Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 12, 1987, 33—46

THE C^2 IMAGE OF A BROWNIAN MOTION IN THE PLANE

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Abstract. We study the image $\varphi(b)$ of a two-dimensional Brownian motion b under a C^2 mapping φ. A decomposition $\varphi(b)=b_\varphi+c_\varphi$ is given. Here c_φ is a slow drift and b_φ is a process much like a Brownian motion. Especially, it has a kind of elliptic local behaviour. The method is to use a nonstandard representation of b given in [4]. This yields a discrete structure describing b_φ — and φ. It is shown that a C^2 mapping φ corresponds to such a structure, and we thus obtain discrete geometrical characterizations of C^2, K-quasiconformal or conformal mappings.

Introduction

If $\varphi\colon G'\to G$ is nonconstant analytic, b is a Brownian motion and $u\colon G'\to\mathbf{R}$ is harmonic, then it is well-known that $\varphi\circ b$ is a generalized Brownian motion (with a new clock but coming out from discs with uniform probability when started from the center); and where φ is injective, $u\circ\varphi^{-1}$ is harmonic (i.e., the value of $u\circ\varphi^{-1}$ at the center of a disc is the average of its values taken on the boundary of the disc). Broadly speaking, one can say that in this case the local behavior of $\varphi\circ b$ and $u\circ\varphi^{-1}$ is circular.

In this paper we look for similar local behavior while φ is only assumed to be C^2. It turns out that there exists in a sense elliptic limiting local behavior. Our method is to use a discrete conformally invariant nonstandard random walk B generating a two-dimensional Brownian motion as presented in [4]. Using B, we define a random walk B_φ generating $\varphi(b)$ by the discrete relation

$$\varphi\big(B(t,\omega)\big)\approx B_\varphi(t,\omega)+C_\varphi(t,\omega),$$

where the drift C_φ is the sum

$$\tfrac{1}{2}\Delta t\ \textstyle\sum_{s=0}^{t-\Delta t}\big[D_{11}\varphi\big(B(t,\omega)\big)-D_{22}\varphi\big(B(t,\omega)\big)\big].$$

This is slow when compared to B_φ and B, because the steps of the latter have length $\sqrt{2\Delta t}\,(\approx 0)$. We study the analogy of harmonic measure connected with $B_\varphi+C_\varphi$ and the local behavior of this random walk.

Finally, we show how φ can be recovered from the discrete structure of B_φ.

doi:10.5186/aasfm.1987.1227

This gives a discrete characterization for C^2-, K-quasiconformal and conformal mappings.

This paper is a continuation of [4]. For a background in nonstandard measure theory and other aspects of nonstandard theory of Brownian motion, see [1], [2] and [3].

1. Basic constructions

We first recall our conformally invariant construction of two dimensional Brownian motion from [4]. Let $H \in \mathbf{N}^* \setminus \mathbf{N}$ and set $N = H!$. Denote $\Delta t = 1/H$ and set $T = \{0, \Delta t, \ldots, H\}$. Let S be the set of the roots of the equation $z^N = 1$. Define

$$\Omega = S^{T \setminus \{0\}} = \text{the set of internal sequences } \big(\omega(\Delta t), \omega(2\Delta t), \ldots, \omega(H)\big).$$

Let $\tilde{\Omega} = (\Omega, \mathscr{D}, P)$ be the Loeb space obtained by giving every $\omega \in \Omega$ the weight $\Delta \bar{P} = 1/|\Omega| = 1/N^H$. Analogously, $\tilde{T} = (T, \mathscr{F}, M)$ will be the Loeb space obtained by giving every $t \in T$ the weight Δt. Recall that this space represents the Lebesgue measure via the map st^{-1}. The internal random walk B is defined in $T \times \Omega$ by

$$B(0, \omega) = 0 \quad \text{and} \quad B(t+\Delta t, \omega) = B(t, \omega) + \sqrt{2\Delta t}\, \omega(t+\Delta t).$$

Linear interpolation extends B to ${}^*[0, H] \times \Omega$. Finally, a standard process b is obtained by $b(t, \omega) = {}^\circ B(t, \omega)$ $\big(=$the standard $x \approx B(t, \omega)\big)$ for $t \in \mathbf{R}^+$ and $\omega \in \Omega$. Denote $B_x(t, \omega) = B(t, \omega) + x$ and $b_x(t, \omega) = b(t, \omega) + x$.

1.1. Proposition ([4]). *The process b: $\mathbf{R}^+ \times \Omega \to \mathbf{R}^2$ is a Brownian motion; for P-almost every ω, $b(\cdot, \omega)$ is continuous and $B(\cdot, \omega)$ is S-continuous.*

We shall work with a bounded domain $G' \subseteqq \mathbf{R}^2$ which, for simplicity, is assumed to contain 0 and to satisfy the following assumptions made also in [4].

(i): There is a continuous function p: $G'' \setminus G' \to \partial G'$ with $p|\partial G = \mathrm{id}$; here G'' is a domain with $G' \cup \partial G' \subseteqq G''$.

(ii) If $x \in {}^*G'$ and ${}^\circ x \in \partial G$, then there are $r_x \approx 0$ and $c_x \approx 1/2$, for which an arc of length $\geqq c_x 2\pi r_x$ of the circle with center x and radius r_x is contained in $-{}^*G'$. Moreover, the function $x \mapsto (r_x, c_x)$ is assumed to be internal.

Discs have these properties; they hold also whenever $\partial G'$ is C^2.

The discrete versions of the interior and boundary of G' were defined in [4] as

$$IG' = \{B(t, \omega) | \, t \in T,\ \omega \in \Omega \text{ and } B(s, t) \in {}^*G' \text{ as } t \geqq s \in T\}$$

$$DG' = \big\{B(T(\omega), \omega) | \, \omega \in \Omega \text{ and } T(\omega) \text{ is defined}\big\},$$

where $T_x(\omega)$ is the smallest $t \in T$ with $B_x(t, \omega) \notin {}^*G'$, when such a one exists, and $T(\omega) = T_0(\omega)$.

Next let φ: $G' \to G$ be C^2 and onto with Jacobian $\neq 0$ everywhere in G', defined in some domain G'' with $G' \subset\subset G''$.

To study the effect of φ, we define the following analogies of the previous notions:

$$B_\varphi(0,\omega)=\varphi(0),$$

$$B_\varphi(t+\Delta t, a)=B_\varphi(t,\omega)+\sqrt{2\Delta t}\,D_\varphi(B(t,\omega))(\omega(t+\Delta t)).$$

We also need another internal process, the *drift* $C_\varphi(t,\omega)=(C_\varphi(t,\omega)_1, C_\varphi(t,\omega)_2)$ as

$$C_\varphi(t,\omega)_i=\tfrac{1}{2}\Delta t\sum_{s=0}^{t-\Delta t}\left[D_{11}{}^*\varphi_i(B(s,\omega))-D_{22}{}^*\varphi_i(B(s,\omega))\right].$$

Then let

$$I_\varphi G=\{B_\varphi(t,\omega)+C_\varphi(t,\omega)|t\in T,\ \omega\in\Omega \text{ and } B_\varphi(s,\omega)+C_\varphi(s,\omega)\in{}^*G \text{ as } t\geqq s\in T\}$$

and

$$D_\varphi G=\{B_\varphi(T_\varphi(\omega),\omega)+C_\varphi(T_\varphi(\omega),\omega)|\omega\in\Omega \text{ and } T_\varphi(\omega) \text{ is defined}\},$$

where $T_\varphi(\omega)$ is the smallest $t\in T$ with $B_\varphi(t,\omega)+C_\varphi(t,\omega)\notin{}^*G$, when such a one exists.

Basically, $B_\varphi+C_\varphi$, $I_\varphi G$ and $D_\varphi G$ are the φ-images of B, IG' and DG'; the only difference is that the steps have been approximated by the derivative of φ and then corrected by a second order term (see Figure 1).

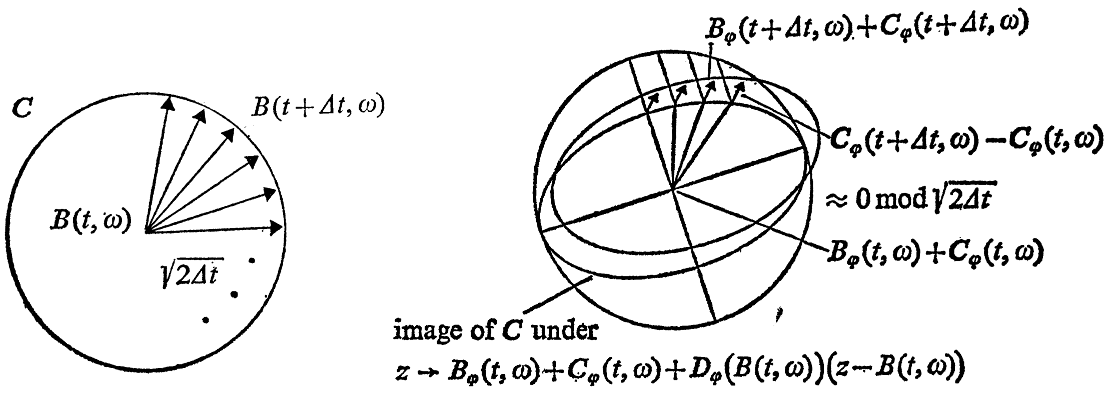

Figure 1

1.2. Theorem. *There is an internal set A with $\bar{P}(A)\approx 1$ and ${}^*\varphi(B(t,\omega))\approx B_\varphi(t,\omega)+C_\varphi(t,\omega)$ for $\omega\in A$ when $t\in T$ is finite and $B(t,\omega)\in IG'\cup DG'$.*

Proof. We use Taylor's formula representing ${}^*\varphi(B(s+\Delta t,\omega))-{}^*\varphi(B(s,\omega))$ as in the proof of Theorem 4.2 in [4].

It suffices to consider the second order term corresponding to φ_i. This has the form

$$\Delta t\left[\omega_1(s+\Delta t)^2 D_{11}{}^*\varphi_i(B(s,\omega))+2\omega_1(s+\Delta t)\omega_2(s+\Delta t)D_{12}{}^*\varphi_i(B(s,\omega))\right.$$
$$\left.+\omega_2(s+\Delta t)^2D_{22}{}^*\varphi_i(B(s,\omega))\right]$$

(where $\omega=(\omega_1,\omega_2)$). If $\omega=\omega(s+\Delta t)$ is taken as a complex number, we have $\omega_1=(\omega+\bar\omega)/2$, $\omega_2=(\omega-\bar\omega)/2$ and $|\omega|^2=\omega\bar\omega$. Thus

$$\omega_1^2=\frac{1}{4}\omega^2+\frac{1}{4}\bar\omega^2+\frac{1}{2}|\omega|^2,$$

$$\omega_1\omega_2=\frac{1}{2}\omega^2-\frac{1}{4}\bar\omega^2,$$

and

$$\omega_2^2=\frac{1}{4}\omega^2+\frac{1}{4}\bar\omega^2-\frac{1}{2}|\omega|^2.$$

As in 4.2 of [4], the sums of terms corresponding to ω^2 or $\bar\omega^2$ are liftings of infinitesimal stochastic integrals of nonanticipating integrands, because $z\to z^2$ and $z\to\bar z^2$ preserve uniform probability measure on $\{z \mid |z|=1\}$.

What remains up to ≈ 0, when the terms corresponding to $s=0,\dots,t-\Delta t$ are summed, is

$$\tfrac{1}{2}\Delta t\sum_{s=0}^{t-\Delta t}\left[D_{11}{}^*\varphi_i(B(s,\omega))-D_{22}{}^*\varphi_i(B(s,\omega))\right].\quad\square$$

Remark. If φ is conformal, complex derivatives are available and it follows from Taylor's formula of second order that the drift c_φ vanishes (see the proof of Theorem 4.2 in [4]).

The internal random walk $B_\varphi+C_\varphi$ on $(\Omega, {}^*\mathcal{P}(\Omega), \bar P)$ can be extended from $T\times\Omega$ to ${}^*[0,H]\times\Omega$ by linear interpolation. Then we get standard processes $b_\varphi\colon \mathbf{R}^+\times\Omega\to\mathbf{R}^2$ and $c_\varphi\colon \mathbf{R}^+\times\Omega\to\mathbf{R}^2$ by

$$b_\varphi(t,\omega)={}^0B_\varphi(t,\omega)\quad\text{and}\quad c_\varphi(t,\omega)={}^0C_\varphi(t,\omega).$$

To define versions of $B_\varphi+C_\varphi$ and $b_\varphi+c_\varphi$ started from a given point, we must be a bit more careful than with B and b. If $x=B_\varphi(t',\omega')+C_\varphi(t',\omega')\in I_\varphi G$, t' finite, and $\omega'\in A$ with A as in 1.2, we set

$$(B_\varphi+C_\varphi)_x(t,\omega)=B_\varphi(t'+t,\omega'')+C_\varphi(t'+t,\omega''),$$

where $\omega''(s)=\omega'(s)$ as $s\leqq t'$ and $\omega''(t'+s)=\omega(s)$ as $s\leqq t$. Thus $B_{\varphi,x}(t,\omega)$ will be defined at least for all finite t (and for all $t\in T$ if $B_\varphi(t'+t,\omega'')$ is understood in the obvious way for all t).

If $x\in G$ is standard and $x\approx x'\in I_\varphi G$, then we define

$$(b_\varphi+c_\varphi)_x(t,\omega)={}^0(B_\varphi(t,\omega)+C_\varphi(t,\omega))_{x'}.$$

Several basic properties of these notions are collected in the following lemma:

1.3. Lemma. (i) *The definition of* $(B_\varphi+C_\varphi)_x$ *does not depend on the choice of* ω'.

(ii) *If* $x \in G$, *then* $x \approx B_\varphi(t, \omega) + C_\varphi(t, \omega)$ *for some finite* $t \in T$ *and some* $\omega \in \Omega$.

(iii) *If* $x, y \in I_\varphi G$ *and* $x \approx y$, *then* $(B_\varphi + C_\varphi)_x(t, \omega) \approx (B_\varphi + C_\varphi)_y(t, \omega)$; *especially* $(b_\varphi + c_\varphi)_x$ *is correctly defined.*

(iv) ${}^0B(t, \omega) \in G'$ *if and only if* ${}^0B_\varphi(t, \omega) \in G$.

(v) *There is a finite constant* k *with* $|C_\varphi(t, \omega)| \leqq k \cdot t$.

(vi) $\varphi(b_x(t, \omega)) = (b_\varphi(t, \omega)) + c_\varphi(t, \omega))_{\varphi(x)}$ *for* P-*a.a.* ω.

(vii) $t_x(\omega) = t_{\varphi, \varphi(x)}(\omega)$ *for* P-*a.a.* ω, *where* t_x *and* $t_{\varphi, \varphi(x)}$ *are the times of the first exits of* b_x *and* $(b_\varphi + c_\varphi)_{\varphi(x)}$ *from* G' *and* G, *respectively.*

(viii) $c_\varphi(t, \omega) = \int_0^t [D_{11}\varphi(b(s, \omega)) - D_{22}\varphi(b(s, \omega))]\, ds$ *for* P-*a.a.* ω.

Proof. Apply the arguments of [4]. Some remarks:

(ii) This follows from the analogous fact about G' and B (i.e., standard properties of Brownian motion).

(iii) Straightforward induction.

(iv) Straightforward induction.

(vi) Use the same for B and the continuity of φ.

(vii) Follows from (v) and (vi).

(viii) Follows from the nonstandard representation of Lebesgue measure in the case $B(\cdot, \omega)$ is continuous. □

The most important part of this is the representation

$$\varphi(b(t, \omega)) = b_\varphi(t, \omega) + c_\varphi(t, \omega).$$

Like $(B_\varphi + C_\varphi)_x$, we can of course define $B_{\varphi,x}$ and $b_{\varphi,x} = {}^0B_{\varphi,x'}$, $x \approx x'$ and x' is of the form $B_\varphi(t', \omega')$. If $t'_{\varphi,x}$ is the time of the first exit of $b_{\varphi,x}$, we have the following result.

1.4. Theorem. *For* $x \in G$, *the process* $b_{\varphi,x}$ *is a martingale, i.e., a Markov process with*

$$x = Eb_{\varphi,x}(t_{\varphi,x}(\omega), \omega)\, dP.$$

Proof. The Markov property is obvious. We prove the other property by proving a similar result (that $B_{\varphi,x}$ is a hypermartingale) for the discrete version. Consider $B_{\varphi,x}$. The sets A_t are defined recursively as follows: $A_0 = \{x\}$ and $A_{t+\Delta t}$ is obtained from A_t by replacing every point

$$B_{\varphi,x}(t, \omega) \in A_t \cap I_\varphi G$$

with all the possible values of $B_{\varphi,x}(t + \Delta t, \omega)$ and taking these to $A_{t+\Delta t}$; if

$$B_{\varphi,x}(t, \omega) \in A_t \cap D_\varphi G,$$

then $B_{\varphi,x}(t, \omega)$ is taken to $A_{t+\Delta t}$. Intuitively, A_t tells where $B_{\varphi,x}$ can be at time t. Clearly,

$$A_H = \{B_{\varphi,x}(T_{\varphi,x}(\omega), \omega) | \omega \in \Omega\}.$$

An induction over t shows that the average over ω of A_t is always x. So

$$EB_{\varphi,x}(T_{\varphi,x}(\omega),\omega)\,d\bar{P} = x. \quad \square$$

A similar proof gives

1.5. Theorem. *For* $x\in G'$,

$$\varphi(x) = E\left[\varphi(b_x(t_x(\omega),\omega)) - \int_0^{t_x(\omega)} \left[D_{11}\varphi(b_x(s,\omega)) - D_{22}\varphi(b_x(s,\omega))\right] ds\right] dP.$$

The process $B_\varphi + C_\varphi$ has the following invariance property.

1.6. Theorem. *If* $\psi\colon G\to G_1$ *is* C^2, *then there is an internal set* $A\subseteq\Omega$ *with* $\bar{P}(A)\approx 1$ *and*

$${}^*\psi(B_\varphi(t,\omega)+C_\varphi(t,\omega)) \approx B_{\psi\circ\varphi}(t,\omega)+C_{\psi\circ\varphi}(t,\omega)$$

for finite $t\in T$ *and* $\omega\in A$. *(The Jacobian of* ψ *is assumed to be* $\neq 0$.*)*

Proof. By 1.2, ${}^*\psi(B_\varphi+C_\varphi)\approx{}^*\psi({}^*\varphi(B))={}^*(\psi\circ\varphi)(B)$ and $B_{\psi\circ\varphi}+C_{\psi\circ\varphi}\approx {}^*(\psi\circ\varphi)(B)$, $\bar{P}$ almost always.

Remark. In this chapter we did not need $D_\varphi G$, wherefore it was enough to assume that φ is defined in G'; in the following chapter we shall need φ also on the boundary.

2. Images of harmonic measures

In [4] we showed how the discrete analogy of Brownian motion provides a pleasant way of looking at harmonic measures. Here we similarly study measures connected to b_φ. We assume throughout this chapter that φ is one to one.

Let $x\in I_\varphi G$. The *internal φ-harmonic measure* $\bar{M}_{\varphi,x}$ on $D_\varphi G$ is defined by the weights

$$\Delta\bar{M}_{\varphi,x}(y) = \bar{P}(\{\omega\,|\,(B_\varphi+C_\varphi)_x(T_{\varphi,x}(\omega),\omega) = y\})$$

for $y\in D_\varphi G$. Here $T_{\varphi,x}$ is the time of the first exit of $(B_\varphi+C_\varphi)_x$. The corresponding Loeb-measure on $D_\varphi G$, $M_{\varphi,x}$, is called the *discrete φ-harmonic measure.* Recall that the internal harmonic measure $\bar{M}_x$, and the discrete harmonic measure $M_{x'}$, $x'\in IG'$, were defined similarly in [4]. By analogy with the construction of harmonic measure $\mu_{x'}$, $x'\in G'$, define the *φ-harmonic measure* $\mu_{\varphi,x}$, $x\in G$, by

$$\mu_{\varphi,x}(C) = M_{\varphi,x}(st^{-1}(C)),$$

whenever the right side is defined.

Here $C\subseteq\partial G$ and $st^{-1}(C)=\{z\in D_\varphi G\,|\,{}^0z\in C\}$. Especially, $\mu_{\varphi,x}(C)$ is defined for Borel sets (see [4]).

Let next $f\colon \partial G\to\mathbf{R}$ be continuous. Actually, it suffices to assume that f has a lifting F with respect to the measures $M_{\varphi,x}$ (see [4]). It corresponds to a continuous function $g=f\circ(\varphi|\partial G')$ defined on $\partial G'$. We can extend ${}^*f$ to $D_\varphi G$ by letting

$$^*f(B_\varphi(T_\varphi(\omega),\omega)+C_\varphi(T_\varphi(\omega),\omega)) = {}^*f(B_\varphi(t,\omega)+C_\varphi(t,\omega)),$$

where $t\in{}^*[T_\varphi(\omega)-\Delta t, T_\varphi(\omega)]$ satisfies $B_\varphi(t,\omega)+C_\varphi(t,\omega)\in{}^*\partial G$.

Remark. Using this trick, one could eliminate the assumption concerning the projection p (also from Chapter 2 of [4]), when continuous boundary values are considered.

After these remarks we define

$$u(x) = \int_{\partial G} f\, d\mu_{\varphi,x} \quad \text{for} \quad x\in G;$$

$$U(x) = \sum_{D_\varphi G} {}^*f\Delta \overline{M}_{\varphi,x} \quad \text{for} \quad x\in I_\varphi G.$$

These are like their analogies in [4] and so is the proof of the following list of basic properties.

2.1. Theorem. (i) *If* $x\in I_\varphi G$, *then* ${}^0U(x)=u({}^0x)$; *especially, u is continuous and U is S-continuous.*

(ii) *If* $x\in I_\varphi G$, *then*

$$U(x) = E\big({}^*f(B_\varphi+C_\varphi)_x(T_{\varphi,x}(\omega),\omega)\big)\,d\overline{P}.$$

(iii) *If* $x\in G$, *then*

$$u(x) = E\big(f(b_\varphi+c_\varphi)_x(t_{\varphi,x}(\omega),\omega)\big)\,dP.$$

(iv) *If* $x\in I_\varphi G$ *and* ${}^0x\in\partial G$, *then* $U(x)\approx f({}^0x)$.

(v) *If* $y\in\partial G$, *then* $u(x)\to f(y)$ *as* $x\to y$ *in* G.

Let V and v be defined as

$$V(x) = \sum_{DG'} {}^*g\Delta\overline{M}_x$$

and

$$v(x) = \int_{\partial G'} g\, d\mu_x,$$

where g is the continuous function on $\partial G'$ defined as $g=f\circ(\varphi|\partial G')$.

2.2. Theorem. (i) *If* $B(t,\omega)\in IG'$, *then*

$$V\big(B(t,\omega)\big) \approx U\big(B_\varphi(t,\omega)+C_\varphi(t,\omega)\big);$$

(ii) $u\circ\varphi=v$;

(iii) $v\circ\varphi^{-1}=u$.

Proof. (i) Because ${}^0B(t',\omega')\in G'$ if and only if ${}^0B_\varphi(t',\omega')+{}^0C_\varphi(t',\omega')\in G$ by 1.3, also

$$x={}^0B(t',\omega')\in\partial G', \quad \text{if and only if} \quad y={}^0B_\varphi(t,\omega)+{}^0C_\varphi(t,\omega)\in\partial G.$$

In this case $\varphi(x)=y$, and hence

$$(+) \qquad {}^*g\big(B(t',\omega')\big)\approx g(x) = f(y) \approx {}^*f\big(B_\varphi(t'',\omega')+C_\varphi(t'',\omega')\big)$$

as $B(t', \omega')\in DG'$ and $B_\varphi(t'', \omega')+C_\varphi(t'', \omega', \in D_\varphi G$. If (+) is applied to the definitions of V and U, we get (i).

Assertions (ii) and (iii) follow from (i).

Remark. The result implies that, given continuous boundary values, $B_\varphi+C_\varphi$ generates a function u which is an extremal with respect to a variational integral I_F corresponding to φ and the Dirichlet integral in G'; hence u is F-harmonic.

3. Local behavior of $\varphi\circ b$

The characteristic feature of b and a harmonic function v is that they are circular in the following sense:

when started from the center of a disc, b comes out with uniform probability;

the value of v at the center of a disc is the (uniform) average of values taken on the boundary.

Moreover, if φ is conformal (or just nonconstant analytic), also $\varphi\circ b$, and $v\circ\varphi^{-1}$ with φ one to one, have the same properties. Here we look for a similar local description of b_φ and $u=v\circ\varphi^{-1}$ in the more general situation considered in this paper.

If E is a domain whose boundary is an ellipse, then the *elliptic measure*, μ_e, on ∂E is obtained from the uniform probability measure on the circumscribed circle by projection along the shorter axis of ∂E. (See the figure below.)

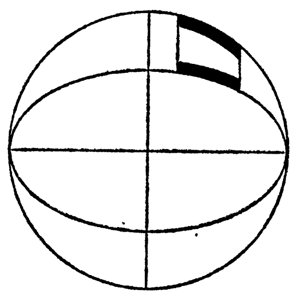

Figure 2

We shall show that if E is the image of the unit disc with center at x under $x'\mapsto D_\varphi(x)(x'-x)$, then μ_e represents the behavior of $\varphi\circ b$ and $u=v\circ\varphi^{-1}$ at $\varphi(x)$ in two senses.

Let $B_\varphi+C_\varphi$ and U be as in the previous chapter. Then the distribution of $B_\varphi(t+\Delta t, \omega)+C_\varphi(t+\Delta t, \omega)$ is like μ_e for $x=B(t, \omega)$, and actually generates μ_e

via an obvious Loeb measure construction. Also,

$$U(y) = \frac{1}{N} \sum_{i=1}^{N} U(y_i),$$

where $y = B_\varphi(t, \omega) + C_\varphi(t, \omega)$ and $y_1, \ldots, y_N$ are the possible values of $B_\varphi(t+\Delta t, \omega) + C_\varphi(t+\Delta t, \omega)$. In other words, $B_\varphi + C_\varphi$ and U behave locally exactly like a discrete version of μ_e. We feel that this form of ellipticity is the more basic one.

The other sense is a limiting one. Some notation is needed for it. Fix $x \in G'$ and let $r > 0$. Denote

$$\begin{aligned} A_r &= {}^*\partial\{x' \mid |x'-x| < r\}, \\ B_r &= {}^*\varphi A_r, \\ C_r &= \{{}^*\varphi(x) + D\,{}^*\varphi(x)(x'-x) \mid x' \in A_r\}. \end{aligned}$$

3.1. Theorem. *If φ is one to one and $u = v \circ \varphi^{-1}$ with v harmonic, then $u(\varphi(x)) = \lim_{r\to 0} \int_{C_r} u\, d\mu_e$.*

Proof. Let $r \approx 0$ and let v be as in Chapter 2. By harmonicity,

$$u(\varphi(x)) = v(x) = \int_{A_r} {}^*v\, d\mu = \int_{B_r} {}^*u\, d\mu_\varphi,$$

where μ is the uniform probability measure on A_r and μ_φ is its image on B_r.

Claim. With notation as above, $\int_{B_r} {}^*u\, d\mu_\varphi \approx \int_{C_r} {}^*u\, d\mu_e$.
If the claim holds, we have

$$\mu(\varphi(x)) \approx \int_{C_r} {}^*u\, d\mu_e.$$

Thus for standard $\varepsilon > 0$ and $r \approx 0$,

$$\left|\mu(\varphi(x)) - \int_{C_r} {}^*u\, d\mu_e\right| < \varepsilon.$$

This implies the result of the theorem by overflow. So it is enough to prove the claim.

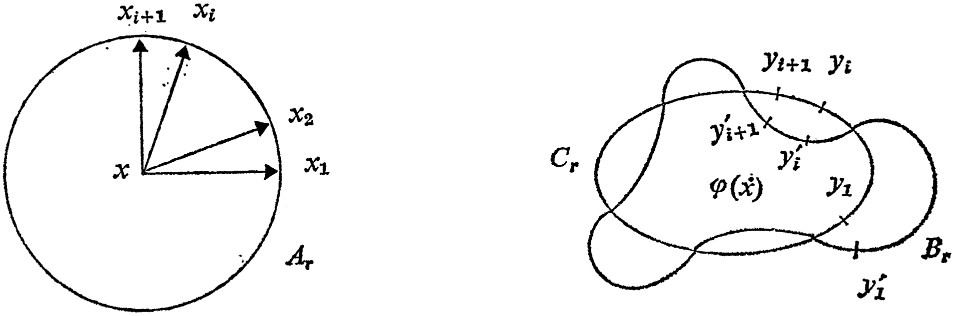

Figure 3

Consider the points $x+rs\in A_r$ for $s\in S$. List these as $x_1, \ldots, x_N$ in the positive direction starting from $x_1=x+r\cdot(1,0)$. Let $y_i=\varphi(x)+D_\varphi(x)(x_i-x)$, and let $y_i'=\varphi(x_1)$ and $y_2, \ldots, y_N$ be chosen such that the μ_φ-measure of the arc of B_r between y_i' and y_{i+1}' equals the μ_e-measure of the arc between y_i and y_{i+1} of C_r. (Actually, $y_i'=\varphi(x_i)$.)

We compare the integrals on the arcs (y_i, y_{i+1}) and (y_i', y_{i+1}'). Because these have the same measure, it is enough to compare the integrands. But since φ is differentiable and $r\approx 0$, we obtain a uniform bound $\delta\approx 0$ not depending on i for $|{}^*u(y)-{}^*u(y')|$, where $y\in\operatorname{arc}(y_i, y_{i+1})$ and $y'\in\operatorname{arc}(y_i', y_{i+1}')$. Thus

$$\left|\int_{C_r} {}^*u\, d\mu_e - \int_{B_r} {}^*u\, d\mu_\varphi\right| \leqq \int_{\operatorname{arc}(y_i, y_{i+1})} d\mu_e \approx 0,$$

where the indices are understood $\bmod N$. This completes the proof.

A standard calculation gives the following corollary which needs some notation. The exits of $b_{\varphi,\varphi(x)}+c_{\varphi,\varphi(x)}$ from C_r generate a φ-subharmonic measure $\mu_{\varphi,\varphi(x),C_r}$ on ∂C_r. This homothetically generates a measure on ∂C_{r_0}, which will be denoted by μ_{r,r_0}.

3.2. Corollary. *If $A\subseteqq\partial C_{r_0}$ is Borel, then*

$$\mu_e(A) = \lim_{r\to 0} \mu_{r,r_0}(A).$$

Theorem 3.1 and Corollary 3.2 are generalizations of a classical result about analytic functions.

4. A construction of a C^2-mapping from its derivative

In this chapter we shall show how φ can be recovered from our representation of the image of a Brownian motion under φ; i.e., essentially from the derivative of φ.

It turns out to be important to be able to discuss all $\omega\in\Omega$, not just almost all as is the case in Theorem 1.2. For this reason we define D_φ to be the sum of those second order terms (corresponding to $s=0, \ldots, t-\Delta t$) omitted in the definition of C_φ. Let G', G and φ be as before.

4.1. Lemma. *If $t\in T$ is finite and $B(t,\omega)\in IG'\cup DG'$, then* ${}^*\varphi(B(t,\omega))\approx B_\varphi(t,\omega)+C_\varphi(t,\omega)+D_\varphi(t,\omega)$.

The assertion follows essentially from the proof of Theorem of 1.2.

The main concept in this chapter is an abstract version of the representation $B_\varphi+C_\varphi+D_\varphi$.

An elliptic structure $\hat{L}$ on G consists of an element $z_0\in G$ and a family of linear mappings L_x, where the mappings adjoin to x, the coefficients of L_x are finite and have finite S-continuous partial derivatives, and which satisfy conditions (i)—(iv) below. Before the conditions we shall define some notation. First an analogy of B_φ:

$$\hat{B}(0, \omega) = z_0;$$

$$\hat{B}(t+\Delta t, \omega) = \hat{B}(t, \omega) + \sqrt{2\Delta t}\, L_{B(t,\omega)}\omega(t+\omega t),$$

when $L_{B(t,\omega)}$ is defined. Similarly, $\hat{C}$ and $\hat{D}$ are analogies of C_φ and D_φ and they are defined as follows: $\hat{C}=(\hat{C}_0, \hat{C}_1)$ and $\hat{D}=(\hat{D}_0, \hat{D}_1)$ where

$$\hat{C}_i(t, \omega) = \tfrac{1}{2}\Delta t \sum_{s=0}^{t-\Delta t} [L_i^{11}(s, \omega) - L_i^{22}(s, \omega)];$$

and

$$\hat{D}_i(t, \omega) = \Delta t \sum_{s=0}^{t-\Delta t} \left(\frac{1}{4} L_i^{11}(s, \omega) + L_i^{12}(s, \omega) + \frac{1}{4} L_i^{22}(s, \omega)\right) \omega(s+\Delta t)^2$$

$$+\Delta t \sum_{s=0}^{t-\Delta t} \left(\frac{1}{4} L_i^{11}(s, \omega) - L_i^{12}(s, \omega) + \frac{1}{4} L_i^{22}(s, \omega)\right) \overline{\omega}(s+\Delta t)^2,$$

when all the terms appearing in the sums are defined; here $L_i^{kl}(s, \omega)$ denotes the (s^-)derivative along the l'th coordinate axis of the coefficient of $L_{B(s,\omega)}$ in the same place as $D_k\varphi_i$ is in the Jacobian of a function φ: $\mathbf{R}^2 \to \mathbf{R}^2$. The conditions are:

(i) if $L_{B(t,\omega)}$ is defined and $t > s \in T$, then $L_{B(s,\omega)}$ is defined;

(ii) if $L_{B(t,\omega)}$ is defined, then $\hat{B}(s, \omega) + \hat{C}(s, \omega) + \hat{D}(s, \omega)$ is defined and lies in $^*G$ when $t+\Delta t \geqq s \in T$;

(iii) for all $x \in G$ there is (t, ω) with $\hat{B}(t, \omega) + \hat{C}(t, \omega) + \hat{D}(t, \omega)$ defined and $\approx x$;

(iv) if $\hat{B}(t, \omega) + \hat{C}(t, \omega) + \hat{D}(t, \omega)$ and $\hat{B}(t', \omega') + \hat{C}(t', \omega') + \hat{D}(t', \omega')$ are defined and $B(t, \omega) \approx B(t', \omega')$, then

$$\hat{B}(t, \omega) + \hat{C}(t, \omega) + \hat{D}(t, \omega) \approx \hat{B}(t', \omega') + \hat{C}(t', \omega') + \hat{D}(t', \omega').$$

The elliptic structure (L_x, z_0) is an *elliptic G'-structure*, if in addition

(v) $\hat{B}(t, \omega) + \hat{C}(t, \omega) + \hat{D}(t, \omega)$ is defined, if and only if $B(t, \omega) \in IG' \cup DG'$.

The name elliptic structure refers to the fact that the image of the unit circle under a linear mapping is an ellipse. (See also Figure 1.)

An elliptic structure is *K-quasiconformal* if the eccentricity of the image of the unit circle under each L_x lies between $1/K$ and K. The structure is *conformal*, if all these ellipses are circles.

4.1. Example. The representation of $\varphi(b(t, \omega))$ in terms of $B_\varphi + C_\varphi (+D_\varphi)$ is an elliptic G-structure. It is K-quasiconformal or conformal, if φ is.

The converse holds in the following sense.

4.2. Theorem. *If $\hat{L}$ is an elliptic G'-structure on G, then there is a C^2-mapping φ: $G' \to G$ onto with*

(i) $\varphi({}^0B(t, \omega)) = {}^0\hat{B}(t, \omega) + {}^0\hat{C}(t, \omega)$ *for all finite* $t \in T$ *and almost all* ω;

(ii) $D\varphi({}^0B(t, \omega)) = {}^0L_{B(t,\omega)}$;

(iii) *if $\hat{L}$ is K-quasiconformal or conformal, then φ is, too.*

Proof. Define

$$\varphi({}^0B(t,\omega)) = {}^0\hat{B}(t,\omega)+{}^0\hat{C}(t,\omega)+{}^0\hat{D}(t,\omega),$$

t finite. Because B comes infinitesimally close to every $x\in G'$, φ is defined for all $x\in G'$. Condition (iv) implies that the definition of φ does not depend on the choice of t and w. We prove assertion (ii); it follows then directly from the assumptions that φ is C^2 and that assertion (iii) holds. Finally, assertion (i) follows from the argument of the proof of Theorem 1.2.

Let $x_1\in G'$ and $x_1\approx x_1'=B(t,\omega)\in IG'$. Thus

$$\varphi(x_1) = \hat{B}(t,\omega)+{}^0\hat{C}(t,\omega)+{}^0\hat{D}(t,\omega).$$

We consider first some fixed $s\in S$ and the half-line $x_1'+hs$, $h\geqq 0$. For $x''=x_1'+1\sqrt{2\Delta t}$ on this half-line, let

$$F(x'') = \hat{B}(t+k\Delta t,\omega')+\hat{C}(t+k\Delta t,\omega')+\hat{D}(t+k\Delta t,\omega'),$$

where $\omega'(t')=\omega(t)$ for $t'\leqq t$ and $\omega'(t')=s$ as $t'>t$. If $x''=x_1'+k\sqrt{2\Delta t}$ and $k\sqrt{2\Delta t}\approx 0$, we have

$$F(x'')-F(x_1') = \sqrt{2\Delta t}\,\textstyle\sum_{i=1}^k L_{x_i'}s+((\hat{C}+\hat{D})(t+k\Delta t,\omega')-(\hat{C}+\hat{D})(t,\omega')),$$

where $x_i'=B(t+(i-1)\Delta t,\omega')$, ω' as above. We first observe that

$$\begin{aligned}\sqrt{2\Delta t}\,\textstyle\sum_{i=1}^k L_{x_i'}s &= k\sqrt{2\Delta t}\,L_{x_i'}s+\sqrt{2\Delta t}\,\textstyle\sum_{i=1}^k (L_{x_i'}-L_{x_i'})s\\ &= k\sqrt{2\Delta t}\,L_{x_i'}+k\sqrt{2\Delta t}\cdot a,\quad a\approx 0\end{aligned}$$

because $x_i'\approx x_i'$, and hence $(L_{x_i'}-L_{x_i'})s\approx 0$, as $k\sqrt{2\Delta t}\approx 0$. Next observe that

$$|(\hat{C}+\hat{D})(t+kt,\omega')-(\hat{C}+\hat{D})(t,\omega')| \leqq \tfrac{1}{2}\Delta t\cdot k\cdot a' = k\sqrt{2\Delta t}\,a'',$$

where a' is finite and $a''\approx 0$. Hence for $h=k\sqrt{2\Delta t}\approx 0$ we have

$$\left|\frac{1}{h}(F(x'')-F(x_1'))-L_{x_i'}s\right| \approx 0.$$

Given standard $\varepsilon>0$, overflow gives a standard $\delta>0$ which satisfies for all $s\in S$,

$$\left|\frac{1}{h}F(x'') = F(x_1')-L_{x_i'}s\right| < \varepsilon$$

as $h=k\sqrt{2\Delta t}<\delta$.

Next consider $x_2\in G'$ with $|x_2-x_1|=h<\delta$. Assume that h is small enough for the line segment joining x_1 and x_2 to lie in G'. There are s and k with

$$x_2-x_1 \approx k\sqrt{2\Delta t}s,$$

Then $L_{x_1'}s \approx (1/h) L_{x_1}(x_2 - x_1)$. Because $\varphi(x_1) = {}^0F(x_1')$ and $\varphi(x_2) = {}^0F(x_1' + k\sqrt{2\Delta ts})$ and h is standard, we obtain

$$\left| \frac{1}{h}(\varphi(x_2) - \varphi(x_1)) - L_{x_1}(x_2 - x_1) \right| < \varepsilon.$$

Hence φ is differentiable and (ii) holds.

4.3. Remark. If $\hat{L}$ is the representation $B_\varphi + C_\varphi$, then the mapping $\varphi_{\hat{L}}$ constructed in the proof of Theorem 4.2 is the original φ.

Part (ii) of Theorem 4.2 implies that to $\hat{L}$ and $\varphi_{\hat{L}}$ corresponds a representation $\hat{I}G$ and $\hat{D}G$ of G and ∂G. They can be defined in terms of $\hat{B} + \hat{C}$ exactly like $I_\varphi G$ and $D_\varphi G$ were defined before. Actually,

$$I_{\varphi_{\hat{L}}} G = \hat{I}G \quad \text{and} \quad D_{\varphi_{\hat{L}}} G = \hat{D}G.$$

In a sense, $\hat{I}G$ and $\hat{D}G$ are the essence of the elliptic structure $\hat{L}$.

The following is a continuous analogy of our notion of an elliptic structure.

A stochastic process (with almost all paths continuous) moving in G is called *elliptic* if it has the property of b_φ stated in Corollary 3.2 for some continuously differential family of linear mappings (i.e., ellipses). Moreover, our elliptic process is *simple* if it is defined on $\mathbf{R}^+ \times \Omega$ where Ω is as in Chapter 1. (See Keisler [3] for a discussion of this kind of restrictions.)

Likewise, a continuous function $\hat{u}\colon G \to \mathbf{R}$ is *elliptic,* if it has the property stated in Theorem 3.1 for some continuously differentiable family of linear mappings.

An elliptic process or function is *K-quasiconformal,* if the eccentricities of all the corresponding ellipses are between $1/K$ and K. If all the ellipses are circles, then the elliptic process or function is *conformal.* In this sense, Brownian motion and harmonic functions are conformal.

4.4. Example. If $\hat{L}$ is an elliptic structure on G, then

$$\hat{b} = {}^0(\hat{B} + \hat{C} + \hat{D})$$

is an elliptic process. Elliptic functions $\hat{u}$ can be defined in terms of expectations of the values of continuous functions in ∂G at the exit points of $\hat{b}$ from G. The same result is achieved by use of the hyperfinite random walk $\hat{B} + \hat{C}$ and a corresponding hyperfinite lifting. Clearly $\hat{b}$ and $\hat{u}$ are K-quasiconformal or conformal, if $\hat{L}$ is.

On the other hand, the family of linear functions corresponding to an elliptic process or function gives rise to something very much like an elliptic structure. But it is possible that condition (iv) fails to hold.

We say that $\hat{b}$ or $\hat{u}$ is *unambiguous,* if condition (iv) holds, too.

Before putting all our results together, we make the following observation.

4.5. Theorem. *If $\hat{L}$ is an elliptic structure on G, then there is a domain G'' for which $\hat{L}$ is G''-elliptic.*

Proof. Denote

$$G'' = \operatorname{int}\{{}^0B(t,\omega)\,|\,(t,\omega)\in\operatorname{dom}\hat{B} \text{ and } {}^0\hat{B}(t,\omega)+{}^0\hat{C}(t,\omega)+{}^0\hat{D}(t,\omega)\in G\}.$$

Our aim is to show that $\hat{B}$ is G''-elliptic. It suffices to show that actually ${}^0B(t,\omega)\in G''$ whenever ${}^0(\hat{B}+\hat{C}+\hat{D})(t,\omega)\in G$. Assume ${}^0(\hat{B}+\hat{C}+\hat{D})(t,\omega)\in G$. Then there is a standard $r>0$ with $z\in G$ for $|(\hat{B}+\hat{C}+\hat{D})(t,\omega)-z|<r$. Consider the points $B(t+t',\omega_s)$ where $\omega_s(t')=\omega(t)$ for $t'<t$ and $\omega_s(t+t')=s$ for some $s\in S$. There is some t_s with $t_s\sqrt{2\Delta t}\not\approx 0$ for which $|(\hat{B}+\hat{C}+\hat{D})(t+t_s,\omega_s)-(\hat{B}+\hat{C}+\hat{D})(t,\omega)|<r$. Let

$$r' = {}^0\min_{s\in S} t_s\sqrt{2\Delta t}.$$

It follows that every x with $|x-{}^0B(t,\omega)|<r'$ is of the form ${}^0B(t+t',\omega_s)$ for some $s\in S$ and $t'<t_s$.

Now we have the following representation theorem.

4.6. Corollary. (i) *If $\hat{L}$ is elliptic on G, then there is a domain G'' and a C^2-mapping $\varphi\colon G''\to G$ onto with properties* (i)—(ii) *of* 4.2. *Moreover, φ is one to one if the implication from right to left holds in condition* (iv) *of the definition of the notion of an elliptic structure.*

(ii) *If $\hat{b}$ is simple, elliptic and unambiguous, then $\hat{b}=b_\varphi$ for a C^2-mapping φ.*

(iii) *If $\hat{u}$ is elliptic and unambiguous and if $\hat{L}$ satisfies the implication from right to left in condition* (iv) *of the definition of an elliptic structure, then there is a domain G'' and a C^2-mapping $\varphi\colon G''\to G$ one to one onto and a harmonic mapping $v\colon G''\to\mathbf{R}$ with $\hat{u}\colon v\circ\varphi^{-1}$.*

In addition, in all these statements, φ is K-quasiconformal or conformal if $\hat{L}$, $\hat{b}$ or $\hat{u}$ is K-quasiconformal or conformal.

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Received 16 August 1985