THE COMPOSITION OF HARMONIC MAPPINGS

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0. Introduction

The function \( f(z) \) is said to be a harmonic mapping if \( f(z) \) is complex-valued harmonic:

\[
f_{zz} = 0.
\]

Special properties of two-dimensional harmonic vectors were first considered by Kneser and Radó [3]. There is now a large literature on the subject.\(^2\) Analytic functions and affine mappings, \( f(z) = \mu z + v \bar{z}, \) (\( \mu, v \) complex constants) constitute the simplest examples. Evidently, \( f(z) \) is a harmonic mapping if and only if

\[
f(z) = A(z) + B(z),
\]

where \( A(z), B(z) \) are analytic.

The current contribution is devoted to some elementary facts about compositions which seem to have escaped attention. It is easy to verify that if \( f \) and \( g \) are harmonic mappings, with domain \( g \supset \text{range} f, \) then \( g \circ f \) is not "in general" harmonic. Trivial exceptions occur when \( f \) (or its conjugate) is analytic and \( g \) an arbitrary harmonic mapping, and when \( f \) is an arbitrary harmonic mapping and \( g \) is affine. That, however, there also exist non-trivial exceptions follows from an example given by Choquet [1] for which

\[
(0.1) \quad g(f(z)) = z,
\]

even though neither \( f \) nor \( g \) are analytic, anti-analytic, or affine. Choquet [1, pp.164—165] credits J. Deny with proving that Choquet's example of harmonic mappings \( f, g \) satisfying (0.1) is essentially unique.

Our object (Theorem 1 and its corollary) is to obtain local descriptions of all harmonic mappings \( f \) with the property that \( g \circ f \) is harmonic for some non-affine harmonic \( g. \) The descriptions are local in the sense that they concern sufficiently

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\(^1\) Work done with support from National Science Foundation Grants MCS—8300248 and INT—8414734.

\(^2\) See [2] for a recent bibliography.
small simply-connected neighborhoods of points where whatever non-constant analytic functions are involved are non-zero. It is perhaps surprising that very explicit descriptions can be obtained. For completeness we have also included a proof of the Choquet—Deny theorem, as [1] does not contain one.

1. Characterization of harmonic decompositions

We can assume, without loss of generality, that \( f(z) \) has the form

\[
(1.1) \quad f(z) = z + B(z),
\]

where \( B(z) \) is analytic, and that \( g(w) \) has the form,

\[
(1.2) \quad g(w) = C(w) + D(w),
\]

where \( C(w), D(w) \) are analytic functions of \( w \). The most general harmonic map has the form \( f \circ \Phi \), where \( \Phi \) is analytic, and \( f \) has form (1.1). The problem is to characterize those functions \( B(z) \) for which there exist \( C(w), D(w) \), with \( C''(w), D''(w) \) not both vanishing, such that \( g \circ f \circ \Phi \) is harmonic. Whether or not \( B \) has this property is of course independent of \( \Phi \).

Let

\[
G(z) = B'(z).
\]

Calculating the laplacian of \( g(f(z)) \) we find that a necessary and sufficient condition that \( g(f(z)) \) is harmonic is that

\[
(1.3) \quad \overline{G(z)} C''(w) = -G(z) \overline{D''(w)} \quad (w = f(z) = z + B(z)).
\]

Our basic problem is to characterize the solutions of this functional equation. One solution is obtained when \( G(z), C''(w), \) and \( D''(w) \) are appropriately related constants.

From now on assume that \( G(z) \) is not a constant\(^8\). Since (1.3) implies that \( |C''(w)| = |D''(w)| \), a necessary condition is that \( D''(w) = \exp(is) C''(w) \) for some real constant \( s \), or

\[
(1.4) \quad \overline{G(z)} \Psi(w) = G(z) \overline{\Psi(w)}, \quad (w = f(z) = z + B(z)),
\]

where

\[
C''(w) = -ie^{-i(s/2)} \Psi(w), \quad D''(w) = -ie^{i(s/2)} \Psi(w).
\]

The condition that there exists a non-zero analytic function \( \Psi(w) \) satisfying (1.4) is necessary and sufficient on \( B(z) \).

\(^8\) In that case there exists a neighborhood in which \( f \) is one-to-one.
To transform (1.4) into a condition involving only $G$, let

$$(1.5) \quad \gamma(w) = G(f^{-1}(w)), \quad (w = f(z) = z + B(z)).$$

We observe the following: A necessary and sufficient condition that there exists a non-vanishing analytic function $\Psi(w)$ satisfying (1.4) is that

$$(1.6) \quad R(w) = \frac{\gamma_{w\bar{w}}}{\gamma} - \frac{\gamma_w\gamma_{\bar{w}}}{\bar{\gamma}^2} = \text{real function of } w.$$ 

The condition (1.6) is necessary, because, by (1.4),

$$\log \gamma(w) + \log \Psi(w) = \log \gamma(w) + \log \overline{\Psi(w)},$$

and if we operate with $\partial^2/\partial\overline{w}\partial w$ on both sides we obtain $R(w) = \overline{R(w)}$. On the other hand, if (1.5) holds then

$$\frac{\partial}{\partial\overline{w}} \left[ \frac{\gamma_w}{\gamma} - \frac{\gamma_{\bar{w}}}{\bar{\gamma}} \right] = 0.$$ 

Therefore,

$$\frac{\partial}{\partial\overline{w}} [\log \gamma - \log \overline{\gamma}] = F_1(w) = \text{analytic function of } w,$$

and, hence,

$$\log \gamma - \log \overline{\gamma} = \int F_1(w) \, dw - F_2(w),$$

where $F_2(w)$ is also an analytic function of $w$. Thus,

$$\frac{\gamma(w)}{\gamma(w)} = \frac{L(w)}{M(w)}, \quad \text{with} \quad L(w) = \exp \int F_1(w) \, dw, \quad M(w) = \exp F_2(w).$$ 

Since it follows that $|L(w)| = |M(w)|$, we have $M(w) = \exp(it)L(w)$ for some real constant $t$. Setting

$$\Psi(w) = e^{it/2}L(w),$$

we obtain $\gamma(w)\overline{\Psi(w)} = \gamma(w)\Psi(w)$. This shows that (1.6) is also sufficient for (1.4).

The next step is to employ (1.5) to compute $R(f(z))$, making use of the relations

$$(1.7) \quad \frac{\partial z}{\partial w} = \frac{f_z}{J} = \frac{1}{J}, \quad \frac{\partial z}{\partial\overline{w}} = -\frac{f_{\overline{z}}}{J} = -\frac{\bar{G}}{J},$$

$$J = |f_z|^2 - |f_{\overline{z}}|^2 = 1 - |G|^2, \quad G = B'.$$

We find that

$$\gamma_w \circ f = G' / J, \quad \gamma_{\overline{w}} \circ f = -\bar{G}G' / J,$$

$$\gamma_{w\bar{w}} \circ f = (-\bar{G}G'' + G|G'|^2 - J\bar{G}G') / J^3.$$ 

Therefore,

$$(1.8) \quad J^3 R(f(z)) = -\frac{\bar{G}G''}{G} + |G'|^2 - J \frac{\bar{G}G''}{G} + J \frac{\bar{G}G''}{G^2}.$$
This suggests defining
\[ R_1(z) = \frac{J^3 R(f(z)) - |G'|^2}{|G|^2}, \]
which is real if and only if \( R(f(z)) \) is real. By (1.8),
\[ (1.9) \quad R_1(z) = S(z) - T(z) \overline{G(z)}, \]
where \( S, T \) are the following analytic functions:
\[ (1.10) \quad S(z) = \frac{G'(z)^2 - G(z) G''(z)}{G(z)^3}, \quad T(z) = \frac{2G'(z)^2 - G(z) G''(z)}{G(z)^3}. \]
Since \( R_1 \) is real we have
\[ (1.11) \quad S - \overline{S} = \overline{G} T - G \overline{T}, \]
and this condition is equivalent to the basic requirement (1.6). Applying \( \partial^2 / \partial \overline{z} \partial z \) to both sides, one sees that (1.11) holds if and only if
\[ (1.12) \quad S(z) = a - \bar{a} G(z), \quad T(z) = c G(z) + \bar{c}, \]
where \( a \) and \( c \) are real constants and \( \bar{c} \) is a complex constant.

Relations (1.10) and (1.12) yield the pair of simultaneous differential equations
\[ G'^2 - G G'' = a G^3 - \bar{a} G^4, \quad 2G'^2 - G G'' = c G^3 + \bar{c} G^4, \]
which constitute necessary and sufficient conditions on \( G \). Subtracting the first equation from the second we obtain the equivalent pair
\[ (1.13) \quad G'^2 = \bar{a} G^4 + (c - a) G^3 + \bar{c} G^2, \quad G'^2 - G G'' = a G^3 - \bar{a} G^4. \]
The key to the situation is that the two equations (1.13) are highly dependent. The first implies
\[ \frac{d}{dz} \left( \frac{G'}{G} \right)^2 = [2\bar{a} G + (c - a)] G'. \]
On the other hand, the second equation of (1.13) implies that
\[ \frac{d}{dz} \left( \frac{G'}{G} \right)^2 = \frac{2G'}{G} (\bar{a} G^3 - a G) = 2(\bar{a} G - a) G'. \]
So we must have \( a = -c \). Conversely, if we set \( a = -c \), then we see, as above, that the first of equations (1.13) implies the second.

We note that if \( \bar{c} = c = a = 0 \), then (1.13) also holds when \( G \) is constant. Thus we have proved the following:

**Theorem 1.** Suppose \( f(z) = z + B(z) \), \( G(z) = B'(z) \). A necessary and sufficient condition that there locally exists a non-affine complex harmonic function \( g(w) \), such
that \( g(f(z)) \) is harmonic is that \( G(z) \) satisfies

\[
G'{}^2 = \alpha^2 G^4 + 2cG^3 + \bar{\alpha}^2 G^2
\]

for some complex constant \( \alpha \) and some real constant \( c \).

\( G(z) \) and \( B(z) \) can be expressed in terms of elementary functions. We distinguish the following separate cases; the functions \( G(z) \) are given up to a translation of the \( z \) plane.

(I) \[ \alpha = c = 0, \quad G(z) = \text{const.} \]

(II) \[ \alpha = 0, \quad c \neq 0, \quad G(z) = \frac{2}{cz^2}. \]

(III) \[ \alpha \neq 0, \quad c = r|\alpha|^2, \quad (r = \text{real const}, \quad r^2 \neq 0, 1), \]

\[
G(z) = -\frac{\bar{\alpha}/\alpha}{r + \sqrt{1 - r^2 \sinh (\bar{\alpha}z)}}.
\]

(IV) \[ \alpha \neq 0, \quad c = 0, \quad G(z) = -\frac{\bar{\alpha}/\alpha}{\sinh (\bar{\alpha}z)}. \]

(V) \[ \alpha \neq 0, \quad c = |\alpha|^2, \quad G(z) = -\frac{\bar{\alpha}}{\alpha} \frac{e^{\bar{\alpha}z}}{e^{\alpha z} - 1}. \]

The above expressions are obtained by integrating the differential equation (1.14) and solving for \( G(z) \). Integrating the functions \( G(z) \), the following classification of harmonic mappings \( f(z) \) is obtained.

**Corollary.** Suppose \( f(z) \) is complex harmonic. A necessary and sufficient condition that there locally exists a non-affine complex harmonic function \( g(w) \), such that \( g(f(z)) \) is harmonic, is that \( f(z) \) is one of the following types, where \( \nu \) is a complex constant, and \( r \) a real constant.

- **Type 0:** \( f(z) = \Phi(z) \), where \( \Phi(z) \) is analytic.
- **Type 1:** \( f(z) = \Phi(z) + \nu\Phi(z), \quad \nu \neq 0. \)
- **Type 2:** \( f(z) = \Phi(z) + \frac{r}{\Phi(z)}, \quad r \neq 0. \)
- **Type 3:** \( f(z) = \Phi(z) + \nu \tanh^{-1} \left( \frac{1}{\sqrt{1 - r^2}} - r \tanh \frac{\Phi(z)}{\nu} \right), \quad r^2 \neq 0, 1. \)
- **Type 4:** \( f(z) = \Phi(z) - \nu \log \tanh \frac{\Phi(z)}{2\nu}. \)
- **Type 5:** \( f(z) = \Phi(z) - \nu \log \left[ \exp \left( \frac{\Phi(z)}{\nu} \right) - 1 \right]. \)

\(^4\) Up to translations and conjugation of \( z \) and \( f(z) \).
The associated non-affine harmonic mappings \( g(w) \) can be determined in principle by integrating (1.6). When \( f(z) \) is of type 1, there is an especially simple solution; namely (up to an additive affine function)

\[
g(w) = \bar{v}w^2 - w\bar{w}^2.
\]

2. Harmonic decompositions of the identity

Choquet—Deny Theorem. Suppose \( f \) is a sense-preserving harmonic homeomorphism and is neither analytic nor affine. A necessary and sufficient condition that \( f^{-1} \) is also harmonic is that

\[
f(z) = \frac{\sigma}{\bar{\alpha}} z + \frac{1}{\bar{\alpha}} \log \left[ \frac{\mu - e^{-\sigma z}}{\mu - e^{-\alpha z}} \right] + \text{const},
\]

where \( \sigma, \alpha, \mu \) are non-zero complex constants, \( |\mu| > \sup_z |e^{-\alpha z}| \).

Proof. While one could proceed to test each of the types 1 to 5 of the corollary of Theorem 1 to see which fulfilled the requirement that \( f^{-1} \) is also one of these types, it is more efficient to take advantage of the symmetry of the relation

\[
g(f(z)) = z,
\]

by means of an independent proof. Assuming

\[
w = f(z) = A(z) + B(z),
\]

where \( A(z), B(z) \) are analytic, and where

\[
J = |A'|^2 - |B'|^2 > 0, \quad B' \neq 0,
\]

in the domain of definition of \( f \), we have, in place of (1.7),

\[
g_w = \frac{A'}{J}, \quad g_w = -\frac{B'}{J}.
\]

By (2.3), (2.4), (2.5) one finds that \( g_{ww} = 0 \) if and only if

\[
A'B'\overline{A''} - |A'|^2 B'\overline{B''} = \overline{A''} - \overline{B''} B' \overline{B''}.
\]

Dividing by \( |A'|^2 |B'|^2 \), (2.6) becomes

\[
\frac{A''}{A'} - \frac{B''}{B'} = \frac{\alpha A' - \beta B'}{A'B'}, \quad \text{where} \quad \alpha(z) = \frac{A''}{A'B'}, \quad \beta(z) = \frac{B''}{A'B'},
\]

or, equivalently,

\[
(\bar{\alpha} + \beta) B' = (\alpha + \beta) A'.
\]

By (2.4), therefore,

\[
\beta = -\bar{\alpha}.
\]
Since $\alpha, \beta$ are both analytic, they must therefore both be constants,
\begin{equation}
A'' = \alpha A' B', \quad B'' = -\bar{\alpha} A' B'.
\end{equation}
Since affine $f$ are excluded, $\alpha$ cannot be 0. Eliminating $B$ between the differential equations (2.8),

$$
\frac{B''}{B'} = -\bar{\alpha} A' = \frac{\alpha A'B''}{A''} = \frac{A'A''-A''^2}{A''} = \frac{A''}{A'} - A''.
$$

Integrating, we obtain
\begin{equation}
A' = \text{const } e^{-\bar{\alpha} z} + \text{const}.
\end{equation}
This is a separable differential equation. For the solution we have
\begin{equation}
A(z) = \frac{\sigma}{\bar{\alpha}} z + \frac{1}{\bar{\alpha}} \log (\mu - e^{-\sigma z}) + \text{const},
\end{equation}
\begin{equation}
A'(z) = \frac{\mu \sigma}{\bar{\alpha}} (\mu - e^{-\sigma z})^{-1}, \quad A''(z) = -\frac{\mu \sigma^2}{\bar{\alpha}} (\mu - e^{-\sigma z})^{-2} e^{-\sigma z}.
\end{equation}

Therefore, by the first equation of (2.8),
\begin{equation}
B'(z) = -\frac{\sigma}{\alpha} \frac{e^{-\sigma z}}{\mu - e^{-\sigma z}}
\end{equation}
and therefore,
\begin{equation}
B(z) = -\frac{1}{\alpha} \log (\mu - e^{-\sigma z}) + \text{const}.
\end{equation}
Conversely, if (2.9), (2.10) hold then (2.7) is satisfied.

The restriction on $\mu$ in the statement of the theorem is to insure that $J > 0$.

We note that the harmonic mapping $f$ of the Choquet—Deny Theorem falls within type 5 of Section 1.

References


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Received 26 November 1985