

ON ESTIMATING THE INVERSE COEFFICIENTS FOR MEROMORPHIC UNIVALENT FUNCTIONS OMITTING A DISC

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1. Introduction

In [2] and [3] the class

$$\Sigma_b = \{H \mid H(z) = z + \sum_0^\infty A_n z^{-n}, \quad |z| > 1, \quad |H(z)| > b \in (0, 1)\}$$

of meromorphic univalent functions was defined and considered. This class was found to have a one-to-one relationship with bounded univalent functions

$$S(b) = \{f \mid f(z) = b(z + \sum_2^\infty a_\nu z^\nu), \quad |z| < 1, \quad |f(z)| < 1, \quad b \in (0, 1)\}$$

so that

$$(1) \quad H(z) = \frac{b}{f\left(\frac{1}{z}\right)}, \quad |z| > 1.$$

Especially the inverse coefficients, i.e., those of the inverse function I of H , were estimated. Denote the inverse function of $w = H(z)$ by

$$z = I(w) = w + \sum_0^\infty E_n w^{-n}.$$

In [2] a FitzGerald—Launonen inequality was written for I . This appeared to be effective especially for the odd indexes. Thus for odd Σ_b -functions all the inverse coefficients were maximized. However, for general Σ_b -functions the FitzGerald—Launonen inequality gave results only for some initial odd inverse coefficients. Moreover, this method needed the maxima of corresponding initial even inverse coefficients. For estimating the latter the variational method appeared to be the most useful. However, technical difficulties prevented proceeding beyond $|E_4|$ in [3].

In the present paper the general result for inverse Σ_b -coefficients is derived by applying the variational method, which appears to work equally well for both odd and even indexes.

2. The variation of I

The relation (1) allows us to utilize the variational formulae true for $S(b)$ -functions. For brevity, adopt directly the notations and results of [4]. Thus (1), p. 137, yields the variational formula of $f \in S(b)$ (here the parameters A_v are not to be mixed with the coefficients A_v of H). Through (1) the variational formula for f is transformed into that for H and further for I :

$$(2) \quad I^*(w) = I(w) + I'(w)K \cdot \varepsilon + O(\varepsilon^2);$$

$$K = \sum_1^N \left\{ A_v \frac{w_v w}{w_v - w} + \bar{A}_v \frac{\bar{w}_v w^2}{b^2 - \bar{w}_v w} + A_v \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2 \frac{I(w)^2}{I'(w)(I(w) - I(w_v))} \right. \\ \left. + \bar{A}_v \left(\frac{\bar{w}_v I'(w_v)}{I(w_v)} \right)^2 \frac{I(w)}{I'(w)(I(w)\bar{I}(w_v) - 1)} \right\}.$$

The expansion form of (2) reads

$$\sum_0^\infty (E_n^* - E_n) w^{-n} = \varepsilon \sum_1^N \left\{ \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2 [A_v (w + (E_0 + I(w_v)) w^0 + \dots) \right. \right. \\ \left. \left. + \bar{A}_v (I(w_v)^{-1} w^0 + \dots)] + A_v (-w_v w^0 + \dots) + \bar{A}_v (-w - b^2 \bar{w}_v^{-1} w^0 + \dots) \right\} + O(\varepsilon^2).$$

On the right side the coefficient of w appears to be

$$(3) \quad \sum_1^N \left\{ A_v \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2 - \bar{A}_v \right\} = 0.$$

This is due to the fact that according to (4), p. 138, in [4]

$$b^* = b \left[1 + \varepsilon \sum_1^N \left\{ A_v \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2 - \bar{A}_v \right\} \right] + O(\varepsilon^2).$$

Hence, for preserving the class \sum_b , i.e., for keeping b constant in the variation, we must require that the parameters A_v are chosen in such a way that (3) holds.

By comparing the coefficients of the above series we obtain the variational formula for the E_v -coefficients. In order to express this in a useful form we introduce a suitable coefficient-notation, i.e., for any series

$$W(t) = \sum \kappa_v t^v$$

we denote the coefficients by the aid of W :

$$\kappa_v = [W(t)]_v.$$

Using this we can write

$$E_n^* - E_n = \varepsilon \sum_1^N (A_v X_v + \bar{A}_v Y_v + A_v Z_v + \bar{A}_v U_v) + O(\varepsilon^2), \quad n = 0, 1, 2, \dots;$$

$$X_v = \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2 \underset{1}{\left[\frac{I(w)}{1 - \frac{I(w_v)}{I(w)}} \right]}_{-n},$$

$$Y_v = \left(\frac{\overline{w_v I'(w_v)}}{I(w_v)} \right)^2 \underset{2}{\left[\frac{\frac{1}{\overline{I(w_v)}}}{1 - \frac{1}{\overline{I(w_v)I(w)}}} \right]}_{-n},$$

$$Z_v = \underset{3}{\left[\frac{-w_v I'(w)}{1 - \frac{w_v}{w}} \right]}_{-n},$$

$$U_v = \underset{4}{\left[\frac{-w I'(w)}{1 - \frac{b^2}{\bar{w}_v w}} \right]}_{-n}.$$

Applying the notation of [4] we write the variational formula as in (5), p. 138:

$$E_n^* = E_n + \varepsilon \sum_1^N (A_v H_v + \bar{A}_v K_v) + O(\varepsilon^2);$$

$$H_v = X_v + Z_v, \quad K_v = Y_v + U_v.$$

(8), p. 140, corresponds to (3) in the abbreviated form

$$\sum_1^N (A_v e_v - \bar{A}_v) = 0; \quad e_v = \left(\frac{w_v I'(w_v)}{I(w_v)} \right)^2.$$

For the extremal $E_n > 0$ the treatment on pp. 140—142 of [4] yields the necessary condition (13), p. 142:

$$H_1 + \bar{K}_1 + \bar{\lambda} e_1 - \lambda = 0$$

⇔

$$X_1 + Z_1 + \bar{Y}_v + \bar{U}_v + \bar{\lambda} e_1 - \lambda = 0.$$

Denote $w_1 = w$. In the expressions $\underset{v}{[\]}$ write $w = t$. Thus, the necessary extremal condition assumes the form

$$\left(\frac{w I'(w)}{I(w)} \right)^2 \underset{1}{[\]} + \underset{3}{[\]} + \left(\frac{w I'(w)}{I(w)} \right)^2 \underset{2}{[\]} + \underset{4}{[\]} + \bar{\lambda} \left(\frac{w I'(w)}{I(w)} \right)^2 - \lambda = 0$$

i.e.,

$$(4) \quad \left(\frac{w I'(w)}{I(w)} \right)^2 \left\{ \underset{1}{\left[\frac{I(t)}{1 - \frac{I(t)}{I(t)}} \right]}_{-n} + \underset{2}{\left[\frac{\overline{I(w)}^{-1}}{1 - \overline{I(w)}^{-1} I(t)^{-1}} \right]}_{-n} + \bar{\lambda} \right\}$$

$$= - \underset{3}{\left[\frac{-w I'(t)}{1 - w t^{-1}} \right]}_{-n} - \underset{4}{\left[\frac{-t I'(t)}{1 - b^2 \bar{w}^{-1} t^{-1}} \right]}_{-n} + \lambda.$$

This is the necessary condition for the inverse \sum_b -function I maximizing $E_n > 0$. The comparison of coefficients in (4) yields in this case $\lambda \in \mathbf{R}$.

The expansions of the functions in the ${}_v\mathbb{I}$ -expressions can be written by the aid of

$$I(t) = t + \sum_0^\infty E_v t^{-v}, \quad I(t)^{-1} = \sum_1^\infty \alpha_v t^{-v} \quad (\alpha_1 = 1)$$

as follows ($I(w) = I$):

$$\begin{aligned} 1\mathbb{I}]: & \frac{I(t)}{1 - I \cdot I(t)^{-1}} = t + (E_0 + I)t^0 + (E_1 + I^2)t^{-1} + (E_2 + \alpha_2 I^2 + I^3)t^{-2} \\ & + (E_3 + \alpha_2 I^2 + 2\alpha_2 I^3 + I^4)t^{-3} + (E_4 + \alpha_4 I^2 + (2\alpha_3 + \alpha_2^2)I^3 + 3\alpha_2 I^4 + I^5)t^{-4} + \dots, \end{aligned}$$

$$\begin{aligned} 2\mathbb{I}]: & \frac{\bar{I}^{-1}}{1 - \bar{I}^{-1}I(t)^{-1}} = \bar{I}^{-1} \cdot t^0 + \bar{I}^{-2}t^{-1} + (\alpha_2 \bar{I}^{-2} + \bar{I}^{-3})t^{-2} \\ & + (\alpha_3 \bar{I}^{-2} + 2\alpha_2 \bar{I}^{-3} + \bar{I}^{-4})t^{-3} + (\alpha_4 \bar{I}^{-2} + (2\alpha_3 + \alpha_2^2)\bar{I}^{-3} + 3\alpha_2 \bar{I}^{-4} + \bar{I}^{-5})t^{-4} + \dots, \end{aligned}$$

$$\begin{aligned} 3\mathbb{I}]: & -\frac{-wI'(t)}{1 - wI(t)^{-1}} = w \cdot t^0 + w^2 t^{-1} + (w^3 - E_1 w)t^{-2} + (w^4 - E_1 w^2 - 2E_2 w)t^{-3} \\ & + (w^5 - E_1 w^3 - 2E_2 w^2 - 3E_3 w)t^{-4} + \dots, \end{aligned}$$

$$\begin{aligned} 4\mathbb{I}]: & -\frac{-tI'(t)}{1 - b^2 \bar{w}^{-1} t^{-1}} = t + b^2 \bar{w}^{-1} \cdot t^0 + (b^4 \bar{w}^{-2} - E_1)t^{-1} \\ & + (b^6 \bar{w}^{-3} - E_1 b^2 \bar{w}^{-1} - 2E_2)t^{-2} + (b^8 \bar{w}^{-4} - E_1 b^4 \bar{w}^{-2} - 2E_2 b^2 \bar{w}^{-1} - 3E_3)t^{-3} \\ & + (b^{10} \bar{w}^{-5} - E_1 b^6 \bar{w}^{-3} - 2E_2 b^4 \bar{w}^{-2} - 3E_3 b^2 \bar{w}^{-1} - 4E_4)t^{-4} + \dots \end{aligned}$$

As an example consider the case $n=4$.

3. The coefficient E_4

The necessary condition (4) for E_4 assumes, according to the above developments, the form ($\lambda \in \mathbf{R}$):

$$\begin{aligned} (5) \quad & \left(\frac{wI'(w)}{I(w)} \right)^2 (I^5 + 3\alpha_2 I^4 + (2\alpha_3 + \alpha_2^2)I^3 + \alpha_4 I^2 + \lambda + E_4 + \bar{\alpha}_4 I^{-2} + \overline{(2\alpha_3 + \alpha_2^2)} I^{-3} \\ & + 3\bar{\alpha}_2 I^{-4} + I^{-5}) = w^5 - E_1 w^3 - 2E_2 w^2 - 3E_3 w + \lambda - 4E_4 - 3\bar{E}_3 b^2 w^{-1} \\ & - 2\bar{E}_2 b^4 w^{-2} - \bar{E}_1 b^6 w^{-3} + b^{10} w^{-4}. \end{aligned}$$

In [1] Netanyahu maximizes the E_v -coefficient for \sum i.e., in the special case $b=0$ by the aid of the variational formula. In agreement with him denote

$$E_v = -\beta_v \quad (v = 0, 1, \dots)$$

and maximize the β_v -coefficients. Because

$$z = I(w) = w - \sum_0^\infty \beta_n w^{-n}$$

and thus

$$\frac{1}{I(w)} = \sum_1^\infty \alpha_v w^{-v} = \frac{1}{w(1 - \sum_0^\infty \beta_v w^{-v-1})},$$

we see that α_v is a polynomial $P_v(\beta_0, \dots, \beta_{v-2})$ with only positive coefficients. For example,

$$\begin{aligned} \alpha_2 &= \beta_0, \\ \alpha_3 &= \beta_0^2 + \beta_1, \\ \alpha_4 &= 2\beta_0\beta_1 + \beta_0^3 + \beta_2. \end{aligned}$$

This remark appears to be useful when estimating coefficients of the differential equation.

Rewrite (5) for the extremal $H(z)$ by using the α_v - and β_v -coefficients. Remember that $E_4 > 0$ so that $\beta_4 = -E_4 < 0$.

$$\begin{aligned} (6) \quad & \left(\frac{zH'}{H} \right)^2 \times \\ & \times (H^5 + \beta_1 H^3 + 2\beta_2 H^2 + 3\beta_3 H + \lambda + 4\beta_4 + 3\bar{\beta}_3 b^2 H^{-1} + 2\bar{\beta}_2 b^4 H^{-2} + \bar{\beta}_1 b^6 H^{-3} + b^{10} H^{-5})^1 \\ & = (z^5 + v_{42} z^4 + v_{43} z^3 + v_{44} z^2 + \lambda - \beta_4 + \bar{v}_{44} z^{-2} + \bar{v}_{43} z^{-3} + \bar{v}_{42} z^{-4} + z^{-5})_2; \\ & v_{42} = 3\alpha_2 = 3\beta_0, \\ & v_{43} = 2\alpha_3 + \alpha_2^2 = 3\beta_0^2 + 2\beta_1, \\ & v_{44} = \alpha_4 = 2\beta_0\beta_1 + \beta_0^3 + \beta_2. \end{aligned}$$

In [3] all the lower coefficients included are maximized essentially by the radial-slit mapping I_r , which satisfies the condition

$$\begin{aligned} (7) \quad & I_r + I_r^{-1} = w - 2(1-b) + b^2 w^{-1}; \\ & I_r(w) = \frac{w + b^2 w^{-1} + 2b - 2}{2} + \frac{w}{2} \times \\ & \times [1 + 4(b-1)w^{-1} + (6b^2 - 8b)w^{-2} + 4b^2(b-1)w^{-3} + b^4 w^{-4}]^{1/2} = w + \sum_0^\infty (-B_v)w^{-v}; \\ & B_0 = 2(1-b), \quad B_1 = 1 - b^2, \\ & B_v = \gamma_{n0} + \gamma_{n1}b + \gamma_{n2}b^2 + \dots + \gamma_{n(n-1)}b^{n-1}; \\ & \gamma_{nv} = (-1)^v \frac{2}{v!} \frac{n-v}{n-v+1} \frac{(2n-v-1)!}{[(n-v)!]^2}. \end{aligned}$$

The numbers B_v are thus the maxima of β_v . Assume here that the inequalities

$$|\beta_v| \leq B_v \quad (v = 0, 1, 2, 3)$$

are true and maximize β_4 by using the necessary condition (6).

As in [3], use the fact that the extremal image-domain has at least one slit, i.e., $()_1=0$ at the starting point $be^{i\psi}$ of the slit and $\lambda+4\beta_4$ is determined by $()_1\cong 0$, $\psi\in[0, 2\pi]$, that is,

$$\begin{aligned} \lambda+4\beta_4 &= \max [-2\operatorname{Re} \{b^5 e^{i5\psi} + \beta_1 b^3 e^{i3\psi} + 2\beta_2 b^2 e^{i2\psi} + 3\beta_3 b e^{i\psi}\}] \\ &= \max [2\operatorname{Re} \{-b^5 e^{i5\psi} + E_1 b^3 e^{i3\psi} + 2E_2 b^2 e^{i2\psi} + 3E_3 b e^{i\psi}\}] \\ &\cong 2(b^5 + |\beta_1| b^3 + 2|\beta_2| b^2 + 3b|\beta_3|) \\ &\cong 2(b^5 + b^3 B_1 + 2b^2 B_2 + 3b B_3) \\ &= 2(15b - 20b^2 + 6b^3). \end{aligned}$$

Equality here holds for $\psi=0$ and $I=I_r$, which has $E_v = -B_v < 0$, when properly rotated. Thus

$$\lambda+4\beta_4 = 2(15b - 20b^2 + 6b^3).$$

$()_2$ has at least one zero $e^{i\varphi}$, the pre-image of the endpoint of the slit. Thus $\beta_4 - \lambda = -5|\beta_4| - (30b - 40b^2 + 12b^3) = -2\operatorname{Re} \{e^{i5\varphi} + v_{42} e^{i4\varphi} + v_{43} e^{i3\varphi} + v_{44} e^{i2\varphi}\}$;
 $5|\beta_4| \cong -(30b - 40b^2 + 12b^3) + 2(1 + |v_{42}| + |v_{43}| + |v_{44}|).$

The numbers $|v_{nv}|$ are maximized with $|\beta_v|$:

$$\begin{aligned} |v_{42}| &\cong 6(1 - b), \\ |v_{43}| &\cong 14 - 24b + 10b^2, \\ |v_{44}| &\cong 14 - 30b + 20b^2 - 4b^3; \\ |\beta_4| &\cong \frac{1}{5} [-(30b - 40b^2 + 12b^3) + 70 - 120b + 60b^2 - 8b^3] \\ &= 14 - 30b + 20b^2 - 4b^3. \end{aligned}$$

Again, the equality belongs essentially to I_r , because this holds for all the lower coefficients used in the triangle inequality estimations. — This result is the one proved in the above manner in [3].

4. The general case

From (4) we obtain for $E_n = -\beta_n > 0$ the necessary condition for the extremal H :

$$\begin{aligned} (8) \quad &\left(\frac{zH'}{H}\right)^2 \times \\ &\times (H^{n+1} + \sum_1^{n-1} v\beta_v H^{n-v} + \lambda + n\beta_n + \sum_{n-1}^1 v\beta_v b^{2n-2v} H^{-n+v} + b^{2n+2} H^{-n-1})_1 \\ &= (z^{n+1} + \sum_2^n v_{nv} z^{n+2-v} + \lambda - \beta_n + \sum_n^2 \bar{v}_{nv} z^{-n-2+v} + z^{-n-1})_2. \end{aligned}$$

The numbers v_{nv} are polynomials $Q_v(\beta_0, \dots, \beta_{v-2})$ with only positive coefficients.

Assume that the lower coefficients $\beta_0, \dots, \beta_{n-1}$ are all maximized by I_r defined by (7). We proceed in maximizing $|\beta_n|$ by imitating the procedure of Section 3. Thus, $()_1$ has at least one zero $be^{i\psi}$ which determines $\lambda+n\beta_n$ so that $()_1 \cong 0$ for $\psi \in [0, 2\pi]$:

$$\begin{aligned} \lambda + n\beta_n &= \max \left[-2\operatorname{Re} \left\{ b^{n+1} e^{i(n+1)\psi} + \sum_1^{n-1} v\beta_v b^{n-v} e^{i(n-v)\psi} \right\} \right] \\ &= \max \left[2\operatorname{Re} \left\{ -b^{n+1} e^{i(n+1)\psi} + \sum_1^{n-1} vE_v b^{n-v} e^{i(n-v)\psi} \right\} \right] \\ &\leq 2(b^{n+1} + \sum_1^{n-1} v|\beta_v| b^{n-v}) \\ &\leq 2(b^{n+1} + \sum_1^{n-1} vB_v b^{n-v}) \\ &= 2 \sum_1^{n-1} C_v b^v; \end{aligned}$$

$$C_v = \frac{(-1)^{v-1}}{(v-1)!} \frac{n-v}{n-v+1} \frac{(2n-v-1)!}{[(n-v)!]^2} = -\frac{v}{2} \gamma_{nv} \quad (v = 1, 2, \dots, n-1).$$

Again, equality holds for $\psi=0$ and for a rotated I_r ; hence

$$\lambda + n\beta_n = 2 \sum_1^{n-1} C_v b^v.$$

$()_2 = 0$ at some $e^{i\varphi}$ and thus

$$\begin{aligned} \beta_n - \lambda &= -(n+1)|\beta_n| - 2 \sum_1^{n-1} C_v b^v \\ &= 2\operatorname{Re} \left\{ e^{i(n+1)\varphi} + \sum_2^n v_{nv} e^{i(n+2-v)\varphi} \right\}; \end{aligned}$$

$$(9) \quad (n+1)|\beta_n| \leq -2 \sum_1^{n-1} C_v b^v + 2(1 + \sum_2^n |v_{nv}|)_0.$$

In order to estimate $()_0$ we return to the general form (4) of the differential equation. The coefficients v_{nv} are defined by $()_1$, i.e., they are those of

$$\frac{I(t)}{1-I \cdot I(t)^{-1}} = I(t) + I + I^2 \cdot I(t)^{-1} + I^3 \cdot I(t)^{-2} + \dots$$

We are actually interested in the non-constant term of the I -polynomials (cf. the expansion of $()_1$). Because the constant term E_v is due to the first term $I(t)$, we may confine ourselves to the expansion

$$\begin{aligned} F(t) &= \frac{I(t)}{1-I \cdot I(t)^{-1}} - I(t) = \frac{I}{1-I \cdot I(t)^{-1}} \\ &= I \cdot t^0 + I^2 \cdot t^{-1} + (\alpha_2 I^2 + I^3) t^{-2} + (\alpha_3 I^2 + 2\alpha_2 I^3 + I^4) t^{-3} \\ &\quad + (\alpha_4 I^2 + (2\alpha_3 + \alpha_2^2) I^3 + 3\alpha_2 I^4 + I^5) t^{-4} + \dots \\ &= I \cdot t^0 + I^2 \cdot t^{-1} + \sum_{n=2}^\infty \left(\sum_{v=2}^n v_{n(n-v+2)} I^v + I^{n+1} \right) t^{-n}. \end{aligned}$$

Because the numbers v_{nv} are polynomials of $\beta_0, \dots, \beta_{v-2}$ with only positive coefficients, we obtain their maxima V_{nv} by writing $\beta_k = B_k$ in those polynomials. Actually,

we need only the maximal sum $u_n = 1 + \sum_{v=2}^n V_{nv}$, which is thus obtained as the coefficient u_n of $F(t)$ with $I=1$ and $I_r(t)$ defined by (7) ($w=t$). Thus we have the expansion

$$\begin{aligned} I_r(t) &= t - B_0 - B_1 t^{-1} - B_2 t^{-2} - \dots \\ \frac{I_r(t)}{I_r(t)-1} &= 1 + t^{-1} + u_2 t^{-2} + u_3 t^{-3} + \dots; \\ u_n &= 1 + \sum_{v=2}^n V_{nv}. \end{aligned}$$

The multiplication of the u_n -expansion by

$$I_r(t) - 1 = t - (1 + B_0) - B_1 t^{-1} - B_2 t^{-2} - \dots$$

yields the recursion formula:

$$\begin{aligned} u_2 &= 1 + B_0, \\ u_3 &= (1 + B_0)u_2 + B_1, \\ &\dots \\ u_n &= (1 + B_0)u_{n-1} + \sum_{v=2}^{n-1} B_{v-1}u_{n-v} \quad (u_1 = 1; n = 2, 3, \dots). \end{aligned}$$

This gives

$$(10) \quad u_n = \sum_{v=0}^{n-1} \frac{(-1)^v}{v!} (n-v) \frac{(2n-v-1)!}{[(n-v)!]^2} b^v = \sum_{v=0}^{n-1} \frac{n-v+1}{2} \gamma_{nv} b^v.$$

According to (9) we now obtain

$$\begin{aligned} (n+1)|\beta_n| &\leq 2 \sum_1^{n-1} \frac{v}{2} \gamma_{nv} b^v + 2 \sum_0^{n-1} \frac{n-v+1}{2} \gamma_{nv} b^v \\ &= (n+1) \sum_0^{n-1} \gamma_{nv} b^v; \end{aligned}$$

$$(11) \quad |\beta_n| \leq \sum_0^{n-1} \gamma_{nv} b^v = B_n.$$

Because the lower coefficients $\beta_0, \dots, \beta_{n-1}$ are all maximized, save the rotation, by I_r , the same holds in (11) and thus for all n . Hence we have generalized the result of Netanyahu [1].

Theorem. For the inverse \sum_b -functions

$$I(w) = w - \sum_0^\infty \beta_v w^{-v}, \quad |w| > 1,$$

the coefficient estimation

$$|\beta_0| \leq 2(1-b), \quad |\beta_1| \leq 1-b^2, \quad |\beta_n| \leq \sum_0^{n-1} \gamma_{nv} b^v \quad (n = 2, 3, \dots);$$

$$\gamma_{nv} = (-1)^v \frac{2}{v!} \frac{n-v}{n-v+1} \frac{(2n-v-1)!}{[(n-v)!]^2} \quad (v = 0, \dots, n-1)$$

holds.

Essentially the only maximizing function is the radial-slit mapping I_r defined by

$$I_r + I_r^{-1} = w - 2(1 - b) + b^2 w^{-1}.$$

The coefficient β_1 has a one-parametric family of maximizing functions I from

$$I + I^{-1} = w - c + b^2 w^{-1}$$

with $c \in [-2(1 - b), 2(1 - b)]$ as a parameter (cf. [3]).

This result generalizes the classical theorem of Netanyahu [1] showing that also in the cases other than $b=0$ the radial-slit mapping preserves its extremal role with respect to the inverse coefficients.

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