = T.$$

Thus $g(x)=B\cdot\Psi(x)$ is the desired map. As we observed in the introduction g is quasiconformal for Ψ is conformal and B is quasiconformal.

The following refinement of the above lemma is what we need in the discrete case to replace Lemma 6 of [T. 3].

3.2 Theorem. *Suppose that G is a discrete group of Möbius transformations of $\bar{\mathbf{R}}^n$ which is the topological conjugate of a group of translations T of rank $n-1$. Then there is a subgroup G^* of finite index in G and*

$$g\in A(n) \rtimes \text{Möb}\,(n)$$

such that

$$gG^*g^{-1} = T.$$

In particular g is quasiconformal.

Proof. Proceeding as in Lemma 3.1 we find that there is a Möbius transformation ψ such that

$$H = \psi G\psi^{-1} \subset E(n).$$

Then by the Bieberbach theorems (see [W, 3.2.8)] after a change of origin, there is a normal subgroup H_1 of finite index in H, a vector subgroup $V\subseteqq \mathbf{R}^n$ and a toral subgroup $O\subset O(n)$, with O acting trivially on V, such that

$$H_1 \subset O \rtimes V.$$

Since T and hence G, H and H_1 are free abelian on $n-1$ generators we see $\dim V= n-1$. That is V is a hyperplane in $\bar{\mathbf{R}}^n$, passing through the origin. Since O acts trivially on a hyperplane and since $O\subset O(n)$ there are at most two elements of O. Namely the identity and possibly reflection in the hyperplane V. This corresponds to the possibilities that the components of $\boldsymbol{R}^n\setminus V$ are fixed or permuted. We can then pass to a subgroup H_2 of H_1 of index at most 2 and so of finite index in H, with $H_2\subset V$. We may choose an orthogonal transformation A, with $A^{-1}V=\boldsymbol{R}^{n-1}$ and set

$$H^* = A^{-1}H_2A.$$

Thus $H^*\subset\boldsymbol{R}^{n-1}$ (as groups), acts on $\boldsymbol{R}^n$ and is free abelian on $n-1$ generators. We may also choose an orthogonal transformation B such that $B^{-1}TB\subset\boldsymbol{R}^{n-1}$, where each $a\in\boldsymbol{R}^{n-1}$ is identified with the translation $x\to x+a$ giving $\boldsymbol{R}^{n-1}$ the usual group structure. Now since H was isomorphic to T (in fact conjugate) H^* is isomorphic to a finite index subgroup of $B^{-1}TB$, but any finite index subgroup of a translation group is isomorphic (in fact affinely conjugate by a change of basis) to that group and so we see H is isomorphic to $B^{-1}TB$. Restricting the actions to the invariant hyperplane $\boldsymbol{R}^{n-1}$ we see $H^*|\boldsymbol{R}^{n-1}$ is isomorphic to the discrete uniform (and hence crystallographic) group $B^{-1}TB|\boldsymbol{R}^{n-1}$. Thus these two groups are affinely conjugate by the Bieberbach theorems, by some $C_1\in A(n-1)$. Since the action of each group is trivial on $(\boldsymbol{R}^{n-1})^\perp$, that is it is obtained by producting the restriction with the identity we see that the groups H^* and $B^{-1}TB$ are affinely conjugate by the mapping

$$C=\begin{pmatrix} C_1 & 0\\ 0 & 1\end{pmatrix}.$$

The result now follows if we set

$$G^* = \psi^{-1}AH^*A\psi.$$

Then

$$g = B^{-1}CA^{-1}\psi.$$

It remains only to observe that A and B were orthogonal so that g is quasiconformal.

Tukia has pointed out that the result is not true if the rank of the translation group is less than $n-1$. For instance if $g(x)=(e^{i\theta}z, y+1)$ where θ is an irrational multiple of π and if $t(x)=x+e_1$ we see the groups $\langle g\rangle$ and $\langle t\rangle$ are topologically conjugate (the quotient spaces $\boldsymbol{R}^n/\langle g\rangle$ and $\boldsymbol{R}^n/\langle t\rangle$ are homeomorphic to $\boldsymbol{R}^{n-1}\times S^1$), but no subgroup of finite index in $\langle g\rangle$ can be affinely conjugate to $\langle t\rangle$. To see this we observe that a finite index subgroup would be generated by some power of g, we would then have for some affine map A and integer n

$$g^n(x) = (e^{in\theta}z, y+n) = A^{-1}tA(x) = x+A^{-1}e_1$$

which is impossible.

3.3. Corollary. *If G is a discrete quasiconformal group acting on $\bar{\boldsymbol{R}}^n$, which is the topological conjugate of a translation group T, of rank $n-1$, then G is the quasiconformal conjugate of a Möbius group only if there is a subgroup G^* of finite index in G which is the quasiconformal conjugate of T. In particular the orbit of a point, $G^*(x)$, must lie in a quasiconformal hyperplane.*

Proof. Suppose G were the conjugate of the Möbius group H, by a quasiconformal homeomorphism $f\colon \bar{R}^n \to \bar{R}^n$. Thus

$$G = fHf^{-1}.$$

Since G is the topological conjugate of a translation group and since G is discrete we see that H is both discrete and also the topological conjugate of a translation group. By Theorem 3.2 there is a subgroup H^* of finite index in H and a $g \in A(n) \rtimes \text{Möb}\,(n)$ so that

$$gH^*g^{-1} = T.$$

If we set $G^* = fH^*f^{-1}$, we see that for $h = gf^{-1}$

$$hG^*h^{-1} = T.$$

The result now follows once we observe that the orbit of a point under the group T must lie in a hyperplane, and so $G^*(x) = h^{-1}Th(x)$ lies in a quasiconformal hyperplane.

We now set out to prove that every discrete subgroup of maximal rank in Tukia's group G (there are many such) is not the quasiconformal conjugate of a Möbius group. This will be essentially due to Theorem 2.14 and Corollary 3.3.

Suppose that G' is a subgroup of G which is both discrete and of maximal rank, $n-1$. Corresponding to G' is the discrete group of translations $T' = F^{-1}G'F$. We now suppose

(*) G' is the quasiconformal conjugate of a Möbius group.

We will show the supposition (*) leads to a contradiction. By Corollary 3.3 there is a subgroup G^* of G' of finite index and a quasiconformal map g, such that

$$g_1^{-1}G^*g_1 = T'.$$

It is not difficult to see that a discrete subgroup of rank $n-1$ of $T = \{x \to x + a\colon a = (a_1, 0, a_3 \ldots a_n)\}$ is affinely conjugate to the discrete subgroup

$$T^* = \langle x \to x + e_i\colon i = 1, 3, 4 \ldots n \rangle,$$

as this is again just a change of basis. Thus we may assume that there is a quasiconformal mapping $g\colon \bar{R}^n \to \bar{R}^n$ such that

$$g^{-1}G^*g = T^*.$$

Define for each integer m the map $h_m\colon R^n \to R^n$ by

$$h_m(x) = (4^m z, 3^m y),$$

for $x = (z, y) \in R^2 \times R^{n-2}$. We set

$$T_m = h_{-m}T^*h_m$$

and

$$G_m = FT_mF^{-1}.$$

Notice that $T_0 = T^*$ and $G_0 = G^*$.

The idea now is that the groups T_m become dense in T as m gets large while up to a conformal scaling the orbit $G_m(0)$ remains the same, so that if $G^*(0)$ lies in a K-quasiconformal hyperplane (as our supposition (*) asserts via Corollary 3.3), then so does $G_m(0)$ for every m. This we show is impossible.

Since $h_{-m}=h_m^{-1}$ we see T_m is a discrete group generated by the maps

$$x \to x+4^{-m}e_1,\ x \to x+3^{-m}e_i,\quad i = 3, 4, \dots n.$$

Thus we set

$$V = \{x = (x_1, x_2, \dots, x_n)\in \mathbf{R}^n\colon\ x_2 = 0\}$$

we see T_m is a group of translations of $\boldsymbol{R}^n$ leaving V invariant. The vectors $4^{-m}e_1$ and $3^{-m}e_i$, $i=3, 4, \dots, n$. span V and all have length no more than 3^{-m}. Hence by the archimedean property of the reals for each $\omega\in V$ there is a sequence of points $\omega_m\in T_m(0)$ such that

$$\omega_m \to \omega,\quad \text{as}\quad m \to \infty.$$

Hence if $s\in S=J\times \boldsymbol{R}^n$, then $w=F^{-1}(s)\in V$. Then accordingly there is a sequence $\omega_m\in T_m(0)$, with $\omega_m\to\omega$. Since $F(0)=0$ we see

$$s_m = F(\omega_m) \to s$$

and that

$$s_m\in FT_mF^{-1}(0) = G_m(0) \subset S.$$

Thus

$$(3.5)\qquad \begin{cases} \text{for each}\quad s\in S,\quad \text{there is a sequence of points}\quad \{s_m\} \\ \text{such that}\quad s_m\in G_m(0)\quad \text{and}\quad s_m\to s\quad \text{as}\quad m\to\infty. \end{cases}$$

Hence $G_m(0)$ becomes dense in S as $m\to\infty$, and so we are able to prove the following.

3.6. Lemma. *There is no finite K such that each $G_m(0)$ lies in a K-quasiconformal hyperplane.*

Proof. Suppose there were such a finite K. Then for each m there would be a K-quasiconformal mapping $f_m\colon \bar{\boldsymbol{R}}^n\to\bar{\boldsymbol{R}}^n$ such that

$$G_m(0) \subset f_m(\mathbf{R}^{n-1}).$$

We may normalize the maps $\{f_m\}$ by auxiliary Möbius transformations and an orthogonal rotation so that

$$f_m(0) = 0,\quad f_m(\infty) = \infty,\quad \tfrac{1}{2} \leqq |f_m(e_1)| \leqq 2$$

and

$$G_m(0) \subset f_m(V).$$

Then there exists a K-quasiconformal self homeomorphism $f\colon \bar{\boldsymbol{R}}^n\to\bar{\boldsymbol{R}}^n$ and a subsequence of the $\{f_m\}$, which we relabel as $\{f_m\}$ again, such that

$$f_m \to f\quad \text{uniformly in}\quad \bar{\boldsymbol{R}}^n.$$

(see [V, Corollary 37.4]). We wish to show that in this case $S\subset f(V)$ which would be impossible by Theorem 2.14. Let $s\in S$. We denote by $q(x, y)$ the chordal distance between two points $x, y\in \bar{R}^n$. By (3.5) there is a sequence $\{s_m\}$ such that

$$s_m \to s \quad \text{and} \quad s_m\in G_m(0).$$

In particular $s_m\in f_m(V)$. Let $\delta>0$. Then by the uniform convergence there is an integer N such that if $m>N$, then for all $x\in \bar{R}^n$

$$q(f_m(x),\ f(x)) < \delta.$$

There is also an integer M such that if $m>M$, then $q(s_m, s)<\delta$.

We then see that if $n>M+N$,

$$\begin{aligned} q(s, f(V)) &\leqq q(s, s_m)+q(s_m, f(V)) \\ &< 2\delta, \end{aligned}$$

since $s_m\in f_m(V)$. Since δ was arbitrary we must conclude

$$S\subseteqq f(V),$$

which we have already observed is impossible due to the fact S is not locally quasiconformally flat. The lemma is proved.

Thus, the preceding lemma says that all the $G_m(0)$ cannot lie in a quasiconformal hyperplane. The following lemma will imply that if $G_0(0)$ lies in a K-quasiconformal hyperplane then so does $G_m(0)$ for all m.

3.7. Lemma. *If $g_m\in G_m$, then $g_m(0)=3^{-m}g_0(0)$ for some $g_0\in G_0$. In particular if Φ_m is the conformal mapping $\Phi_m(x)=3^{-m}x$, then*

$$G_m(0) = \Phi_m^{-1} G_0 \Phi_m(0) = 3^m G_0(0).$$

Proof. If $g_m\in G_m$, then there is a $t\in T^*$ such that

$$g_m = Fh_{-m}th_m F^{-1}.$$

We accordingly choose $g_0=FtF^{-1}$.

If $t(x)=x+(a_1, 0, a_3, \ldots, a_n)$, then

$$\begin{aligned} 3^m g_m(0) &= 3^m Fh_{-m}t(0) \quad (\text{since } F(0) = h_m(0) = 0) \\ &= 3^m F(4^{-m}a_1, 0, 3^{-m}a_3, \ldots, 3^{-m}a_n) \\ &= 3^m(F_1(4^{-m}a_1, 0), 3^{-m}a_3, \ldots, 3^{-m}a_n) \end{aligned}$$

by the definition of F (see (2.12)). Now $(4^{-m}a_1, 0)\in R\hookrightarrow R^2$ and $F_1((4^{-m}a_1, 0))=f_1(4^{-m}a_1)$. We then observe from Lemma 2.4

$$3^m f_1(4^{-m}a_1) = f_1(a_1).$$

Thus

$$\begin{aligned}3^m g_m(0) &= (f_1(a_1), a_3, \dots, a_n)\\ &= (F_1(a_1, 0), a_3, \dots, a_n)\\ &= Ft(0) = FtF^{-1}(0) = g_0(0).\end{aligned}$$

The lemma is now proved.

Recalling that $G_0=G^*$, we see from the above lemma that if $G^*(0)$ lies in a K-quasiconformal hyperplane, then so does $G_m(0)$ for all m. Thus by Lemma 3.6 this cannot be the case. That is $G^*(0)$ lies in no quasiconformal hyperplane. But according to our supposition (*) and Corollary 3.3 $G^*(0)$ does lie in such a hyperplane. We must conclude the supposition (*) is false. We have then obtained the following.

3.8. Theorem. *Every discrete subgroup of maximal rank $n-1$, in Tukia's group G is not the quasiconformal conjugate of any Möbius group.*

Thus in every dimension n greater than two there is a properly discontinuous quasiconformal group G acting on $\boldsymbol{R}^n$ which is not the quasiconformal conjugate of any Möbius group.

In fact Lemmas 3.6 and 3.7 show that the orbit of the origin under any discrete subgroup of maximal rank in Tukia's group cannot lie in a quasiconformal hyperplane. Since in our construction of the map F_* (see (2.12)) we ensured that $F_*|V=F|V$, see (2.2), we see every discrete subgroup of maximal rank in the group $H=F_*TF_*^{-1}$ is also not the quasiconformal conjugate of any Möbius group.

3.9. Corollary. *Every discrete subgroup of maximal rank in the group H is not the quasiconformal conjugate of any Möbius group.*

From (4) in [T. 1] we see that

$$\frac{1}{M}|a|^\alpha \leqq |f_1(x+a)-f_1(x)| \leqq M|a|^\alpha.$$

Thus the image of any uniformly discrete subset of $\boldsymbol{R}$ (see below) is uniformly discrete in J. It then follows that the orbit of the origin under any maximal rank discrete subgroup is uniformly discrete in S (since the corresponding orbit will be uniformly discrete in V). We may choose an arbitrarily coarse discrete subgroup (i.e. large fundamental domain) and we then find

3.10. Corollary. *For each $M>0$ there is a uniformly discrete set $L\subset S=J\times\boldsymbol{R}^{n-2}$, that is for each $x\in L$*

$$\operatorname{dist}(x, L\setminus\{x\}) > M,$$

and L lies in no K-quasiconformal hyperplane for any K.

This is essentially due to the fact that any uniformly discrete subset of the $(n-1)$-cell S is dense at infinity, where S is not quasiconformally flat. Notice that the set

$$L = f_1(Z)\times(Z)^{n-2}$$

is the orbit of the origin under the group $FT^*F^{-1}\subset G$ and so cannot lie in any quasiconformal hyperplane. There is a natural P. L. map of R^{n-1} with these points as vertices which we construct as follows. Let

$$P_i\colon [0, 1] \to J_i', \quad i = 0, 1, 2 \ldots \quad \text{be the natural P. L. map}$$

outlined in Tukia's construction. Notice that in the construction each vertex at each stage is retained. If $j\in[4^i, 4^{i+1}]\cap Z$, then the point $p_i(4^{-i}j)$ is a vertex of J_i', and so therefore of J_j' for all $j\geqq i$. Since

$$f_1(j) = 3^i f(4^{-i}j) = 3^i p_i(4^{-1}j)$$

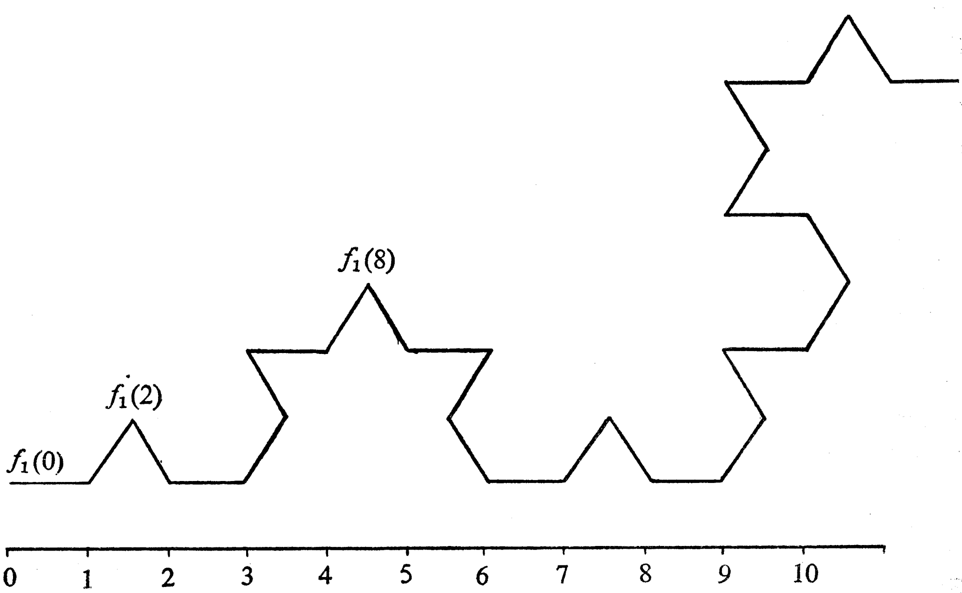
we see that $f(j)$ is a vertex of $3^iJ_i'$. We then define $P\colon R\to R^2$ as

$$P(x) = 3^i p_i(4^{-i}x)$$

for $x\in[4^i, 4^{i+1}]$ and $P(-x)=-P(x)$ It is not difficult to see that P is a well defined P. L. embedding. $P(R)$ is illustrated below, it is merely the natural P. L. map connecting the integer spaced vertices of the non-rectifiable arc J. A more compact formula for P is given by

$$P(x) = (1-r)f_1(n)+rf_1(n+1)$$

if $x=n+r$, $0<r\leqq 1$.



We see P is a P. L. embedding with the following two properties:

(1) $|P(n)-P(n+1)| = 1$.

(2) The angle between successive vertices is $\pi/3$. We then set

$$P_1(x) = (P(t), y)$$

where $x=(t, 0, y)\in R^2\times R^{n-2}$. So $P_1: V\to R^n$ is a P. L. map having $f_1(Z)\times(R)^{n-2}$ as its only corners. It is rather easy to see that $Q^n=P_1(V)\subset R^n$ is locally quasiconformally flat, in fact since the angle is $\pi/3$ and P_1 is an isometry on each $[n, n+1]\times R^{n-2}$ the map

$$(\gamma, \theta, y) \to (\gamma, \theta/3, y)$$

shows that Q^1 is locally 3^{n-1}-quasiconformally flat. Q^n cannot be quasiconformally flat for that would imply that Q^n and hence $f_1(Z)\times(Z)^{n-2}$ lay in a quasiconformal hyperplane which we know is impossible. We can do even better than this. It is easy to see one may smooth out the corners of our above example in such a way as to ensure that the vertices lie in the smoothed hyperplane. Such a smooth hyperplane is $(1+\varepsilon)$-locally quasiconformally flat for each positive ε, (since the projection from the tangent plane at a point, down to the hyperplane is almost conformal in a sufficiently small neighbourhood of that point). This smooth hyperplane cannot be quasiconformally flat as we have seen. Gehring [G] and Tukia [T. 1] have similar examples. The difference between their examples and ours is that ours is uniformly nice at every point, while Gehring's example oscillates wildly at the origin and Tukia's examples are low dimensional and exhibit rather peculiar behaviour at one point where it is the limit of Fox-Artin spheres. We note that the P. L. map $P: R\to R^2$ previously constructed is also quasisymmetric, (this is easy to see locally, while globally it follows since $P(n)=f_1(n)$ for all $n\in Z$, and since f_1 is quasisymmetric) and so has a quasiconformal extension $\bar{P}: \bar{R}^2\to\bar{R}^2$, which is quasiconformal. Then the components of $\bar{R}^2\setminus P(R)$ are uniform domains (see [M]). Thus the map $\bar{P}\times\mathrm{Id}: R^n\to R^n$ is a topological flattening of Q^n, (in fact since $\bar{P}(\infty)=\infty$ we see that this gives a flattening of the $(n-1)$-sphere $\bar{Q}^n$ in $\bar{R}^n$), that is $\bar{P}\times\mathrm{Id}\,|V=P_1$. Notice that as noted in [T. 3, p. 158] if $D\subset R^2$ is unbounded and uniform, then $D\times R^{n-2}$ is uniform. Hence both components of $R^n\setminus Q^n$ are uniform domains. We now see the interesting aspect of examples such as ours. The $(n-1)$-sphere $\bar{Q}^n$ is locally quasiconformally flat except at one point and not quasiconformally flat (it is topologically flat). While such examples as these are not surprising for low dimensions $(n\leqq 3)$, it is unusual for dimension $n\geqq 4$. For there flatness off a discrete set is a removable condition. That is if a sphere is locally flat except at a discrete subset, then it is flat [D., Corollary 3A. 5.]. Such a result is evidently not true in the quasiconformal category as the above examples point out. We summarize the above discussion in the following (c.f. [G, Example 2]).

3.11. Theorem. *For every $n \geqq 3$ there is a topologically flat $(n-1)$-sphere in $\bar{R}^n$, which is locally $(1+\varepsilon)$-quasiconformally flat except at one point and not quasiconformally flat. Further both the components of the complement of this sphere can be chosen to be uniform domains.*

The importance of the conclusion that the complementary components of such a sphere be uniform domains (which they are not in Gehring's example) stems from my paper [M] where I tried to find high dimensional counterexamples to the decomposition theorem for uniform domains (see [G. O.]). I found such an example in dimension 3, where the idea was to force bad behaviour at one point. Such examples as indicated in Theorem 3.11 seem to suggest that higher dimensional counterexamples may also be found using similar techniques.

4. Fuchsian and non-elementary examples

We now wish to construct from our previous examples, Fuchsian and nonelementary examples of discrete quasiconformal groups which are not the quasiconformal conjugates of Möbius groups. We denote by U^n the upper half-space

$$U^n = \{x = (x_1, x_2, \dots, x_n) \in \bar{R}^n \colon x_n > 0\}.$$

In this section a Fuchsian group will be a discrete subgroup of $\text{Möb}^+(n)$, the group of orientation preserving Möbius transformations of $\bar{R}^n$, acting on U^n as a group of hyperbolic isometries. A quasiconformal Fuchsian group is a discrete orientation preserving quasiconformal group of U^n. The restriction to the orientation preserving case is merely for simplicity, the general case follows easily, see [T. 4, Remark F2]. We identify the boundary of U^n with $\bar{R}^{n-1}$.

If G is a K-quasiconformal Fuchsian group, then G extends naturally by reflection to a K-quasiconformal group $\bar{G}$ of $\bar{R}^n$, see [V, 32.5]. The group $\bar{G}|\bar{R}^{n-1}$ is then a K-quasiconformal group of $\bar{R}^{n-1}$. When $n=3$, $\bar{G}|R^2$ is quasiconformally conjugate to a subgroup of $\text{Möb}^+(2)$, (this group is often denoted $\text{PSL}\,(2C)/\{\pm I\}$), by a K-quasiconformal mapping $f_1\colon \bar{R}^2 \to \bar{R}^2$, see [T. 2]. Now every such quasiconformal mapping can be extended to a K'-quasiconformal mapping $f\colon \bar{R}^3 \to \bar{R}^3$, where K' depends only on K and $n=3$ (this extension theorem is in fact true in all dimensions and is quite deep, see [T. V.]). Thus the group fGf^{-1} is a KK'^2-quasiconformal group of U^3 with conformal (or Möbius) boundary values, i.e.

$$f\bar{G}f^{-1}|\bar{R}^2 \subset \text{Möb}^+(2).$$

Hence in dimension three every quasiconformal Fuchsian group is quasiconformally conjugate to a quasiconformal Fuchsian group with conformal boundary values. Consequently the nice situation in dimension two affects the behaviour of three dimensional quasiconformal Fuchsian groups. It then seems that three dimensional quasiconformal Fuchsian groups that are not the quasiconformal conjugates of Fuchsian groups may be difficult to find. Indeed Tukia and I propose the following

4.1. Conjecture. *Every three dimensional quasiconformal Fuchsian group is the quasiconformal conjugate of a Fuchsian group.*

It seems very likely that this is the case. The obvious candidate for the conjugacy is the Poincaré extension of the conformal boundary group, indeed if the associated Möbius group has finite volume then up to a Möbius transformation of U^3, this is the only possible candidate by Mostow rigidity.

If we assume no torsion, then a quasiconformal Fuchsian group acts properly discontinuously and effectively (see [G. M.]) so the quotient space will be a manifold. If we then obtain a topological conjugacy to a Möbius group we can obtain a quasiconformal conjugacy by appealing to the L. Q. C.-Hauptvermutung in the compact case, (see the next section). We may obtain such a topological conjugacy by using some three manifold theory in some special cases, for instance if the quotient manifold is a compact Haken manifold, see [H]. Notice that the quotient of the associated Möbius group acting on U^3 is always a hyperbolic manifold of constant negative curvature. We will take another opportunity to discuss this conjecture in more detail.

For higher dimensions we can easily construct discrete quasiconformal Fuchsian groups which are not the quasiconformal conjugates of Möbius groups as follows. Let G' be any discrete subgroup of maximal rank in Tukia's group. Then G' is not the quasiconformal conjugate of any Möbius group and G' is a Lipschitz group, that is there is an L such that each $g\in G'$ is L-bilipschitz in the euclidean metric, see Theorem 2.13. The group $H=G'\times \mathrm{Id}$ acting on U^{n+1} by

$$h(X)=(g'(x),t) \quad X=(x,t)\in U^{n+1},\ h\in H,$$

is also a Lipschitz group, and hence a quasiconformal Fuchsian group. Suppose for contradiction that there was a quasiconformal mapping $f\colon U^{n+1}\to U^{n+1}$ such that fHf^{-1} was a Fuchsian group. We observe that f extends to a quasiconformal mapping $\bar f\colon \bar U^{n+1}\to \bar U^{n+1}$. Also the Fuchsian group fHf^{-1} is naturally defined on $\bar U^{n+1}$. We then see

$$\bar f|R^n\circ H|R^n\circ \bar f^{-1}|R^n=\bar f|R^n\circ G'\circ \bar f^{-1}|R^n$$

is a Möbius group. With our choice of G' this is impossible. We have shown

4.2. Theorem. *For every $n\geqq 4$, there is a discrete quasiconformal Fuchsian group which is not the quasiconformal conjugate of any Fuchsian (Möbius) group.*

Indeed we may observe that any Möbius group, not necessarily Fuchsian, conjugate to H would have to be free abelian on $n-2$ generators, and every element would have to be parabolic. Thus such groups would be conjugate into $E(n)$ by a Möbius transformation.

We now set

$$H' = F_* \circ T \circ F_*^{-1},$$

where F_* is the map of (2.12) and T is the group of translations generated by $x \to x + e_1$; $x \to x + e_i$, $i = 3, 4 \ldots n$.

By Corollary 3.9, H' is not the quasiconformal conjugate of any Möbius group. Now

$$F_*(x) = \big(F_2(z), y\big) \quad x = (z, y) \in R^2 \times R^{n-2},$$

and

$$F_2 | \textstyle\bigcup_{m \in Z} B^2\big((m, 1), \delta\big) \quad \text{is a translation on each component.}$$

So therefore is $F_* | E$, where

$$E = \textstyle\bigcup_{m \in Z} B^2\big((m, 1), \delta\big) \times R^{n-2}.$$

We next observe that if $t \in T$, then $t(E) = E$. Thus if the euclidean ball

$$B = F_*\big(B^n((0, 1, 0, \ldots, 0), \delta)\big),$$

then $h|B$ is a translation for every $h \in H'$. Actually we have found a subset B, of the fundamental domain, on which H' is conformal. We will see from the following combination theorem that this is all that is necessary to construct quite complicated quasiconformal groups. The main point is that if we combine a quasiconformal group which is not the quasiconformal conjugate of a Möbius group, with any other group, then the combination cannot be the quasiconformal conjugate of a Möbius group either. I am indebted to P. Tukia for suggesting the following version of the classical combination theorem, see [F, Section 25, Theorem 13].

4.3. Theorem. *Let G_i, $i \in I$, be a family of K-quasiconformal groups of $\bar{R}^n$ each with fundamental domain D_i. Suppose for each $i \in I$ there is an open set $B_i \subset D_i$ such that*

$$g_i \text{ is conformal near } \bar{B}_i \text{ for all } g_i \in G_i,$$

and that if $i \neq j$, then

$$\bar{R}^n \setminus B_i \subset B_j.$$

Then the group

$$G = \langle G_i \colon i \in I \rangle,$$

the group generated by the G_i, is the free product of the G_i and is a K-quasiconformal group and acts discontinuously in the open set

$$\operatorname{int}\big(\textstyle\bigcup_{i \in I} D_i\big).$$

Proof. We first show that the group is K-quasiconformal. Let $g \in G$. Then we can write g in the form

$$g = g_n \circ g_{n-1} \circ \ldots \circ g_2 \circ g_1$$

where no g_j is the identity and no two consecutive g_j and g_{j+1} belong to the same group G_i. For each $j = 1, 2, \ldots, n$ we set $B_j^* = B_i$ and $D_j^* = D_i$, where $i \in I$ is an index such that $g_j \in G_i$.

We consider the sequence

$$x \to g_1(x) \to g_2 g_1(x) \to \ldots \to g(x).$$

There are three cases.

(1) $x \in \bar{B}_1^*$. In this case g_1 is conformal near x and since $g_1(B_1^*) \cap B_1^* = \emptyset$, $g_1(x) \in B_2^*$. Thus we are reduced to considering this case again with the element $g_n \circ g_{n-1} \circ \ldots \circ g_2$ with x replaced by $g_1(x)$. Recursively we see that g is conformal near x.

(2) $x \notin B_1^*$ and $g_1(x) \notin \bar{B}_1^*$. In this case g_1 is K-quasiconformal near x and $g_1(x) \in \bar{R}^n \setminus \bar{B}_1^* \subset B_2^*$. Then from case (1) we see that the element $g_n \circ g_{n-1} \circ \ldots \circ g_2$ is conformal near $g_1(x)$ so that g is K-quasiconformal near x.

(3) $x \notin B_1^*$ and $g_1(x) \in B_1^*$. In this case g_1 must be conformal near x for g_1^{-1} is conformal near $g_1(x)$. Thus we need only examine the three cases again with the element $g_n \circ g_{n-1} \circ \ldots \circ g_2$ and x replaced by $g_1(x)$.

Hence each $g \in G$ is K-quasiconformal. We may also see in a similar manner that G acts discontinuously in $\operatorname{int}(\cap D_i)$, this helps too to see that G is the free product. Thus we assume that $g \in G$ has the above form and that the B_i are again appropriately relabeled. If $\operatorname{int}(\cap D_i)$ is empty there is nothing to prove (this may indeed be the case, in order then to see that G is the free product we need only observe that we could assume that the index set I was finite so that $\operatorname{int}(\cap D_i)$ was not empty), otherwise let $x \in \operatorname{int}(\cap D_i)$. Then $x \in D_1^*$, so that

$$g_1(x) \in R^n \setminus D_1^* \subset R^n \setminus B_1^* \subset B_2^* \subset D_2^*.$$

Similarly, $g_2 g_1(x) \in D_3^*$, and so on. Finally we see $g(x) \in R^n \setminus D_n^*$, so that

$$g(x) \notin \textstyle\bigcap_{i \in I} D_i,$$

for $g(x)$ does not lie in D_n^*. The proof is complete.

Returning to our construction we recall that we have shown that the group

$$H' = F_* T_* F^{-1},$$

was not the quasiconformal conjugate of any Möbius group and that each $h \in H'$ was conformal on the euclidean ball $B = F_*(B^n((0, 1, 0, \ldots, 0), \delta))$. A fundamental domain for the group T is easily seen to be

$$D = \{(x_1, x_2, \ldots, x_n) \colon 0 < x_i < 1,\ i = 1, 3, 4 \ldots n\}.$$

Thus a fundamental domain for the group H' is then $D' = F_*(D)$. Notice that $B \subset D'$.

Next let Q be any Möbius group with fundamental domain containing $R^n \setminus B$. There are many such groups, for instance the group generated by inversion in the sphere $S^{n-1}((0, 1, 0 \ldots 0), \delta/2)$ or more generally a Schottky group generated by reflections in spheres lying inside B. Then from Theorem 4.3. the group

$$G = \langle H', Q \rangle,$$

the group freely generated by H' and Q, will be a quasiconformal group. Since infinity cannot be fixed by Q, as it is interior to a fundamental domain, we see that the group cannot be elementary for infinity is a limit point for the group H', so that $Q(\infty)=\{q(\infty): q\in Q\}$ is also contained in the limit set. This set must then be an uncountable perfect set, see [G. M.]. The group G could not be the quasiconformal conjugate of a Möbius group since it contains the group H' which is not quasiconformally conjugate to any Möbius group.

The group G will be discrete since it acts discontinuously in the region

$$D'\cap\bar{R}^n\setminus B\supset D'\setminus B\neq\emptyset.$$

Actually it is not difficult to see what a fundamental domain for G should be, although proving that is rather complicated and immaterial to our considerations. Also we could extend H' to a quasiconformal Fuchsian group, say $H_0=H'\times\mathrm{Id}$, (this is possible since H' is actually a Lipschitz group from Section 2). If Q_0 is a Möbius Fuchsian group with fundamental domain containing $U^n\setminus B^{n+1}((0,1,0\dots0),\delta/2)$, then we see, as above, that the group $G_0=\langle H_0, Q_0\rangle$ is a quasiconformal Fuchsian group, since each element of H_0 and Q_0 preserves U^{n+1}, which is both discrete and non-elementary. By a process of infinite combination of such groups, or if we combined H_0 with a sufficiently complicated Fuchsian group, we could have arranged that the limit set was quite complicated, perhaps even all of $\bar{R}^n=\partial U^{n+1}$. However we could not have arranged that a fundamental domain for G_0 was compact in U^{n+1} for then by Tukia's result [T. 4] the boundary group $G_0|R^n$ would be quasiconformally conjugate to a Möbius group, which we know is impossible. We summarize the above discussion in the following two theorems,

4.4. Theorem. *For every $n\geqq 3$ there is a discrete quasiconformal group of $\bar{R}^n$ which is non-elementary and not the quasiconformal conjugate of any Möbius group.*

4.5. Theorem. *For every $n\geqq 4$ there is a discrete quasiconformal Fuchsian group of U^n which is non-elementary and not the quasiconformal conjugate of any Fuchsian (Möbius) group.*

5. Quasiconformal groups and the L. Q. C. Hauptvermutung

If we consider properly discontinuous, free actions G on R^n, then there is a natural manifold to consider, namely the quotient space R^n/G. If G is a K-quasiconformal group then R^n/G will have a natural quasiconformal structure of bounded dilation K. If H is another properly discontinuous free action on R^n, then G is conjugate to H if and only if there is a homeomorphism $F: R^n/G\to R^n/H$ such that the following diagram commutes,

$$\begin{array}{ccc} R^n & \xrightarrow{F} & R^n \\ /G\downarrow & \circ & \downarrow /H \\ R^n/G & \xrightarrow[f]{} & R^n/H, \end{array}$$

where F is the homeomorphism such that $FGF^{-1}=H$. We also note that this is true with U^n replacing $\boldsymbol{R}^n$. The L. Q. C. Hauptvermutung says that if two quasiconformal manifolds are homeomorphic, then there is in fact a locally quasiconformal homeomorphism between them, see [T. V.], if the dimension is not four, or in the case the manifolds have boundary, the dimension should not be four or five.

In the case that H is a discrete subgroup of rank $n-1$ of the translation group T earlier described, and G is the corresponding subgroup of Tukia's group, then H and G are topologically conjugate, by construction, and so the manifolds $\boldsymbol{R}^n/G$ and $\boldsymbol{R}^n/H$ are homeomorphic. They both have K-quasiconformal structures, in fact $\boldsymbol{R}^n/H$ has a conformal structure, hence by the L. Q. C. Hauptvermutung they are locally quasiconformally homeomorphic. They cannot be quasiconformally homeomorphic since this quasiconformal homeomorphism would lift to a quasiconformal mapping conjugating the groups, which we have shown is impossible. The usual example of such manifolds is of course the unit ball and $\boldsymbol{R}^n$, each of these has a 1-quasiconformal structure but there is no quasiconformal mapping between them, see [V, Theorem 17]. In the special case that G and H are cocompact, that is $\boldsymbol{R}^n/G$ and $\boldsymbol{R}^n/H$ are compact, topological conjugacy implies quasiconformal conjugacy, if $n\neq 4$, for a locally quasiconformal mapping on a compact manifold is quasiconformal. We have then observed

5.1. Theorem. *If G and H are co-compact K-quasiconformal groups acting properly discontinuously and freely on $\boldsymbol{R}^n, n\neq 4$ (resp. U^n) which are topologically conjugate, then in fact there is a quasiconformal self mapping of $\boldsymbol{R}^n$ (resp. U^n) such that*

$$fGf^{-1} = H.$$

The problem of topological conjugacy of isomorphic groups is a difficult one. Farrell and Hsiang [F. H.] have shown that if G is a co-compact properly discontinuous free action on $\boldsymbol{R}^n$ which is isomorphic to $\boldsymbol{Z}^n$ as groups, then G is topologically conjugate to the standard action of $\boldsymbol{Z}^n$ on $\boldsymbol{R}^n$, that is as a rank n translation group.

We thus obtain the following corollary which in some sense asserts that the groups which we have constructed (the discrete quasiconformal $\boldsymbol{Z}^{n-1}$ actions on $\boldsymbol{R}^n$ which are not the quasiconformal conjugates of the standard $\boldsymbol{Z}^{n-1}$ action) are the best possible.

5.2. Corollary. *If G is a cocompact K-quasiconformal group acting on $\boldsymbol{R}^n$, $n\neq 4$, properly discontinuously and freely, and which is isomorphic to $\boldsymbol{Z}^n$ as a group, then G is quasiconformally conjugate to the standard $\boldsymbol{Z}^n$ action on $\boldsymbol{R}^n$.*

Farrell and Hsiang have also obtained some results on groups acting cocompactly on U^n, which would then imply some results in the quasiconformal category. It is true that every n-manifold ($n\neq 4$ without boundary and $n\neq 4, 5$ with boundary) has a local quasiconformal structure, see [T. V.]. It would be very interesting to know just when this structure is quasiconformal, that is when is there an atlas of K-quasiconformal coordinate charts? This is always the case if the manifold is compact and Kuusalo, [K], has shown that every L. Q. C.-2-manifold has a conformal atlas. Another interesting question is to decide when the quasiconformal universal cover of a manifold is U^n? Note that U^n and $\boldsymbol{R}^n$ are quasiconformally inequivalent. Such manifolds can be of fairly general type, for instance all the constant negative curvature manifolds. However such manifolds must have some inherent geometry since the fundamental group must then act as a quasiconformal Fuchsian group and such actions must be reasonably nice. Indeed in the case $n=3$ and the cocompact case for all n (as we had earlier remarked), the fundamental group must in fact be quasiconformally conjugate to a subgroup of Möb $(n-1)$, see [T. 4]. If $n=3$, Möb$^+$ (2) can be thought of as PSL $(2, \boldsymbol{C})/\{\pm I\}$ and such manifolds are known as hyperbolic manifolds (conjugation into PSL $(2, \boldsymbol{C})$ is a stronger conclusion than the usual requirement that there be a discrete and faithful embedding into this group). It seems clear that the geometry and structure of quasiconformal Fuchsian groups, as well as general quasiconformal groups, may play an important role in the general theory of geometric structures on manifolds in all dimensions, including the case $n=3$.

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