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ON MAXIMAL AND MINIMAL QUASISYMMETRIC FUNCTIONS ON AN INTERVAL

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1. Introduction

Let $k \geqq 1$. In recent papers [HH, H 1, H 2], W. Hayman and A. Hinkkanen have investigated the maximal distortion under a normalized k-quasisymmetric function f of the real line. They denote the family of all such functions, normalized by $f(-1)=-1$ and $f(1)=1$, by $N_0(k)$. Hayman and Hinkkanen have found bounds for the growth of $f \in N_0(k)$, and Hinkkanen has constructed piecewise linear functions $f_0 \in N_0(k)$ for which $f_0(x)=\max\{f(x) \mid f \in N_0(k)\}$ for infinitely many x.

In this note, we consider the related problem of the distortion under a k-quasisymmetric self-map of $[0, 1]$, i.e., under a function f: $[0, 1] \to [0, 1]$ satisfying $f(0)=0$, $f(1)=1$, and

$$(1) \qquad (f(x)-f(x-t))/k \leqq f(x+t)-f(x) \leqq k(f(x)-f(x-t))$$

for all x and t such that $[x-t, x+t] \subset [0, 1]$. We denote the family of all such functions by QS (k). In particular, we investigate whether the functions introduced by R. Salem and studied by K. Goldberg [S, G] are best possible majorants or minorants of the functions in QS (k). In the last section, we apply our results to the dilatation estimates of the Beurling—Ahlfors extensions. — In a sense, the distortion problem is a one-dimensional analogue of the problem of distortion under a K-quasiconformal self-map of the unit disc.

If $f \in N_0(k)$, then g, defined by $g(x)=(f(2x-1)+1)/2$ for $x \in [0, 1]$, is in QS (k). All functions in QS (k) are not restrictions of k-quasisymmetric functions of the real line. (See the remark after Proposition 3.) On the other hand, boundedness of the domain and range of the mappings rules out a behaviour which condition (1) allows for mappings of the real line:

Proposition 1. *If $f \in$ QS (k), then f is a homeomorphism.*

Proof. If $k>1$, a subtraction of the right and left hand sides of (1) shows that f is nondecreasing. It is easy to see that in this case f is continuous: let $c \in (0, 1)$ and assume $\lim_{x \to c^-} f(x)=a<b=\lim_{x \to c^+} f(x)$. Given an $\varepsilon<(b-a)/k$, there is a δ such that $f(x)<b+\varepsilon$ for $c<x<c+\delta$. Choose x and t so that $x-t<c<x<$

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$x+t<c+\delta$ to obtain a contradiction with (1). Similarly one can see that f is continuous at 0 and 1. Consequently, f is a homeomorphism.

This argument does not work if $k=1$ (cf. [G]; in fact, functions satisfying (1) but defined on an unbounded interval need not be monotonic). However, assuming $k=1$ and, say, $f(x)=a>x$ for an $x>1/2$, we obtain from (1) that for some n we have $0<y=2^n x-2^n+1<1/2$ and $f(y)=2^n a-2^n+1>y$. Since $f(y+p2^{-q}(x-y))=f(y)+p2^{-q}(f(x)-f(y))$ for all integers $q\geqq 1$ and $0\leqq p\leqq 2^q$, there is a $z<1/2$ such that $f(z)>1/2$. But then $f(2z)>1$, which is not possible. So, if $k=1$, then $f(x)=x$ for all x.

Beurling and Ahlfors [BA] were the first to ask for $M_k(x)=\max\{f(x)|f\in \mathrm{QS}(k)\}$ and $m_k(x)=\min\{f(x)|f\in \mathrm{QS}(k)\}$, since the computations needed for the estimation of the maximal dilatation of their extension of a quasisymmetric function to a quasiconformal self-map of the half-plane depended on upper and lower bounds of the integral

$$I=\int_0^1 f(x)\,dx$$

for $f\in \mathrm{QS}(k)$. However, the needed inequalities $1/(1+k)\leqq I\leqq k/(1+k)$ were easy to establish without direct knowledge of m_k or M_k ([BA, p. 137]; see [Le] for a slightly sharpened version).

Because f and g, where $g(x)=1-f(1-x)$, either both are in $\mathrm{QS}(k)$ or are not in $\mathrm{QS}(k)$, $m_k(x)=1-M_k(1-x)$ for all $x\in[0,1]$.

2. Salem's functions

K. Goldberg [G] observed the connection between m_k, M_k and the completely singular homeomorphisms of $[0,1]$ introduced by R. Salem [S]. Setting $\lambda=\lambda(k)=1/(1+k)$, $\mu=\mu(k)=k\lambda$, we define the upper Salem function $P=P_k$ inductively for points with a finite dyadic representation by $P(0)=0$, $P(1)=1$, and by

$$P((2j+1)/2^n)=\lambda P(j/2^{n-1})+\mu P((j+1)/2^{n-1}) \tag{2}$$

for $n=1,2,\ldots;\ j=0,1,\ldots,2^{n-1}-1$, and for the rest of $[0,1]$ by continuity. The lower Salem function p_k is similarly defined by exchanging λ and μ in (2), and $p_k(x)=1-P_k(1-x)$ for all x [G]. Since it follows easily from (1) that every $f\in \mathrm{QS}(k)$ satisfies

$$\mu f(a)+\lambda f(b)\leqq f((a+b)/2)\leqq \lambda f(a)+\mu f(b) \tag{3}$$

for $0\leqq a<b\leqq 1$, we see that

$$p_k(x)\leqq m_k(x),\quad M_k(x)\leqq P_k(x) \tag{4}$$

for all $x\in[0, 1]$. The examples

$$f(x) = \begin{cases} 4\mu^2 x, & x\in[0, 1/4] \\ \mu^2+4\mu\lambda(x-1/4), & x\in[1/4, 3/4] \\ \mu+\mu\lambda+4\lambda^2(x-3/4), & x\in[3/4, 1], \end{cases}$$

$$g(x) = \begin{cases} 4\lambda^2 x, & x\in[0, 1/4] \\ \lambda^2+4\mu\lambda(x-1/4), & x\in[1/4, 3/4] \\ \lambda+\mu\lambda+4\mu^2(x-3/4), & x\in[3/4, 1], \end{cases}$$

show that there are points x for which $M_k(x)=P_k(x)$ or $m_k(x)=p_k(x)$ (in the examples, $x=j/4, j=0, 1, 2, 3, 4$). On the other hand, if $k>1$, then $P=P_k\notin \mathrm{QS}\,(k')$ for any k', since

$$(P(1/2+1/2^n)-P(1/2))/(P(1/2)-P(1/2-1/2^n)) = k^{n-2}$$

for $n\geqq 1$.

A direct consequence of Goldberg's observation is

Proposition 2. *M_k and m_k are continuous and strictly increasing.*

Proof. Only the statements concerning M_k have to be proved. Clearly, M_k is non-decreasing. Assume, for instance, that $\lim_{x\to c^+} M_k(x)>M_k(c)$. Then there exist a $\delta>0$, and sequences (x_j), $x_j>c$, $\lim_{j\to\infty} x_j=c$, and (f_j), $f_j\in\mathrm{QS}\,(k)$ such that $f_j(x_j)> M_k(c)+\delta$. Set $g_j(t)=f_j(tx_j)/f_j(x_j)$. Then $g_j\in\mathrm{QS}\,(k)$, and $p_k(c/x_j)\leqq g_j(c/x_j)= f_j(c)/f_j(x_j)\leqq M_k(c)/(M_k(c)+\delta)<1$, in contradiction with $\lim_{j\to\infty} p_k(c/x_j)=1$.

Similarly, if $M_k(a)=M_k(b)$, $a<b$, we find (f_j), $f_j\in\mathrm{QS}\,(k)$, such that $f_j(a)> M_k(a)-1/j$. Define $g_j\in\mathrm{QS}\,(k)$ by $g_j(x)=(f_j(x+(1-x)a)-f_j(a))/(1-f_j(a))$. Then $g_j((b-a)/(1-a))\to 0$, in contradiction with (4) and the fact that $p_k(c)>0$ for $c>0$.

3. Piecewise linear quasisymmetric functions

We shall investigate the possibility of equality in (4); our previous example shows that equality is true for certain values of x at least.

An obvious device for proving an equality $M_k(x)=y$ is to construct a piecewise linear function $f\in\mathrm{QS}\,(k)$ satisfying $f(x)=y$.

The following lemma, which is proved in [HH, p. 64], facilitates the proof that a given piecewise linear function is in $\mathrm{QS}\,(k)$:

Lemma 1. *Let $S\subset[0, 1]$, $\{0, 1\}\subset S$, be a discrete set and $f\colon [0, 1]\to[0, 1]$, $f(0)=0$, $f(1)=1$, be continuous on $[0, 1]$ and linear on each interval in $[0, 1]\setminus S$. Then $f\in\mathrm{QS}\,(k)$ if and only if (1) is true for all x, t such that $\{x-t, x, x+t\}\cap S$ has at least two elements.*

Remark. Even if S is finite, the number of relevant intersections to be checked grows with the cardinality of S: in fact, if S has n elements, then there are at least $n^2/2-3n/2+1$ and at most $3n^2/2-11n/2+5$ intersections to be checked.

As a corollary to Lemma 1, we obtain

Lemma 2. *Let $f\in \mathrm{QS}(k)$ be piecewise linear and $f(1/2)=\mu$.*

(a) *If f is linear on $[1/2, 1]$, and $g(x)=\mu f(2x)$ for $x\in[0, 1/2]$, $g(x)=f(x)$ for $x\in[1/2, 1]$, then $g\in \mathrm{QS}(k)$.*

(b) *If f is linear on $[0, 1/2]$, and $g(x)=f(x)$ for $x\in[0, 1/2]$, $g(x)=\mu+\lambda f(2x-1)$ for $x\in[1/2, 1]$, then $g\in \mathrm{QS}(k)$.*

Proof. (a) For any homeomorphism h, we denote the quotient

$$\big(h(x+t)-h(x)\big)/\big(h(x)-h(x-t)\big)$$

by q_n. Denote by S_f (S_g) the set of points where f (g) is not linear. Choose x, t such that two of the points $x-t, x, x+t$ are in S_g. Then two of the points $2x-2t$, $2x$, $2x+2t$ are in S_f and $q_g(x, t)=q_f(2x, 2t)$, except when $x+t=1$. In this case $x=t=1/2$ and $q_g(x, t)=1/k$. The proof for (b) is similar.

If we apply Lemma 2 repeatedly to f, $f(x)=2\mu x$, $x\in[0, 1/2]$, $f(x)=\mu+2\lambda(x-1/2)$, $x\in[1/2, 1]$, and observe that $x\to 1-f(1-x)$ is in $\mathrm{QS}(k)$ together with f, we immediately get infinitely many points x at which $M_k(x)=P_k(x)$:

Proposition 3. *For every natural number n,* $M_k(1/2^n)=\mu^n=P_k(1/2^n)$,

$$M_k(1-1/2^n)=1-\lambda^n=P_k(1-1/2^n);\ m_k(1/2^n)=p_k(1/2^n)=\lambda^n,$$

$$m_k(1-1/2^n)=p_k(1-1/2^n)=1-\mu^n.$$

Remark. The construction above gives an example of functions $f\in \mathrm{QS}(k)$ with no k-quasisymmetric extension to the real line. Indeed, if $f(1/2^n)=\mu^n$, $f(1/2)=\mu$ and f is linear on $[1/2, 1]$, then a k-quasisymmetric extension of f would have

$$f(-1/2^n)\geqq -\mu^{n-1}\lambda,\quad f(1+1/2^n)\leqq 1+\mu/2^n,$$

and $q_f(1/2, 1/2+1/2^n)\leqq(1/k)\big(1+\mu/(\lambda 2^n)\big)/(1+\mu^{n-2}\lambda)<1/k$ for n large enough. (Since $\mu>1/2$, the numerator tends to 0 faster than the denominator.) — Of course, every $f\in \mathrm{QS}(k)$ has a k_1-quasisymmetric extension to $\mathbf{R}$, with $k_1>k$ depending on k only.

Further evidence supporting the hypothesis $P_k=M_k$ is obtained from the piecewise linear functions which are linear on intervals in $[0, 1]\setminus S$, $S=\{j/8\mid j=0, 1, \ldots, 8\}$, and agree with P_k or p_k on S. A direct check gives

Proposition 4. *For $j=0, 1, \ldots, 8$, $M_k(j/8)=P_k(j/8)$, $m_k(j/8)=p_k(j/8)$.*

4. The points 1/3 and 2/3

From (3), we obtain for $f \in \mathrm{QS}(k)$

$$\mu f(1/3)+\lambda \leqq f(2/3) \leqq \lambda f(1/3)+\mu,$$
$$\lambda f(2/3) \leqq f(1/3) \leqq \mu f(2/3).$$

Solving, we get

$$\lambda^2/(1-\mu\lambda) \leqq f(1/3) \leqq \mu^2/(1-\mu\lambda), \tag{5}$$

$$\lambda/(1-\mu\lambda) \leqq f(2/3) \leqq \mu/(1-\mu\lambda). \tag{6}$$

Using (2) and the representation

$$1/3 = 1/4+\sum_{n=2}^{\infty}(1/2^{2n-1}-1/2^{2n}),$$

we obtain

$$P_k(1/3) = \mu^2 \sum_{n=0}^{\infty}(\mu\lambda)^n = \mu^2/(1-\mu\lambda).$$

Similarly, the other three upper and lower bounds in (5) and (6) are the respective values of the upper and lower Salem functions.

However, the equality $M_k(1/3)=P_k(1/3)$ does not hold. In fact, setting $f(j/12)=a_j$ for brevity, we obtain from (3) that $a_4 \leqq \mu a_8$, $a_8 \leqq \lambda a_7+\mu a_9$ or $a_4/\mu-\mu a_9 \leqq \lambda a_7$ and $a_4 \leqq \lambda a_3+\mu a_5$. Since $\mu a_5+\lambda a_7 \leqq a_6$ we get

$$a_4-\lambda a_3+a_4/\mu-\mu a_9 \leqq a_6.$$

Inserting $a_3 \leqq M_k(1/4)=\mu^2$, $a_6 \leqq M_k(1/2)=\mu$, $a_9 \leqq M_k(3/4)=\mu+\mu\lambda$ we obtain

$$f(1/3) = a_4 \leqq \mu^2(1+\mu+2\mu\lambda)/(1+\mu). \tag{7}$$

It is easy to check that the right hand side in (7) is indeed strictly less than $P_k(1/3)$ as soon as $k>1$.

A similar argument improves the right hand side of (6) to

$$f(2/3) \leqq \mu(1+\lambda+\lambda\mu+2\mu\lambda^2)/(1+\lambda) < P_k(2/3). \tag{8}$$

Denote the upper bounds in (7) and (8) by $\varkappa=\varkappa(k)$ and $\nu=\nu(k)$, respectively. Using the fact that g, $g(x)=1-f(1-x)$, belongs to $\mathrm{QS}(k)$ together with f, we get the lower bounds $1-\nu \leqq f(1/3)$, $1-\varkappa \leqq f(2/3)$.

We observe the strict inequalities $\varkappa<\mu\nu$ and $\nu<\lambda\varkappa+\mu$. These imply that the simultaneous equations $q_f(x, t)=k$ and $q_f(x+t, t)=k$ or $q_f(x, t)=1/k$ and $q_f(x+t, t)=1/k$ cannot hold for any k-quasisymmetric function.

To prove that $M_k(1/3)=\varkappa$, $M_k(2/3)=\nu$, we construct piecewise linear functions $f, g \in \mathrm{QS}(k)$ such that $f(1/3)=\varkappa$, $g(2/3)=\nu$.

Proposition 5. $M_k(1/3)=1-m_k(2/3)=\varkappa$, $M_k(2/3)=1-m_k(1/3)=\nu$.

Proof. In view of the restrictions described above, any $f \in \mathrm{QS}(k)$ satisfying $f(1/3)=\varkappa$ must also satisfy $f(1/4)=\mu^2$, $f(1/2)=\mu$, $f(3/4)=\mu+\mu\lambda$, $f(2/3) \geqq \varkappa/\mu$,

$f(5/12)=\varkappa/\mu-\mu\lambda$, and $f(7/12)=(\varkappa/\mu-\mu^2)/\lambda-\mu^2$. In addition, (1) requires $f(5/6)\geqq f(5/12)/\mu$. Finally, (1) with $x=1/3$, $t=1/6$ requires $f(1/6)\geqq\varkappa-k(\mu-\varkappa)$, and with $x=1/3$, $t=1/4$ requires $f(1/12)\geqq(1+k)\varkappa-(k/\lambda)\big(\varkappa/\mu-\mu^2(1+\lambda)\big)$. Define a piecewise linear function f, linear in $[0,1]\setminus S$, $S=\{0, 1/12, ..., 10/12, 1\}$ by replacing all the inequality signs in the above conditions by equalities; a check by Lemma 1 shows that f is in $\mathrm{QS}(k)$. Similarly, a piecewise linear g satisfying $g(2/3)=\nu$, $g(1/6)=\mu^2/(1+\lambda)$, $g(1/4)=\mu^2$, $g(1/3)=\mu\nu$, $g(5/12)=\mu-\mu\lambda^2/(1+\lambda)$, $g(1/2)=\mu$, $g(7/12)=\mu+\mu^2\lambda/(1+\lambda)$, $g(3/4)=\mu(1+\lambda)$, $g(5/6)=\mu+\mu\lambda(1+2\lambda)/(1+\lambda)$, $g(11/12)=(1+\mu\lambda+\mu\lambda^2)/(1+\lambda)$ is in $\mathrm{QS}(k)$.

Corollary. *If $k>1$, the set of x for which $M_k(x)=P_k(x)$ is nowhere dense, and $M_k(x)<P_k(x)$ for infinitely many x with a finite dyadic representation.*

Proof. If $M_k(x)=P_k(x)$ on an interval, then $M_k(x)=P_k(x)$ on an interval whose endpoints are $m/2^n$ and $(m+1)/2^n$. But $M_k\big((m+1/3)/2^n\big)<P_k\big((m+1/3)/2^n\big)$.

Propositions 2 and 5 as well as the continuity of P_k also imply that there are intervals on which $M_k<P_k$. Majorants of M_k, sharper than P_k, can be constructed in a way similar to the construction of Salem's function but using both (3) and the inequalities $f(2a/3+b/3)\leqq(1-\varkappa)f(a)+\varkappa f(b)$, $f(a/3+2b/3)\leqq(1-\nu)f(a)+\nu f(b)$ on subintervals of $[0, 1]$.

By using Proposition 5, we can also easily prove

Proposition 6. *If $k>1$, M_k is not in* $\mathrm{QS}(k)$.

Proof. We show that no f which satisfies $f(1/3)=\varkappa$ and $f(2/3)=\nu$ is in $\mathrm{QS}(k)$. Assuming the contrary, we get $\varkappa\leqq\lambda f(1/4)+\mu f(5/12)$, $\nu\leqq\lambda f(7/12)+\mu f(3/4)$ and hence $\varkappa+\nu\leqq\mu^2\lambda+\mu^2(1+\lambda)+\mu f(5/12)+\lambda f(7/12)\leqq\mu+2\mu^2\lambda+\mu^2$. Inserting the formulas for $\varkappa$ and ν and simplifying, we get the contradiction $\mu\leqq 1/2$.

5. On the dilatation of the Beurling—Ahlfors extension

The Beurling—Ahlfors extension F of $f\in\mathrm{QS}(k)$ is defined in

$$T=\{z=x+iy|0\leqq x\leqq 1,\ 0<y\leqq\min\{x, 1-x\}\}$$

by $2F(z)=\alpha_0(z)+\alpha_1(z)+ir\big(\alpha_0(z)-\alpha_1(z)\big)$ where

$$\alpha_j(z)=\int_0^1 f\big(x+(-1)^j yt\big)\,dt,\quad j=0,1,\quad z=x+iy,$$

and $r>0$. By [BA] and [Le], r can be chosen so that the maximal dilatation K_F of F is at most $\min\{k^{3/2}, 2k-1\}$. On the other hand, for $k>12$ there are examples of k-quasisymmetric functions f defined on the whole real line for which every Beurling—Ahlfors extension has maximal dilatation at least $3k/2$ [Le] and for large k even larger than $1.587k$ [Li].

Restricting ourselves to $k>7$, we find functions $f\in\mathrm{QS}(k)$ for which $K_F>8k/5$ for every r. To this end, let $f(j/8)=P_k(j/8)$, $j=0, \ldots, 8$, and let f be linear on each interval $[j/8, (j+1)/8]$ (cf. Proposition 4). Passing to g, $g(x)=1-f((1-x)/2)/\mu$, and computing the dilatation D of the Beurling—Ahlfors extension of g at the point i as in [Le, p. 139], we arrive at $D>8k/5$ for $k>7$ and $\lim_{k\to\infty} D/k=1+49/64=1.765625$. Observe, however, that f has no k-quasisymmetric extension to the real line. Such an extension ought to satisfy $f(9/8)\leqq 1+\mu\lambda^2$ and also $f(5/4)\geqq f(5/8)/\mu=1+\mu\lambda$. Since $f(7/8)=1-\lambda^3$, this would imply that $f(5/4)-f(9/8)=k(f(9/8)-f(1))=k^2(f(1)-f(7/8))$, which clearly contradicts the remark preceding Proposition 5.

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