CONVEX INCREASING FUNCTIONS PRESERVE
THE SUB-F-EXTREMALITY

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1. Introduction

Suppose that \( u: G \to (a, b) \) is a subharmonic function and that \( f: (a, b) \to \mathbb{R} \) is an increasing convex function. It is relatively easy to check that \( v = f \circ u \) satisfies the mean-value inequality and hence \( v \) is again subharmonic in \( G \). The purpose of this note is to give a generalization of this fact in a non-linear potential theory developed by S. Granlund, P. Lindqvist and O. Martio in [GLM 1—3]; see also [LM 1—2].

1.1. Theorem. Suppose that \( G \) is an open set in \( \mathbb{R}^n \) and that \( u: G \to (a, b) \) is a sub-F-extremal in \( G \). If \( f: (a, b) \to \mathbb{R} \) is an increasing convex function, then \( f \circ u \) is a sub-F-extremal in \( G \).

The cases \( a = -\infty \) and \( b = +\infty \) are not excluded.

In the classical potential theory Theorem 1.1 is due to M. Brelot [B, p. 16]. He makes use of an approximation method. There is another proof for this result, based on integral means, in the book of T. Radó [R, p. 15]; see also [HK, p. 46]. However, such a method is not available in our case. In a non-linear potential theory P. Lindqvist and O. Martio have recently proved a special case of Theorem 1.1 in [LM 2].

2. Sub-F-extremals

Suppose that \( G \) is an open set in \( \mathbb{R}^n \) and that \( F: G \times \mathbb{R}^n \to \mathbb{R} \) satisfies the following assumptions:

(a) For each open set \( D \subset \subset G \) and \( \varepsilon > 0 \), there is a compact set \( C \subset D \) with \( m(D \setminus C) < \varepsilon \) and \( F|C \times \mathbb{R}^n \) is continuous.

(b) For a.e. \( x \in G \) the function \( h \mapsto F(x, h) \) is strictly convex and differentiable in \( \mathbb{R}^n \).

(c) There are \( 0 < \alpha \leq \beta < \infty \) such that for a.e. \( x \in G \)

\[ \alpha |h|^n \leq F(x, h) \leq \beta |h|^n, \quad h \in \mathbb{R}^n. \]
(d) For a.e. \( x \in G \)

\[
F(x, \lambda h) = |\lambda|^n F(x, h), \quad \lambda \in \mathbb{R}, \quad h \in \mathbb{R}.
\]

An alternative characterization of the functions \( F \) satisfying (a)–(d) is given in [K]. Note that the exponent \( n \) in (c) and (d) is essential for applications in conformal geometry; cf. [GLM 1–2]. It is not essential for Theorem 1.1; see Remark 4.4.

A function \( u \in C(G) \cap \text{loc} W^1_n(G) \), i.e., \( u \) is ACL\(^n\) in \( G \), is called an \( F \)-extremal in \( G \) if for all domains \( D \subset G \)

\[
I_F(u, D) = \inf_{v \in F_u} I_F(v, D),
\]

where

\[
I_F(v, D) = \int_D F(x, \nabla v(x)) \, dm(x)
\]

is the variational integral with the kernel \( F \) and

\[
F_u = \{v \in C(\overline{D}) \cap W^1_n(D) \mid v = u \text{ in } \partial D\}.
\]

A function \( u \) is an \( F \)-extremal if and only if \( u \in C(G) \cap \text{loc} W^1_n(G) \) is a weak solution of the Euler equation

\[
\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0.
\]

An upper semi-continuous function \( u : G \to \mathbb{R} \cup \{-\infty\} \) is called a sub-\( F \)-extremal if \( u \) satisfies the \( F \)-comparison principle in \( G \), i.e., if \( D \subset G \) is a domain and \( h \in C(\overline{D}) \) is an \( F \)-extremal in \( D \), then \( h \geq u \) in \( \partial D \) implies \( h \geq u \) in \( D \). A function \( u : G \to \mathbb{R} \cup \{\infty\} \) is a super-\( F \)-extremal if \( -u \) is a sub-\( F \)-extremal. For basic properties of \( F \)-extremals and sub-\( F \)-extremals we refer to [GLM 2].

A sub-\( F \)-extremal \( u : G \to \mathbb{R} \) is called a regular sub-\( F \)-extremal if \( u \) is ACL\(^n\), i.e., \( u \in C(G) \cap \text{loc} W^1_n(G) \). Note that it follows from [GLM 3, Theorem 4.1] that a sub-\( F \)-extremal \( u : G \to \mathbb{R} \) is regular if \( u \in C(G) \). In what follows we shall make use of the following lemma.

2.1. Lemma. Let \( u \) be an ACL\(^n\)-function in \( G \). Then \( u \) is a regular sub-\( F \)-extremal in \( G \) if and only if for all non-negative \( \varphi \in C_0(G) \cap W^1_{n,0}(G) \)

\[
\int_G \nabla_h F(x, \nabla u) \cdot \nabla \varphi \, dm \equiv 0.
\]

Proof. The result follows from [GLM 2, Theorem 5.17] via an easy approximation procedure.

2.2. Lemma. Suppose that \( u : G \to \mathbb{R} \cup \{-\infty\} \) is a sub-\( F \)-extremal in \( G \) and that \( D \subset \subset G \) is a domain. Then there exists a decreasing sequence of regular sub-\( F \)-extremals \( u_i \) in \( D \) which are continuous in \( \overline{D} \) such that \( \lim_{i \to \infty} u_i = u \) in \( D \). Moreover, for each \( \varepsilon > 0 \) the sequence \( u_i \) can be chosen such that \( \sup_D u_i \equiv \sup_D u + \varepsilon \).

Proof. This follows from [GLM 3, Section 4].
2.3. Lemma. Suppose \( u : G \to \mathbb{R} \cup \{-\infty\} \) to be a function such that for each domain \( D \subset\subset G \) there exist a decreasing sequence of sub-F-extremals \( u_i \) in \( D \) with \( \lim_{i \to \infty} u_i = u \) in \( D \). Then \( u \) is a sub-F-extremal in \( G \).

Proof. Clearly \( u \) is upper semi-continuous in \( G \). Let \( D \subset\subset G \) be a domain and \( h \in C(\overline{D}) \) an F-extremal in \( D \) such that \( h \geq u \) in \( \partial D \). Choose a domain \( D' \subset\subset G \) and a decreasing sequence \( u_i \) of sub-F-extremals in \( D' \) such that \( D \subset\subset D' \) and \( \lim_{i \to \infty} u_i = u \) in \( D' \). Fix \( \varepsilon > 0 \). Since for every \( x \in \partial D \) there is \( i_x \in \mathbb{N} \) such that

\[
\left| u_i(x) - h(x) \right| < \varepsilon,
\]

and the upper semi-continuity of \( u_i - h - \varepsilon \), together with the compactness of \( \partial D \), implies

\[
u_i \leq u_{i_0} \leq h + \varepsilon
\]

in \( \partial D \) for \( i \geq i_0 \). Thus the sub-F-extremality of \( u_i \) yields

\[
u_i \leq h + \varepsilon
\]

in \( D \) for \( i \geq i_0 \). Hence

\[
u \leq h + \varepsilon
\]

in \( D \). Letting \( \varepsilon \to 0 \) we obtain the desired result.

We close this section with the following lemma, leaving its easy proof to the reader.

2.4. Lemma. Suppose that \( u_i : G \to \mathbb{R} \cup \{-\infty\} \) is a sequence of sub-F-extremals in \( G \) with \( u_i \to u \) uniformly on compact subsets of \( G \). Then \( u \) is a sub-F-extremal in \( G \).

3. Approximation lemmas

Let \( a, b \in [-\infty, \infty) \). We say that a function \( f : (a, b) \to \mathbb{R} \) preserves the (regular) sub-F-extremality in \( G \) if for each domain \( D \subset\subset G \) \( f \circ u \) is a (regular) sub-F-extremal in \( D \) whenever \( u : D \to (a, b) \) is a (regular) sub-F-extremal.

3.1. Lemma. Let \( f : (a, b) \to \mathbb{R} \) be an increasing continuous function. Suppose that for each \( \delta > 0 \) there is a sequence of functions \( f_i : (a+\delta, b-\delta) \to \mathbb{R} \), \( f_i \) preserving the regular sub-F-extremality in \( G \) for each \( i \) and \( f_i \to f \) uniformly on compact subsets of \( (a+\delta, b-\delta) \). Then \( f \) preserves the sub-F-extremality in \( G \).

Proof. Let \( G' \subset G \) be a domain. Let \( u_0 : G' \to (a, b) \) be a sub-F-extremal. Clearly \( f \circ u_0 \) is upper semi-continuous in \( G' \). Let \( D \subset\subset G' \) be a domain, and write

\[
d_0 = \max_{x \in D} u_0(x) < b.
\]

Let \( 0 < 3\varepsilon < b - d_0 \). Then \( u = u_0 + \varepsilon \) is again a sub-F-extremal in \( G' \) and \( u(x) = \)
$u_0(x)+\varepsilon>a+\varepsilon/2$ in $G'$. By Lemma 2.2 we may choose a decreasing sequence of sub-$F$-extremals $u_i \in C(D)$ in $D$ such that $\lim_{i \to \infty} u_i = u$ in $D$ and that

$$\sup_{x \in D} u_i(x) \leq d_0 + 2\varepsilon < b - \varepsilon.$$ 

Observe that $u_i: D \to (a+\varepsilon/2, b-\varepsilon/2)$ is a regular sub-$F$-extremal in $D$ and choose a sequence $r_k$ such that $r_k \to f$ uniformly on compact subsets of $(a+\varepsilon/2, b-\varepsilon/2)$ and that $r_k \circ u_i$ is a sub-$F$-extremal in $D$ for each $k$ and $i$. Fix $i$. Then $r_k \circ u_i \to f \circ u_i$ uniformly on compact subsets of $D$ as $k \to \infty$. Now Lemma 2.4 implies that $f \circ u_i$ is a sub-$F$-extremal in $D$ and $f \circ u_i$ is a decreasing sequence of sub-$F$-extremals in $D$ which tends to $f \circ u$. Then $f \circ u = f \circ (u_0 + \varepsilon)$ is a sub-$F$-extremal in $D$. Letting $\varepsilon \to 0$ Lemma 2.3 yields that $f \circ u_0$ is a sub-$F$-extremal in $G'$.

Next we shall consider smooth convex functions $f$.

3.2. Lemma. Suppose $f: (a, b) \to \mathbb{R}$ to be a convex increasing function such that $f \in C^2(a, b)$. Then $f$ preserves the regular sub-$F$-extremality in $G$.

Proof. Let $D \subset G$ be a domain. Suppose that $u: D \to (a, b)$ is a regular sub-$F$-extremal. Clearly $v = f \circ u \in C(D) \cap \text{loc} W^1_1(D)$. Choose $\varphi \in C^\infty_0(D)$, $\varphi \equiv 0$. It suffices to show that

$$\int_D \nabla_h F(x, \nabla u) \cdot \nabla \varphi \, dm \leq 0;$$

see Lemma 2.1. Let $\psi(x) = f'(u(x))^{n-1} \varphi(x)$. Then $\psi \in C^1_0(D) \cap W^1_1(D)$ and $\psi \equiv 0$. Furthermore,

$$(3.3) \quad \nabla \psi(x) = (n-1)f''(u(x))f'(u(x))^{n-2} \varphi(x)\nabla u(x) + f'(u(x))^{n-1} \nabla \varphi(x)$$

a.e. in $D$. Since the homogeneity assumption (d) of $F$ implies for a.e. $x \in G$

$$\nabla_h F(x, \lambda h) = |\lambda|^{n-2} \lambda \nabla_h F(x, h)$$

for $\lambda \in \mathbb{R}$ and $h \in \mathbb{R}^n$, we obtain

$$(3.4) \quad \nabla_h F(x, \nabla v(x)) = \nabla_h F(x, f'(u(x))\nabla u(x)) = f'(u(x))^{n-1} \nabla_h F(x, \nabla u(x))$$

a.e. in $D$. Now (3.4) and (3.3), together with the regular sub-$F$-extremality of $u$, yield

$$0 \equiv \int_D \nabla_h F(x, \nabla u) \cdot \nabla \psi \, dm = (n-1) \int_D f''(u) f'(u)^{n-2} \varphi \nabla_h F(x, \nabla u) \cdot \nabla u \, dm$$

$$+ \int_D f'(u)^{n-1} \nabla_h F(x, \nabla u) \cdot \nabla \varphi \, dm \equiv \int_D \nabla_h F(x, \nabla u) \cdot \nabla \varphi \, dm,$$

since $\nabla_h F(x, \nabla u) \cdot \nabla u \equiv 0$ a.e. in $D$ by the assumptions (b) and (c).
4. Final steps

4.1. Proof for Theorem 1.1. We make a convolution approximation. Choose $\eta_j \in C_c^\infty(\mathbb{R})$, $\eta_j \equiv 0$ such that $\text{spt} \eta_j = [-1/j, 1/j]$ and $\int_\mathbb{R} \eta_j(y) \, dy = 1$. For $x \in (a + 1/j, b - 1/j)$ define

$$f_j(x) = \int_\mathbb{R} f(x - y) \eta_j(y) \, dy,$$

where $f(x - y) \eta_j(y) = 0$ if $y \notin \text{spt} \eta_j$. In the light of Lemma 3.2, $f_j$ preserves the regular sub-$F$-extremality in $G$. Hence Lemma 3.1 implies the desired result.

4.2. Remark. Suppose that $f: [a, b] \rightarrow [-\infty, \infty)$ is convex and increasing. If $f(a) = -\infty$, then $f(t) = -\infty$ for each $t \in [a, b)$ or $f(t) > -\infty$ for each $t \in (a, b]$. Thus we can use Theorem 1.1 to prove the following slight extension: Suppose that $f: [a, b] \rightarrow [-\infty, \infty)$ is convex and increasing and that $u: G \setminus \{-\infty\}$ is a sub-$F$-extremal in $G$ with $u(G) \subseteq [a, b]$. Then $f \circ u$ is a sub-$F$-extremal in $G$.

4.3. Remark. It is clear that $f \circ u$ is a super-$F$-extremal if $u$ is and if $f$ is concave and increasing. Also, it is evident that if $v$ is a sub-$F$-extremal and if $f$ is concave and decreasing, then $f \circ v$ is a super-$F$-extremal.

4.4. Remark. Theorem 1.1 holds also when the exponent $n$ in (c) and (d) is replaced by a more general constant $p \in (1, \infty)$. For $p \neq n$ the proof is similar to that above. Observe that the continuity of the solution to an obstacle problem, needed in the proofs of 2.1 and 2.2, is established in [MZ].

4.5. Corollary. Suppose that $u: G \rightarrow \mathbb{R}$ is an $F$-extremal in $G$ and $p \in [1, \infty)$. Then $|u|^p$ is a sub-$F$-extremal in $G$.

Proof. Note that $|u| = \max(u, -u)$ is a sub-$F$-extremal and $t \rightarrow t^p$, $t \geq 0$, is convex and increasing. The result follows from Remark 4.2.

4.6. Corollary. Suppose that $u$ is a non-negative super-$F$-extremal in $G$. Then $u^p$ is a super-$F$-extremal for $p \in [0, 1]$.

We close with a remark on quasiregular mappings, which have an important role in a non-linear potential theory; see [GLM 2, Sections 6 and 7]. Suppose that $G$ and $G'$ are domains in $\mathbb{R}^n$ and $f: G \rightarrow G'$ is a quasiregular mapping. Then

$$F(x, h) = \begin{cases} |f'(x)|^{-1} h^n & \text{if } J(x, f) \neq 0, \\ |h|^n & \text{otherwise,} \end{cases}$$

satisfies (a)–(d) in $G$; see [GLM 2, 6.4]. Here $J(x, f)$ stands for the Jacobian determinant of $f$ at $x$ and $A^*$ denotes the adjoint of the linear map $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Now log $|f|$ is a sub-$F$-extremal in $G$ [GLM 2, Theorem 7.10]. Thus, using the convex increasing function $e^{tp}$ for $p \geq 0$, we obtain

4.7. Corollary. Suppose that $f$ and $F$ are as above. Then $|f|^p$ is a sub-$F$-extremal in $G$ for $p \geq 0$. 


References


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